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Bi-Periodic (p, q)-Fibonacci and Bi-Periodic (p, q)-Lucas Sequences

Yasemin TAŞYURDU *[1], Naime Şeyda TÜRKOĞLU[1]

Abstract

In this paper, we define bi-periodic (p, q)-Fibonacci and bi-periodic (p, q)-Lucas sequences, which generalize Fibonacci type, Lucas type, bi-periodic Fibonacci type and bi-periodic Lucas type sequences, using recurrence relations of (p, q)-Fibonacci and (p, q)-Lucas sequences. Generating functions and Binet formulas that allow us to calculate the nth terms of these sequences are given and the convergence properties of their consecutive terms are examined. Also, we prove some fundamental identities of bi-periodic (p, q)-Fibonacci and bi-periodic (p, q)-Lucas sequences conform to the well-known properties of Fibonacci and Lucas sequences.

Keywords: Bi-periodic Fibonacci numbers**,** Fibonacci number, generalized Fibonacci numbers, bi-periodic Lucas numbers**,** Lucas number

1. INTRODUCTION

Fibonacci sequence, $\{F_n\}_{n\in\mathbb{N}}$ is introduced by recurrence relation $F_n = F_{n-1} + F_{n-2}$ with initial terms $F_0 = 0$, $F_1 = 1$ for $n \geq 2$. The most interesting applications of this sequence have been on its generalizations, also called families of Fibonacci sequence. For instance, Lucas sequence, $\{L_n\}_{n\in\mathbb{N}}$ is introduced by recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial terms $L_0 = 2$, $L_1 = 1$ for $n \geq 2$ using different the initial terms and recurrence relation similar to the Fibonacci sequence [1]. Then, k-Fibonacci sequence by recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ with initial terms $F_{k,0} = 0$, $F_{k,1} = 1$ and k-Lucas sequence by recurrence relation $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ with initial terms $L_{k,0} = 2$, $L_{k,1} = k$ are determined according to parameter k [2, 3]. For more details on generalizations, see [4-7].

Further generalizations of the Fibonacci and Lucas sequences are presented according to parameters p and q. For integers $p, q \geq 1$, the (p, q)-Fibonacci sequence is presented by recurrence relation

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, \; n \geq 2 \quad (1)$$

with initial terms $F_{p,q,0} = 0, F_{p,q,1} = 1$, and the (p, q)-Lucas sequence is presented by recurrence relation

$$L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2}, \; n \geq 2 \quad (2)$$

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with initial terms $L_{p,q,0}=2, L_{p,q,1}=p$. The Binet formulas, which are the general formulas for the nth terms of these two sequences, are given by

$$F_{p,q,n}=\frac{\sigma_1^n-\sigma_2^n}{r_1-r_2}$$

$$L_{p,q,n}=\sigma_1^n+\sigma_2^n$$

where $\sigma_1=\frac{p+\sqrt{p^2+4q}}{2}$ and $\sigma_2=\frac{p-\sqrt{p^2+4q}}{2}$. Also, main properties of these sequences are presented by generalizing with the Binet formulas [8-11].

As other generalizations of the Fibonacci and the Lucas sequences, bi-periodic Fibonacci sequence, $\{q_n\}$ is defined by

$$q_n=\begin{cases}aq_{n-1}+q_{n-2}, & \text{if } n \text{ is even}\\ bq_{n-1}+q_{n-2}, & \text{if } n \text{ is odd}\end{cases} \quad n\geq 2$$

with initial terms $q_0=0,\ q_1=1$ [12], and bi-periodic Lucas sequence, $\{l_n\}$ is defined by

$$l_n=\begin{cases}bl_{n-1}+l_{n-2}, & \text{if } n \text{ is even}\\ al_{n-1}+l_{n-2}, & \text{if } n \text{ is odd}\end{cases} \quad n\geq 2$$

with initial terms $l_0=2,\quad l_1=a$ where a and b are any two nonzero real numbers [13]. Also, a matrix related to the bi-periodic Fibonacci sequence is defined and some interesting identities for this sequence are given [14, 15]. Then, these studies lead to the discovery of many studies, called bi-periodic sequences, using recurrence relations and initial terms of special sequences of integers, such as Jacobsthal, Jacobsthal-Lucas, Pell and Pell-Lucas, sequences, etc. In [16], some properties of bi-periodic Horadam sequences are given by generalizing the known properties related to the bi-periodic Fibonacci and the bi-periodic Lucas sequences. In [17, 18], bi-periodic Jacobsthal and bi-periodic Jacobsthal-Lucas sequences are defined, and then in [19, 20], bi-periodic Pell and bi-periodic Pell-Lucas sequences are defined.

The aim of this study is to define new generalizations of the bi-periodic Fibonacci type and the bi-periodic Lucas type sequences, which we shall call bi-periodic (p,q)-Fibonacci and bi-periodic (p,q)-Lucas sequences, using recurrence relations of the (p,q)-Fibonacci and the (p,q)-Lucas sequences, respectively. It is to present general formulas and well-known identities conform to sequences of integers for these sequences. It is also to give special cases of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences and generalize all the results.

2. MAIN RESULTS

In this section, new generalizations of both the Fibonacci type sequences, Lucas type sequences and the (p,q)-Fibonacci sequence, the (p,q)-Lucas sequence, called bi-periodic (p,q)-Fibonacci sequence and bi-periodic (p,q)-Lucas sequence, are presented taking into account that the recurrence relations of the (p,q)-Fibonacci sequence and the (p,q)-Lucas sequences, respectively. Generating functions, Binet formulas, some basic properties as well as the Catalan's identity, Cassini's identity, d'Ocagne's identity for these sequences are obtained.

