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Stability and Bifurcation Analysis For An OSN Model with Delay

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Abstract

In this research, we propose and study an online social network mathematical model with delay based on two innovative assumptions: (1) newcomers are entering community as either potential online network users or that who are never interested in online network at constant rates, respectively; and (2) it takes a certain time for the active online network users to start abandoning the network. The basic reproduction R_0, the user-free equilibrium(UFE) P_0, and the user-prevailing equilibrium(UPE) P^* are identified. The analysis of local and global stability for those equilibria is carried out. For the UPE P^*, using the delay τ as the Hopf bifurcation parameter, the occurrence of Hopf bifurcation is investigated. The conditions are established that guarantee the Hopf bifurcation occurs as τ crosses the critical values. Numerical simulations are provided to illustrate the theoretical results.

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1. Introduction

Online Social Networks (OSNs) have become increasingly important and imperative for people to receive and spread information in the most recent two decades. The creation of Facebook, Twitter, Instagram, and other online social media networks has made information exchanging and spreading easy and it has changed people's daily life dramatically. OSNs allow people to present, interact, and connect themselves virtually with others. The popularity of OSNs, especially in the new high technology-oriented generations, has radically changed so many things such as education, election, information sharing, etc. It has become

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evidently urgent that a better understanding of how OSNs can influence a society's political and economical environments, even just people's daily life, will benefit all interested parties involved. Mathematical models have been developed to serve to better understand the OSN dynamics and have provided a deep insight on how people's opinion/behaviour is affected by social networks, see, for example [2, 3, 5, 6, 7, 4, 8, 11, 9, 10, 1]. Many efforts have been made in the development of models for studying OSN dynamics leveraging SIR/SEIR disease type models. Readers are referred to [12, 13, 14, 15, 16] and references therein for some classic results and advances on SIR/SEIR mathematical models. In this spirit, the total population, denoted by $N(t)$ at time t, is divided into three sub-classes representing three major populations in the OSN dynamics. They are potential OSN users, active OSN users, and people who are opposed to OSNs, with sizes denoted by $x(t), y(t)$, and $z(t)$ at time t, respectively. Cannarella and Spechler [2] proposed the *infectious recovery* SIR type model

$$\begin{cases} x' = -\alpha xy/N, \\ y' = \alpha xy/N - \eta yz/N, \\ z' = \eta yz/N, \end{cases}$$

to analyze user adoption and abandonment of OSNs. This model was further extended to ordinary, fractional, and stochastic differential equation models in [3, 5, 6] respectively. The optimal control problem for an ordinary differential equation model was also studied in [7]. The reader is referred to [2, 3, 5, 6, 7] for the details of the model development and meanings of the parameters. Particularly, Graef et al. [5] studied the following OSN model with demography

$$\begin{cases} x' = \Lambda - \alpha xy - \mu x, \\ y' = \alpha xy - \eta yz - (\mu + \delta)y, \\ z' = \eta yz + \delta y - \mu z, \end{cases} \tag{1}$$

to examine the user adoption and abandonment dynamics of OSNs. Both local and global stability analyses were carried out. Motivated by the research mentioned above and some unique characteristics and the complexity of OSNs such as the fact that there are people who are never interested in online social networks and active online network users may lose their interest after some time, the investigators propose a new dynamic mathematical model to incorporate these characteristics to study the OSN dynamics. Potential users become active after contacting with active users. Active users become opposed to OSNs either after contacting with those people who are opposed to OSNs or lose their interest after a period of time. The interaction and dynamics between these three compartments can be described by the following system of differential equations

$$\left\{ \begin{array}{rcl} x' & = & A - \alpha xy - \mu x, \\ y' & = & \alpha xy - \eta yz - \delta y(t-\tau) - \mu y, \\ z' & = & B + \eta yz + \delta y(t-\tau) - \mu z, \end{array} \right. \tag{2}$$

where the parameters $A > 0$ and $B \geq 0$ represent the rates that newcomers come into the community as either potential users or as people who are never interested in OSNs. $\alpha > 0$ denotes the contact rate between the potential and active OSN users; $\mu > 0$ is the death rate for all people; $\eta > 0$ is the contact rate between active users and people who are opposed to OSNs; $\delta > 0$ is the transferring rate describing the rate the active users lose their interest and become opposing to OSNs, and $\tau \geq 0$ is the time delay that represents the time for active users to starting abandoning the network. It is notable that when $B = \tau = 0$, System (2) covers System (1) as a special case.