Definition 1. For integers $p,q\geq 1$ and any two nonzero real numbers a and b, the bi-periodic (p,q)-Fibonacci sequence, say $\{F_n(p,q)\}_{n\epsilon\mathbb{N}}$, is defined by the recurrence relation

$$F_n(p,q)=\begin{cases}apF_{n-1}(p,q)+qF_{n-2}(p,q), & \text{if } n \text{ is even}\\ bpF_{n-1}(p,q)+qF_{n-2}(p,q), & \text{if } n \text{ is odd}\end{cases} \quad (3)$$

with initial terms $F_0(p,q)=0,\ F_1(p,q)=1$ for $n\geq 2$, and the bi-periodic (p,q)-Lucas sequence, say $\{L_n(p,q)\}_{n\epsilon\mathbb{N}}$, is defined by the recurrence relation

$$L_n(p,q) = \begin{cases} bpL_{n-1}(p,q) + qL_{n-2}(p,q), & \text{if } n \text{ is even} \\ apL_{n-1}(p,q) + qL_{n-2}(p,q), & \text{if } n \text{ is odd} \end{cases} \quad (4)$$

with initial terms $L_0(p,q) = 2,\ L_1(p,q) = ap$ for $n \geq 2$. The nth bi-periodic (p,q)-Fibonacci number is denoted by $F_n(p,q)$ and the nth bi-periodic (p,q)-Lucas number is denoted by $L_n(p,q)$.

From Definition 1, both sequences are as follows

$$\{F_n(p,q)\}_{n\epsilon\mathbb{N}} = \{0,1, ap, abp^2 + q, a^2bp^3 + 2apq, a^2b^2p^4 + 3abp^2q + q^2, a^3b^2p^5 + 4a^2bp^3q + 3apq^2, a^3b^3p^6 + 5a^2b^2p^4q + 6abp^2q^2 + q^3, a^4b^3p^7 + 6a^3b^2p^5q + 10a^2bp^3q^2 + 4apq^3, \dots\}$$

and

$$\{L_n(p,q)\}_{n\epsilon\mathbb{N}} = \{2, ap, abp^2 + 2q, a^2bp^3 + 3apq, a^2b^2p^4 + 4abp^2q + 2q^2, a^3b^2p^5 + 5a^2bp^3q + 5apq^2, a^3b^3p^6 + 6a^2b^2p^4q + 9abp^2q^2 + 2q^3, a^4b^3p^7 + 7a^3b^2p^5q + 14a^2bp^3q^2 + 7apq^3, a^4b^4p^8 + 8a^3b^3p^6q + 20a^2b^2p^4q^2 + 16abp^2q^3 + 2q^4, \dots\}$$

respectively.

Alternative recurrence relations can be given for the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q) Lucas sequences. For integers $p,q \geq 1$ and any two nonzero real numbers a and b, the bi-periodic (p,q)-Fibonacci sequence is given by

$$F_n(p,q) = a^{1-\xi(n)}b^{\xi(n)}pF_{n-1}(p,q) + qF_{n-2}(p,q) \quad (5)$$

with initial terms $F_0(p,q) = 0, F_1(p,q) = 1$ for $n \geq 2$, and the bi-periodic (p,q)-Lucas sequence is given by

$$L_n(p,q) = a^{\xi(n)}b^{1-\xi(n)}pL_{n-1}(p,q) + qL_{n-2}(p,q) \quad (6)$$

with initial terms $L_0(p,q) = 2,\ L_1(p,q) = ap$ for $n \geq 2$, and where $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e.,

$$\xi(n) = \begin{cases} 0, & if\ n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

respectively. Then we have the quadratic equation $x^2 - pabx - qab = 0$ with roots $\sigma = \frac{pab+\sqrt{p^2a^2b^2+4qab}}{2}$ and $\rho = \frac{pab-\sqrt{p^2a^2b^2+4qab}}{2}$ for the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences.

Note that the roots σ and ρ have the following relations

$$\sigma + \rho = pab$$

$$\sigma\rho = -qab$$

$$\sigma - \rho = \sqrt{p^2a^2b^2 + 4qab}$$

$$p\sigma + q = \frac{\sigma^2}{ab}$$

$$p\rho + q = \frac{\rho^2}{ab}.$$

Definition 1 generalizes many the bi-periodic Fibonacci type and the bi-periodic Lucas type sequences. Special cases of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences obtained according to parameters p,q are presented in the Table 1. Since the all results given throughout the study are provided for all the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences, the values given in Table 1 can be used in the relevant theorem or corollary for any bi-periodic sequences which are the bi-periodic Fibonacci type sequences and the bi-periodic Lucas type sequences.

Table 1 Special cases of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas numbers

p	***q***	***Symbol***	***Generalized nth bi-periodic number***
1	1	$F_n(1,1)=q_n$	q_n, nth bi-periodic Fibonacci number [12]
k	1	$F_n(k,1)=q_{k,n}$	$q_{k,n}$, nth bi-periodic k-Fibonacci number
1	1	$L_n(1,1)=l_n$	l_n, nth bi-periodic Lucas number [13]
k	1	$L_n(k,1)=l_{k,n}$	$l_{k,n}$, nth bi-periodic k-Lucas number
1	2	$F_n(1,2)=J_n$	J_n, nth bi-periodic Jacobsthal number [17]
k	2	$F_n(k,2)=J_{k,n}$	$J_{k,n}$, nth bi-periodic k-Jacobsthal number
1	2	$L_n(1,2)=c_n$	c_n, nth bi-periodic Jacobsthal-Lucas number [18]
k	2	$L_n(k,2)=c_{k,n}$	$c_{k,n}$, nth bi-periodic k-Jacobsthal-Lucas number
2	1	$F_n(2,1)=P_n$	P_n, nth bi-periodic Pell number [19]
2	k	$F_n(2,k)=P_{k,n}$	$P_{k,n}$, nth bi-periodic k-Pell number
2	1	$L_n(2,1)=Q_n$	Q_n, nth bi-periodic Pell-Lucas number [20]
2	k	$L_n(2,k)=Q_{k,n}$	$Q_{k,n}$, nth bi-periodic k-Pell-Lucas number

Using Definition 1, some identities for the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas numbers are given in the following lemma.

Lemma 2. The bi-periodic (p,q)-Fibonacci sequence, $\{F_n(p,q)\}_{n\in\mathbb{N}}$ and the bi-periodic (p,q)-Lucas sequence, $\{L_n(p,q)\}_{n\in\mathbb{N}}$ satisfy the following identities

i. $F_{2n}(p,q)=(abp^2+2q)F_{2n-2}(p,q)-q^2F_{2n-4}(p,q)$

ii. $F_{2n+1}(p,q)=(abp^2+2q)F_{2n-1}(p,q)-q^2F_{2n-3}(p,q)$

iii. $L_{2n}(p,q)=(abp^2+2q)L_{2n-2}(p,q)-q^2L_{2n-4}(p,q)$

iv. $L_{2n+1}(p,q)=(abp^2+2q)L_{2n-1}(p,q)-q^2L_{2n-3}(p,q).$

Proof. Using the equation (3)

i.
$$F_{2n}(p,q)=apF_{2n-1}(p,q)+qF_{2n-2}(p,q)$$
$$=ap\big(bpF_{2n-2}(p,q)+qF_{2n-3}(p,q)\big)+qF_{2n-2}(p,q)$$
$$=(abp^2+q)F_{2n-2}(p,q)+apqF_{2n-3}(p,q)$$
$$=(abp^2+q)F_{2n-2}(p,q)+qF_{2n-2}(p,q)-q^2F_{2n-4}(p,q)$$
$$=(abp^2+2q)F_{2n-2}(p,q)-q^2F_{2n-4}(p,q)$$

ii.
$$F_{2n+1}(p,q)=bpF_{2n}(p,q)+qF_{2n-1}(p,q)$$
$$=bp\big(apF_{2n-1}(p,q)+qF_{2n-2}(p,q)\big)+qF_{2n-1}(p,q)$$
$$=(abp^2+q)F_{2n-1}(p,q)+bpqF_{2n-2}(p,q)$$
$$=(abp^2+q)F_{2n-1}(p,q)+qF_{2n-1}(p,q)-q^2F_{2n-3}(p,q)$$
$$=(abp^2+2q)F_{2n-1}(p,q)-q^2F_{2n-3}(p,q)$$

and the i. and ii. are proved. Similarly, the proofs of iii. and iv. can be proved using equation (4).

Now we introduce generating functions of the sequences $\{F_n(p,q)\}_{n\epsilon\mathbb{N}}$ and $\{L_n(p,q)\}_{n\epsilon\mathbb{N}}$. Let the generating function of the sequence $\{F_n(p,q)\}_{n\epsilon\mathbb{N}}$ be $F(x)$ and let the generating function of the sequence $\{L_n(p,q)\}_{n\epsilon\mathbb{N}}$ be $L(x)$. Then, we get the following

$$F(x) = \sum_{n=0}^{\infty} F_n(p,q)x^n$$

$$L(x) = \sum_{n=0}^{\infty} L_n(p,q)x^n$$

where $F_n(p,q)$ is the nth bi-periodic (p,q)-Fibonacci number and $L_n(p,q)$ is the nth bi-periodic (p,q)-Lucas number. By the following theorem, the generating functions of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences are given.

Theorem 3. The generating functions of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences are

$$F(x) = \frac{x + apx^2 - qx^3}{1 - (abp^2 + 2q)x^2 + q^2x^4}$$

$$L(x) = \frac{2 + apx - (abp^2 + 2q)x^2 + apqx^3}{1 - (abp^2 + 2q)x^2 + q^2x^4}$$

respectively.

Proof. Let $F(x)$ be the generating function of the sequence $\{F_n(p,q)\}$. Then

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} F_n(p,q)x^n \\ &= F_0(p,q) + F_1(p,q)x + F_2(p,q)x^2 \\ &\quad + \cdots + F_n(p,q)x^n + \cdots \end{aligned}$$

If we divide the generating function such that the sum of the even subscript terms is $F_Ç(x)$ and the sum of the odd subscript terms is $F_T(x)$. Therefore,

$$F_Ç(x) = F_0(p,q) + F_2(p,q)x^2 + \sum_{i=2}^{\infty} F_{2i}(p,q)x^{2i} \quad (7)$$