In this article, we will perform a detailed analysis for System (2). First, after a basic reproduction number is identified, the existence of equilibrium points is established based on the basic reproduction number. Then the local and global stability when $\tau = 0$ is studied and we show that when the basic reproduction number $R_0 \leq 1$, the unique user-free equilibrium P_0 is globally asymptotically stable, whereas $R_0 > 1$, P_0 becomes unstable and the unique user-prevailing equilibrium P^* is globally asymptotically stable. Next, using τ as the bifurcation parameter, we investigate the Hopf bifurcations at the unique user-prevailing equilibrium point when $R_0 > 1$. Conditions and critical values are obtained so that the Hopf bifurcation will occur as τ passes through the critical values under some given conditions.

We need the following result due to Ruan and Wei [17].

Lemma 1.1. *Consider the exponential polynomial*

$$P(\lambda, e^{-\lambda\tau}) = p(\lambda) + q(\lambda)e^{-\lambda\tau}$$

where p and q are real polynomials such that $\deg(q) < \deg(p)$ and $\tau \geq 0$. As τ varies, the total number of zeros of $P(\lambda, e^{-\lambda\tau})$ on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

The rest of this article is organized as follows. In Section 2, the reproduction number R_0 is identified and the existence of equilibrium points are established based on the reproduction number. Local and global stability of equilibrium points when $\tau = 0$ is studied in Sections 3 and 4. The conditions for occurrence of Hopf bifurcation at UPE as the delay τ passes through critical numbers are investigated in Section 5. Numerical simulations are provided to demonstrate our theoretical results in Section 6. The manuscript ends up with a brief discussion in Section 7.

2. Equilibrium points and basic reproduction number

In this section, we derive a basic reproduction number R_0 for System (2). Then equilibrium points are determined based on R_0. To find the equilibrium points of System (2), we need to solve the following system of equations

$$\begin{cases} A - \alpha xy - \mu x &= 0, \\ \alpha xy - \eta yz - \delta y - \mu y &= 0, \\ B + \eta yz + \delta y - \mu z &= 0. \end{cases} \tag{3}$$

It's easy to show that the system has a unique user-free equilibrium $P_0 = (A/\mu, 0, B/\mu)$ and it exists for all parameter values. Now we will try to find user-prevailing equilibrium(UPE) point(s) $P^* = (x^*, y^*, z^*)$, where $y^* > 0$.

From the second and third equations of System (3), we can express x and y in terms of z as

$$x = \frac{\mu + \delta}{\alpha} + \frac{\eta}{\alpha} z, \tag{4}$$

and

$$y = \frac{\mu z - B}{\delta + \eta z}. \tag{5}$$

Rewrite the first equation of (3) as

$$A = x(\alpha y + \mu)$$

and plug x and y given in (4) and (5) into this equation, we arrive at the equation

$$A = \left(\frac{\mu + \delta}{\alpha} + \frac{\eta}{\alpha} z\right)\left(\mu + \frac{\alpha(\mu z - B)}{\delta + \eta z}\right),$$

which is equivalent to

$$A\alpha(\delta + \eta z) = \mu\eta(\alpha + \eta)z^2 + [\mu(\mu + \delta)(\alpha + \eta) + \eta(\mu\delta - B\alpha)]z + (\mu + \delta)(\mu\delta - B\alpha).$$

Define

$$f(z) = A\alpha(\delta + \eta z) \tag{6}$$

and

$$g(z)=\mu\eta(\alpha+\eta)z^2+[\mu(\mu+\delta)(\alpha+\eta)+\eta(\mu\delta-B\alpha)]z+(\mu+\delta)(\mu\delta-B\alpha). \tag{7}$$

Obviously, z^* is a positive root of the equation

$$f(z)=g(z)$$

and x^* and y^* can be found using expressions (4) and (5). From (5), it's easy to see that for y^* to be positive, it requires that $z^*>B/\mu$.

Let R_0 be the basic reproduction defined by

$$R_0=\frac{A\alpha}{B\eta+\mu(\mu+\delta)} \tag{8}$$

which gives the expected number of active users directly generated by one active user in a population where all individuals are potential users to network. The derivation of R_0 can be performed by the next generation matrix method given by van den Driessche and Watmough [18]. If we rewrite System (2) as

$$\begin{pmatrix} y' \\ z' \\ x' \end{pmatrix}=\begin{pmatrix} \alpha xy \\ 0 \\ 0 \end{pmatrix}-\begin{pmatrix} \eta yz+\delta y+\mu y \\ -B-\eta yz-\delta y+\mu z \\ -A+\alpha xy+\mu x \end{pmatrix}=\mathcal{F}-\mathcal{V}.$$