If both sides of equation (7) are multiplied by $-(abp^2 + 2q)x^2$ and q^2x^4, then we obtain

$$-(abp^2 + 2q)x^2F_Ç(x) = -(abp^2 + 2q) \sum_{i=0}^{\infty} F_{2i}(p,q)x^{2i+2} \quad (8)$$

and

$$q^2x^4F_Ç(x) = q^2 \sum_{i=0}^{\infty} F_{2i}(p,q)x^{2i+4} \quad (9)$$

From the equations (7), (8) and (9), we have

$$\begin{aligned} &(1 - (abp^2 + 2q)x^2 + q^2x^4)F_Ç(x) \\ &= F_0(p,q) + F_2(p,q)x^2 + \sum_{i=2}^{\infty} F_{2i}(p,q)x^{2i} \\ &\quad -(abp^2 + 2q)\sum_{i=0}^{\infty} F_{2i}(p,q)x^{2i+2} \\ &\quad +q^2\sum_{i=0}^{\infty} F_{2i}(p,q)x^{2i+4} \\ &= apx^2 + \sum_{i=2}^{\infty} F_{2i}(p,q)x^{2i} \\ &\quad -(abp^2 + 2q)\sum_{i=2}^{\infty} F_{2i-2}(p,q)x^{2i} \\ &\quad +q^2\sum_{i=2}^{\infty} F_{2i-4}(p,q)x^{2i} \\ &= apx^2 + \sum_{i=2}^{\infty} \big(F_{2i}(p,q) \\ &\qquad -(abp^2 + 2q)F_{2i-2}(p,q) \\ &\qquad +q^2F_{2i-4}(p,q)\big)x^{2i}. \end{aligned}$$

Using Lemma 2, i., generating function of the bi-periodic (p,q)-Fibonacci sequence with even subscript terms is obtained as

$$F_{Ç}(x)=\frac{apx^2}{1-(abp^2+2q)x^2+q^2x^4}.$$

Now let consider the sum of the odd subscript terms in the generating function. Therefore,

$$F_T(x)=F_1(p,q)x+F_3(p,q)x^3 + \sum_{i=2}^{\infty}F_{2i+1}(p,q)x^{2i+1} \quad (10)$$

If both sides of equation (10) are multiplied by $-(abp^2+2q)x^2$ and q^2x^4, then we obtain

$$-(abp^2+2q)x^2F_T(x) = -(abp^2+2q)\sum_{i=0}^{\infty}F_{2i+1}(p,q)x^{2i+3} \quad (11)$$

and

$$q^2x^4F_T(x)=q^2\sum_{i=0}^{\infty}F_{2i+1}(p,q)x^{2i+5} \quad (12)$$

From the equations (10), (11) and (12), we have

$$(1-(abp^2+2q)x^2+q^2x^4)F_T(x)$$

$$=F_1(p,q)x+F_3(p,q)x^3$$

$$+\sum_{i=2}^{\infty}F_{2i+1}(p,q)x^{2i+1}$$

$$-(abp^2+2q)x^3F_1(p,q)$$

$$-(abp^2+2q)\sum_{i=1}^{\infty}F_{2i+1}(p,q)x^{2i+3}$$

$$+q^2\sum_{i=0}^{\infty}F_{2i+1}(p,q)x^{2i+5}$$

$$=x+(abp^2+q)x^3$$

$$+\sum_{i=2}^{\infty}F_{2i+1}(p,q)x^{2i+1}$$

$$-(abp^2+2q)x^3$$

$$-(abp^2+2q)\sum_{i=2}^{\infty}F_{2i-1}(p,q)x^{2i+1}$$

$$+q^2\sum_{i=2}^{\infty}F_{2i-3}(p,q)x^{2i+1}$$

$$=x+(abp^2+q)x^3-(abp^2+2q)x^3$$

$$+\sum_{i=2}^{\infty}\Big(F_{2i+1}(p,q)-(abp^2+2q)F_{2i-1}(p,q)$$

$$+q^2F_{2i-3}(p,q)\Big)x^{2i+1}.$$

Using Lemma 2, ii., generating function of the bi-periodic (p,q)-Fibonacci sequence with odd subscript terms is obtained as

$$F_T(x)=\frac{x-qx^3}{1-(abp^2+2q)x^2+q^2x^4}.$$

From $F(x)=F_{Ç}(x)+F_T(x)$, the generating function of the bi-periodic (p,q)-Fibonacci sequence is obtained as

$$F(x)=\frac{x+apx^2-qx^3}{1-(abp^2+2q)x^2+q^2x^4}.$$

Similarly, the generating function of the bi-periodic (p,q)-Lucas sequence using Lemma 2, iii. and iv., is obtained as

$$L(x)=\frac{2+apx-(abp^2+2q)x^2+apqx^3}{1-(abp^2+2q)x^2+q^2x^4}.$$

Thus, the proof is completed.

Now we give Binet formulas that allow us to calculate the nth terms of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences with the following theorem.

Theorem 4. The Binet formulas for the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences are given by

$$F_n(p,q)=\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\frac{\sigma^n-\rho^n}{\sigma-\rho}$$

$$L_n(p,q) = \left(\frac{a^{\xi(n)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)(\sigma^n + \rho^n)$$

where $\sigma = \frac{pab+\sqrt{p^2a^2b^2+4qab}}{2}$, $\rho = \frac{pab-\sqrt{p^2a^2b^2+4qab}}{2}$ and $\xi(n) = n - 2\left[\frac{n}{2}\right]$.