At the user-free equilibrium P_0, we have

$$D\mathcal{F}(P_0)=\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \qquad D\mathcal{V}(P_0)=\begin{pmatrix} V & 0 \\ J_3 & J_4 \end{pmatrix},$$

where

$$F=\begin{pmatrix} \frac{A\alpha}{\mu} & 0 \\ 0 & 0 \end{pmatrix}, \qquad V=\begin{pmatrix} \frac{B\eta}{\mu}+\delta+\mu & 0 \\ -\frac{B\eta}{\mu}-\delta & \mu \end{pmatrix}$$

and

$$J_3=(\mu, A\alpha/\mu), \qquad J_4=0.$$

It follows that

$$FV^{-1}=\frac{1}{B\eta+\mu(\mu+\delta)}\begin{pmatrix} A\alpha & 0 \\ 0 & 0 \end{pmatrix}$$

and R_0 is the leading eigenvalue of FV^{-1}, which is the R_0 given in (8). We then have the following results regarding z^* in terms of R_0.

Lemma 2.1. *Let $\bar{z}=B/\mu$, f, g, and R_0 be defined by (6) – (8) respectively. Then*

(1) *$z^*=\bar{z}$ if and only if $f(\bar{z})=g(\bar{z})$ if and only if $R_0=1$.*
(2) *$z^*>\bar{z}$ if and only if $f(\bar{z})>g(\bar{z})$ if and only if $R_0>1$.*

Proof. Note that f is a linear function with $f(0)=A\alpha\delta$ and g is a quadratic function with $g(0)=(\mu+\delta)(\mu\delta-B\alpha)$. It can be shown that if $R_0\geq 1$ then $f(0)>g(0)$. Therefore, the equation $f(z)=g(z)$ has a unique positive root z^* with $f(z)>g(z)$ when $0\leq z<z^*$ and $f(z)<g(z)$ when $z>z^*$. Calculation shows

$$f(\bar{z})-g(\bar{z})=\frac{(B\eta+\delta\mu)(A\alpha-B\eta-\mu(\delta+\mu))}{\mu}.$$

The lemma follows immediately from this equation. □

Remark 2.2. *Note that if $R_0 < 1$, then either the equation $f(z) = g(z)$ has a unique solution $z^* < \bar{z}$ or it has no solutions.*

The discussion above results in the following result.

Theorem 2.3. *Let f, g, and R_0 be defined by (6) – (8) respectively. If $R_0 \leq 1$, then System (2) has a unique user-free equilibrium $P_0 = (A/\mu, 0, B/\mu)$ and it exists for all parameter values. If $R_0 > 1$, then System (2) has two equilibria: P_0 and a unique user-prevailing equilibrium $P^* = (x^*, y^*, z^*)$, where z^* is the unique positive root of the equation $f(z) = g(z)$ such that $z^* > B/\mu$, and x^* and y^* are given by (4) and (5).*

3. Local stability of equilibrium points with no delay

In this section, we study local stability of P_0 and P^* when delay $\tau = 0$. The Jacobian matrix of System (2) at an equilibrium $P = (x, y, z)$ is

$$J = \begin{pmatrix} -\alpha y - \mu & -\alpha x & 0 \\ \alpha y & -\delta - \mu + \alpha x - \eta z & -\eta y \\ 0 & \delta + \eta z & \eta y - \mu \end{pmatrix},$$

whose characteristic equation is

$$(\lambda + \mu)(\lambda^2 + a\lambda + b) = 0, \tag{9}$$

where

$$\begin{aligned} a &= 2\mu + \alpha y + \eta z - \alpha x - \eta y + \delta, \\ b &= \mu(\delta + \alpha y + \eta z - \alpha x - \eta y) + \alpha\eta y(x + z - y) + \delta\alpha y. \end{aligned}$$

It is clear that $\lambda_1 = -\mu$ is a negative root of Equation (9). The other two roots of Equation (9) are determined by the equation

$$\lambda^2 + a\lambda + b = 0. \tag{10}$$

At P_0, Equation (10) becomes

$$(\lambda + \mu)\left(\lambda + \mu + \frac{B\eta}{\mu} - \frac{A\alpha}{\mu} + \delta\right) = 0.$$

It follows that $\lambda_2 = -\mu < 0$ and $\lambda_3 = \frac{A\alpha}{\mu} - \frac{B\eta}{\mu} - \mu - \delta$. It is straightforward to see that $\lambda_3 < 0$ if $R_0 < 1$, $\lambda_3 = 0$ if $R_0 = 1$, and $\lambda_3 > 0$ if $R_0 > 1$.