Proof. We complete the proof by induction method on n. The result is obviously valid for $n = 0,1$. Suppose that result is true for $n \in \mathbb{N}$, namely

$$F_n(p,q) = \left(\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\frac{\sigma^n - \rho^n}{\sigma - \rho}.$$

Using equation (5) and the hypothesis of induction, we shall show that it is true for $n+1$. Then, we have

$$F_{n+1}(p,q) = a^{1-\xi(n+1)}b^{\xi(n+1)}pF_n(p,q) + qF_{n-1}(p,q)$$

$$= a^{1-\xi(n+1)}b^{\xi(n+1)}p\left(\left(\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\frac{\sigma^n - \rho^n}{\sigma - \rho}\right) + q\left(\left(\frac{a^{1-\xi(n-1)}}{(ab)^{\left\lfloor\frac{n-1}{2}\right\rfloor}}\right)\frac{\sigma^{n-1} - \rho^{n-1}}{\sigma - \rho}\right)$$

$$= \frac{a^{1-\xi(n+1)}\sigma^{n-1}}{\sigma - \rho}\left(\frac{abp\sigma}{a^{\xi(n)}b^{1-\xi(n+1)}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}} + \frac{abq}{(ab)^{\left\lfloor\frac{n-1}{2}\right\rfloor+1}}\right) - \frac{a^{1-\xi(n+1)}\rho^{n-1}}{\sigma - \rho}\left(\frac{abp\rho}{a^{\xi(n)}b^{1-\xi(n+1)}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}} + \frac{abq}{(ab)^{\left\lfloor\frac{n-1}{2}\right\rfloor+1}}\right)$$

$$= \frac{a^{1-\xi(n+1)}\sigma^{n-1}}{\sigma - \rho}\left(\frac{ab(p\sigma + q)}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right) - \frac{a^{1-\xi(n+1)}\rho^{n-1}}{\sigma - \rho}\left(\frac{ab(p\rho + q)}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)$$

$$= \frac{a^{1-\xi(n+1)}\sigma^{n-1}}{\sigma - \rho}\left(\frac{\sigma^2}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right) - \frac{a^{1-\xi(n+1)}\rho^{n-1}}{\sigma - \rho}\left(\frac{\rho^2}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)$$

$$= \left(\frac{a^{1-\xi(n+1)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)\frac{\sigma^{n+1} - \rho^{n+1}}{\sigma - \rho}$$

where $p\sigma + q = \frac{\sigma^2}{ab}$, $p\rho + q = \frac{\rho^2}{ab}$ and $\xi(n) + \left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor\frac{n+1}{2}\right\rfloor$, $1 - \xi(n+1) + \left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor\frac{n+1}{2}\right\rfloor$.

Similarly, the Binet formula for the bi-periodic (p,q)-Lucas sequence can be obtained using equation (6) and induction method on n. This completes the proof.

Theorem 5. The limit of the ratio of consecutive terms of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences is

i. $\lim_{n\to\infty} \frac{F_{2n+1}(p,q)}{F_{2n}(p,q)} = \frac{\sigma}{a}$

ii. $\lim_{n\to\infty} \frac{F_{2n}(p,q)}{F_{2n-1}(p,q)} = \frac{\sigma}{b}$

iii. $\lim_{n\to\infty} \frac{L_{2n+1}(p,q)}{L_{2n}(p,q)} = \frac{\sigma}{b}$

iv. $\lim_{n\to\infty} \frac{L_{2n}(p,q)}{L_{2n-1}(p,q)} = \frac{\sigma}{a}$

where $F_n(p,q)$ is the nth bi-periodic (p,q)-Fibonacci number and $L_n(p,q)$ is the nth bi-periodic (p,q)-Lucas number.

Proof. Using Binet formula of the bi-periodic (p,q)-Fibonacci sequence given in Theorem 4, we have

i. $\lim_{n\to\infty}\frac{F_{2n+1}(p,q)}{F_{2n}(p,q)}$

$$= \lim_{n\to\infty}\frac{\left(\frac{a^{1-\xi(2n+1)}}{(ab)^{\left\lfloor\frac{2n+1}{2}\right\rfloor}}\right)\left(\frac{\sigma^{2n+1}-\rho^{2n+1}}{\sigma-\rho}\right)}{\left(\frac{a^{1-\xi(2n)}}{(ab)^{\left\lfloor\frac{2n}{2}\right\rfloor}}\right)\left(\frac{\sigma^{2n}-\rho^{2n}}{\sigma-\rho}\right)}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{(ab)^n}\left(\frac{\sigma^{2n+1}-\rho^{2n+1}}{\sigma-\rho}\right)}{\frac{a}{(ab)^n}\left(\frac{\sigma^{2n}-\rho^{2n}}{\sigma-\rho}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{a}\frac{\sigma^{2n+1}\left(1-\left(\frac{\rho}{\sigma}\right)^{2n+1}\right)}{\sigma^{2n}\left(1-\left(\frac{\rho}{\sigma}\right)^{2n}\right)}$$

$$= \frac{\sigma}{a}$$

and

ii. $\lim_{n\to\infty}\frac{F_{2n}(p,q)}{F_{2n-1}(p,q)}$

$$= \lim_{n\to\infty}\frac{\left(\frac{a^{1-\xi(2n)}}{(ab)^{\left\lfloor\frac{2n}{2}\right\rfloor}}\right)\left(\frac{\sigma^{2n}-\rho^{2n}}{\sigma-\rho}\right)}{\left(\frac{a^{1-\xi(2n-1)}}{(ab)^{\left\lfloor\frac{2n-1}{2}\right\rfloor}}\right)\left(\frac{\sigma^{2n-1}-\rho^{2n-1}}{\sigma-\rho}\right)}$$

$$= \lim_{n\to\infty}\frac{\frac{a}{(ab)^n}\left(\frac{\sigma^{2n}-\rho^{2n}}{\sigma-\rho}\right)}{\frac{1}{(ab)^{n-1}}\left(\frac{\sigma^{2n-1}-\rho^{2n-1}}{\sigma-\rho}\right)}$$

$$= \lim_{n\to\infty}\frac{a}{ab}\frac{\sigma^{2n}\left(1-\left(\frac{\rho}{\sigma}\right)^{2n}\right)}{\sigma^{2n-1}\left(1-\left(\frac{\rho}{\sigma}\right)^{2n-1}\right)}$$

$$= \frac{\sigma}{b}$$

where $|\rho| < \sigma$ and $\lim_{n\to\infty}\left(\frac{\rho}{\sigma}\right)^n = 0$. Similarly, the proofs of iii. and iv. can be proved using Binet formula of the bi-periodic (p,q)-Lucas sequence given in Theorem 4. This completes the proof.

Theorem 6. Negative subscript terms of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences are obtained as

$$F_{-n}(p,q) = -(-q)^{-n}F_n(p,q)$$

$$L_{-n}(p,q) = (-q)^{-n}L_n(p,q)$$

respectively.

Proof. Using Binet formulas of the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences given in Theorem 4, we obtain

$$F_{-n}(p,q) = \left(\frac{a^{1-\xi(-n)}}{(ab)^{\left\lfloor\frac{-n}{2}\right\rfloor}}\right)\frac{\sigma^{-n}-\rho^{-n}}{\sigma-\rho}$$

$$= (-1)\left(\frac{a^{1-\xi(-n)}}{(ab)^{\left\lfloor\frac{-n}{2}\right\rfloor}}\right)\frac{\sigma^{n}-\rho^{n}}{(-qab)^n(\sigma-\rho)}$$

$$= (-1)(-q)^{-n}\left(\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\frac{\sigma^{n}-\rho^{n}}{\sigma-\beta}$$

$$= -(-q)^{-n}F_n(p,q)$$

and

$$L_{-n}(p,q) = \left(\frac{a^{\xi(-n)}}{(ab)^{\left\lfloor\frac{-n+1}{2}\right\rfloor}}\right)(\sigma^{-n}+\rho^{-n})$$

$$= \left(\frac{a^{\xi(-n)}}{(ab)^{\left\lfloor\frac{-n+1}{2}\right\rfloor}}\right)\frac{\sigma^{n}+\rho^{n}}{(-qab)^n}$$

$$= (-q)^{-n}\left(\frac{a^{\xi(n)}}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\right)(\sigma^{n}+\rho^{n})$$

$$= (-q)^{-n}L_n(p,q)$$

where $\sigma\rho = -qab$. Thus, the proof is completed.

Now we present some basic identities for the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences, such as Catalan's identity, Cassini's identity and d'Ocagne's identity.

Theorem 7. (Catalan's Identity) Let n and r be nonnegative integers. For $n \geq r$, we have

i. $$a^{\xi(n-r)}b^{1-\xi(n-r)}F_{n-r}(p,q)F_{n+r}(p,q) - a^{\xi(n)}b^{1-\xi(n)}F_n^2(p,q) = -(-q)^{n-r}a^{\xi(r)}b^{1-\xi(r)}F_r^2(p,q)$$

and

ii. $$a^{1-\xi(n+r)}b^{1+\xi(n+r)}L_{n-r}(p,q)L_{n+r}(p,q) - a^{1-\xi(n)}b^{1+\xi(n)}L_n^2(p,q) = (-q)^{n-r}a^{\xi(r)}b^{1-\xi(r)}(p^2ab^2 + 4bq)F_r^2(p,q)$$

where $F_n(p,q)$ is the nth bi-periodic (p,q)-Fibonacci number and $L_n(p,q)$ is the nth bi-periodic(p,q)-Lucas number.

Proof. i. Using Binet formula of the bi-periodic (p,q)-Fibonacci sequence given in Theorem 4, we obtain

$$a^{\xi(n-r)}b^{1-\xi(n-r)}F_{n-r}(p,q)F_{n+r}(p,q) - a^{\xi(n)}b^{1-\xi(n)}F_n^2(p,q)$$

$$= a^{\xi(n-r)}b^{1-\xi(n-r)}\left(\frac{a^{1-\xi(n-r)}}{(ab)^{\left\lfloor\frac{n-r}{2}\right\rfloor}}\right)\left(\frac{a^{1-\xi(n+r)}}{(ab)^{\left\lfloor\frac{n+r}{2}\right\rfloor}}\right)\left(\frac{\sigma^{n-r}-\rho^{n-r}}{\sigma-\rho}\right)\left(\frac{\sigma^{n+r}-\rho^{n+r}}{\sigma-\rho}\right)$$

$$-a^{\xi(n)}b^{1-\xi(n)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left(\frac{a^{1-\xi(n)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\right)\left(\frac{\sigma^{n}-\rho^{n}}{\sigma-\rho}\right)\left(\frac{\sigma^{n}-\rho^{n}}{\sigma-\rho}\right)$$

$$= \frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{\left\lfloor\frac{n-r}{2}\right\rfloor+\left\lfloor\frac{n+r}{2}\right\rfloor}}\left(\frac{\sigma^{2n}-\sigma^{n-r}\rho^{n+r}-\rho^{n-r}\sigma^{n+r}+\rho^{2n}}{(\sigma-\rho)^2}\right)$$

$$-\frac{a^{2-\xi(n)}b^{1-\xi(n)}}{(ab)^{2\left\lfloor\frac{n}{2}\right\rfloor}}\left(\frac{\sigma^{2n}-2\sigma^n\rho^n+\rho^{2n}}{(\sigma-\rho)^2}\right)$$

$$= \frac{a^{2-\xi(n-r)}b^{1-\xi(n-r)}}{(ab)^{n-\xi(n-r)}}\left(\frac{\sigma^{2n}-(\sigma\rho)^{n-r}(\sigma^{2r}+\rho^{2r})+\rho^{2n}}{(\sigma-\rho)^2}\right)$$

$$-\frac{a^{2-\xi(n)}b^{1-\xi(n)}}{(ab)^{n-\xi(n)}}\left(\frac{\sigma^{2n}-2(\sigma\rho)^n+\rho^{2n}}{(\sigma-\rho)^2}\right)$$

$$= \frac{a}{(ab)^{n-1}}\left[\frac{-(\sigma\rho)^{n-r}(\sigma^{2r}+\rho^{2r})+2(\sigma\rho)^n}{(\sigma-\rho)^2}\right]$$

$$= \frac{-a(\sigma\rho)^{n-r}}{(ab)^{n-1}}\left(\frac{\sigma^r-\rho^r}{\sigma-\rho}\right)^2$$

$$= \frac{-a(-qab)^{n-r}}{(ab)^{n-1}}\frac{(ab)^{2\left\lfloor\frac{r}{2}\right\rfloor}}{a^{2-2\xi(r)}}F_r^2(p,q)$$

$$= -(-q)^{n-r}\frac{a(ab)^{2\left\lfloor\frac{r}{2}\right\rfloor}}{(ab)^{\xi(r)+2\left\lfloor\frac{r}{2}\right\rfloor-1}a^{2-2\xi(r)}}F_r^2(p,q)$$

$$= -(-q)^{n-r}a^{\xi(r)}b^{1-\xi(r)}F_r^2(p,q)$$

where $\xi(n) = n - 2\left\lfloor\frac{n}{2}\right\rfloor$ and $\left\lfloor\frac{n-r}{2}\right\rfloor + \left\lfloor\frac{n+r}{2}\right\rfloor = n - \xi(n-r)$. Similarly, the proof of ii. can be proved using Binet formula of the bi-periodic (p,q)-Lucas sequence given in Theorem 4. This completes the proof.

Theorem 8. (Cassini's Identity) Let n be nonnegative integer. Then, we have

i. $$\left(\frac{a}{b}\right)^{\xi(n-1)}F_{n-1}(p,q)F_{n+1}(p,q) - \left(\frac{a}{b}\right)^{\xi(n)}F_n^2(p,q) = -(-q)^{n-1}\frac{a}{b}$$

and

ii. $\left(\frac{b}{a}\right)^{\xi(n+1)} L_{n-1}(p,q)L_{n+1}(p,q)$

$-\left(\frac{b}{a}\right)^{\xi(n)} L_n^2(p,q) = (-q)^{n-1}(p^2ab+4q).$

Proof. The proof can be seen in an obvious way by taking $r=1$ in the Catalan's identity.

Theorem 9. (d'Ocagne's Identity) Let n and r be nonnegative integers. For $n \geq r$, we have

i. $a^{\xi(nr+n)}b^{\xi(nr+r)}F_n(p,q)F_{r+1}(p,q)$

$-a^{\xi(nr+r)}b^{\xi(nr+n)}F_{n+1}(p,q)F_r(p,q)$

$= (-q)^r a^{\xi(n-r)}F_{n-r}(p,q)$

and

ii. $a^{\xi(nr+n)}b^{\xi(nr+r)}L_{n+1}(p,q)L_r(p,q)$

$-a^{\xi(nr+r)}b^{\xi(nr+n)}L_n(p,q)L_{r+1}(p,q)$

$= (-q)^r a^{\xi(n-r)}(p^2ab+4q)F_{n-r}(p,q)$

where $F_n(p,q)$ is the nth bi-periodic (p,q)-Fibonacci number and $L_n(p,q)$ is the nth bi-periodic(p,q)-Lucas number.

Proof. i. Using Binet formula of the bi-periodic (p,q)-Fibonacci sequence given in Theorem 4, we obtain

$$a^{\xi(nr+n)}b^{\xi(nr+r)}F_n(p,q)F_{r+1}(p,q)$$

$$-a^{\xi(nr+r)}b^{\xi(nr+n)}F_{n+1}(p,q)F_r(p,q)$$

$$= a^{\xi(nr+n)}b^{\xi(nr+r)}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(r+1)}}{(ab)^{\lfloor\frac{r+1}{2}\rfloor}}\right)$$

$$\left(\frac{\sigma^n-\rho^n}{\sigma-\rho}\right)\left(\frac{\sigma^{r+1}-\rho^{r+1}}{\sigma-\rho}\right)$$

$$-a^{\xi(nr+r)}b^{\xi(nr+n)}\left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\right)\left(\frac{a^{1-\xi(r)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\right)$$

$$\left(\frac{\sigma^{n+1}-\rho^{n+1}}{\sigma-\rho}\right)\left(\frac{\sigma^r-\rho^r}{\sigma-\rho}\right)$$

$$= \frac{ab^{\xi(nr+r)}a^{1-\xi(n)-\xi(r+1)+\xi(nr+n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{r+1}{2}\rfloor}}$$

$$\left(\frac{\sigma^{n+r+1}-\sigma^n\rho^{r+1}-\rho^n\sigma^{r+1}+\rho^{n+r+1}}{(\sigma-\rho)^2}\right)$$

$$-\frac{ab^{\xi(nr+n)}a^{1-\xi(n+1)-\xi(r)+\xi(nr+r)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor+\lfloor\frac{r}{2}\rfloor}}$$

$$\left(\frac{\sigma^{n+r+1}-\sigma^{n+1}\rho^r-\rho^{n+1}\sigma^r+\rho^{n+r+1}}{(\sigma-\rho)^2}\right)$$

$$= \frac{ab^{\xi(nr+r)}a^{\xi(n-r)-\xi(nr+n)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r}}$$

$$\left(\frac{\sigma^{n+r+1}+\rho^{n+r+1}-(\sigma\rho)^r(\sigma\rho^{n-r}+\rho\sigma^{n-r})}{(\sigma-\rho)^2}\right)$$

$$-\frac{ab^{\xi(nr+n)}a^{\xi(n-r)-\xi(nr+r)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r}}$$

$$\left(\frac{\sigma^{n+r+1}+\rho^{n+r+1}-(\sigma\rho)^r(\sigma^{n-r+1}+\rho^{n-r+1})}{(\sigma-\rho)^2}\right)$$

$$= \frac{ab^{\xi(nr+r)}a^{\xi(nr+r)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+r)+r}}$$

$$\left(\frac{\sigma^{n+r+1}+\rho^{n+r+1}-(\sigma\rho)^r(\sigma\rho^{n-r}+\rho\sigma^{n-r})}{(\sigma-\rho)^2}\right)$$

$$-\frac{ab^{\xi(nr+n)}a^{\xi(nr+n)}}{(ab)^{\frac{n-r-\xi(n-r)}{2}+\xi(nr+n)+r}}$$

$$\left(\frac{\sigma^{n+r+1}+\rho^{n+r+1}-(\sigma\rho)^r(\sigma^{n-r+1}+\sigma^{n-r+1})}{(\sigma-\rho)^2}\right)$$

$$= \frac{a(ab)^{-r}}{(ab)^{\frac{n-r-\xi(n-r)}{2}}}$$

$$\left(\frac{(\sigma\rho)^r(-\sigma\rho^{n-r}-\rho\sigma^{n-r}+\sigma^{n-r+1}+\rho^{n-r+1})}{(\sigma-\rho)^2}\right)$$

$$= \frac{a(ab)^{-r}}{(ab)^{\lfloor\frac{n-r}{2}\rfloor}}\left(\frac{(-qab)^r(\sigma-\rho)(\sigma^{n-r}-\rho^{n-r})}{(\sigma-\rho)^2}\right)$$

$$= \frac{a(-q)^r}{(ab)^{\left\lfloor \frac{n-r}{2} \right\rfloor}} \left(\frac{\sigma^{n-r} - \rho^{n-r}}{\sigma - \rho} \right)$$

$$= (-q)^r a^{\xi(n-r)} F_{n-r}(p,q)$$

where

$$1 - \xi(n-r) = \xi(n) + \xi(r+1) - 2\xi(nr+n)$$
$$= \xi(n+1) + \xi(r) - 2\xi(nr+r)$$

$$\xi(n-r) = \xi(nr+n) + \xi(nr+r)$$

$$\frac{n-r-\xi(n-r)}{2} + \xi(nr+r) + r = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{r+1}{2} \right\rfloor$$

$$\frac{n-r-\xi(n-r)}{2} + \xi(nr+n) + r = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{r}{2} \right\rfloor$$

$$\frac{n-r-\xi(n-r)}{2} = \left\lfloor \frac{n-r}{2} \right\rfloor.$$

Similarly, the proof of ii. can be proved using Binet formula of the bi-periodic (p,q)-Lucas sequence given in Theorem 4. This completes the proof.

3. CONCLUSION AND SUGGESTION

The generalizations and applications of the Fibonacci and the Lucas sequences have been presented in many ways. In this paper, the bi-periodic (p,q)-Fibonacci and the bi-periodic (p,q)-Lucas sequences, which generalize well-known Fibonacci, the k-Fibonacci, the Lucas, the k-Lucas, the Jacobsthal, the k-Jacobsthal, the Jacobsthal-Lucas, the k-Jacobsthal-Lucas, the Pell, the k-Pell, the Pell-Lucas, the k-Pell-Lucas sequences as well as the bi-periodic Fibonacci, the bi-periodic k-Fibonacci, the bi-periodic Lucas, the bi-periodic k-Lucas, the bi-periodic Jacobsthal, the bi-periodic k-Jacobsthal, the bi-periodic Jacobsthal-Lucas, the bi-periodic k-Jacobsthal-Lucas, the bi-periodic Pell, the bi-periodic k-Pell, the bi-periodic Pell-Lucas, the bi-periodic k-Pell-Lucas sequences, are defined. Binet formulas that allow us to calculate the nth terms of these sequences and some properties of their consecutive terms are given. Also generating functions, Catalan's identity, Cassini's identity, and d'Ocagne's identity are obtained.

It would be interesting to study these sequences in matrix theory. More general formulas that allow us to calculate the nth terms of these sequences and relations like the well-known relations between the Fibonacci and the Lucas sequences can be explored.

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The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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