At P^*, a and b are given by, in terms of z^*, using (4) and (5)

$$\begin{aligned} a &= \tfrac{\alpha(\mu z^* - B) + B\eta + \mu\delta}{\delta + \eta z^*}, \\ b &= \tfrac{(\mu z^* - B)(\eta(\delta + \eta z^*)^2 + \alpha(\delta^2 + (B + \eta(z^*)^2) + \delta(2\eta z^* + \mu)))}{(\delta + \eta z^*)^2}. \end{aligned}$$

From the discussion above, we know that if $R_0 > 1$, then $z^* > B/\mu$, or $\mu z^* - B > 0$. Therefore, we have $a > 0$, $b > 0$. The two roots of Equation (10) have negative real parts. We thus obtain the following local stability results.

Theorem 3.1. *Let R_0 be defined in (8). If $R_0 < 1$, P_0 is locally asymptotically stable; if $R_0 = 1$, P_0 is neutrally stable; and if $R_0 > 1$, P_0 becomes unstable, and P^* emerges and it is locally asymptotically stable.*

4. Global stability of equilibrium points with no delay

In this section, we study the global stability of P_0 and P^* when delay $\tau = 0$. Note that $P_0 = (A/\mu, 0, B/\mu)$ and $P^* = (x^*, y^*, z^*)$ where $z^* > 0$ is the unique positive root of the equation $f(z) = g(z)$, and x^* and y^* are given by (4) and (5).

A standard argument, see, for example [19, 20], can be used to show that the first octant $\mathbb{R}^3_+ = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$ of $\mathbb{R}^3$ is positively invariant with respect to System (2), and the system has a unique solution with any initial value starting from inside $\mathbb{R}^3_+$. The first equation of System (2) clearly shows that if $x(0) \leq A/\mu$, then $x(t) \leq A/\mu$ for all $t > 0$. Furthermore, if we add the three equations of System (2) together, we can get

$$(x + y + z)' = A + B - \mu(x + y + z).$$

We, therefore, proved that the set

$$\Omega = \{(x, y, z) \in \mathbb{R}^3_+ : x \leq A/\mu,\ x + y + z \leq (A + B)/\mu\}$$

is a global attractor and it's positively invariant with respect to System (2). We will use $\overset{\circ}{\Omega}$ to denote the interior of Ω. Now, we are ready to prove the global stability of P_0 when $R_0 \leq 1$.

Theorem 4.1. *Assume $R_0 \leq 1$. Then P_0 is globally asymptotically stable in Ω.*

Proof. First, if $y(0) = 0$, from System (2) it's easy to see that $y(t) = 0$ for all $t > 0$ and $x(t) \to A/\mu$ and $z(t) \to B/\mu$ as $t \to \infty$, i.e. $(x(t), y(t), z(t)) \to P_0$.

Next, assume $y(0) > 0$ and $z(0) > B/\mu$. The second equation of System (2) implies

$$y' = y(\alpha x - \eta z - \delta - \mu) < y(\alpha A/\mu - \eta B/\mu - \delta - \mu) \leq 0$$

since $R_0 \leq 1$. It's not difficult to show that in this case $y(t) \to 0$ as $t \to \infty$. As a result, we get $x(t) \to A/\mu$ and $z(t) \to B/\mu$, i.e. $(x(t), y(t), z(t)) \to P_0$.

Finally, assume $y(0) > 0$ and $z(0) \leq B/\mu$. Define a Lyapunov function $V_1 : \Omega \to \mathbb{R}$ by

$$V_1(x, y, z) = \frac{1}{A}\left[\frac{\mu x}{A} - 1 - \ln\left(\frac{\mu x}{A}\right)\right] + \frac{\mu y}{A^2} + \frac{B}{A^2}\left[\frac{\mu z}{B} - 1 - \ln\left(\frac{\mu z}{B}\right)\right]. \tag{11}$$

It is easy to verify that for any $(x, y, z) \in \Omega$, $V_1(x, y, z) \geq 0$ and $V_1(P_0) = 0$. A direct computation, clearly we can assume that $x, z > 0$ from the first and third equation of System (2), leads to

$$\begin{aligned} V_1'(x, y, z) &= \frac{\mu}{A}\left(\frac{\mu}{A} - \frac{1}{x}\right)\left(\frac{A}{\mu} - x\right) + \frac{\mu B}{A^2}\left(\frac{\mu}{B} - \frac{1}{z}\right)\left(\frac{B}{\mu} - z\right) \\ &\quad + \frac{\alpha y}{A} - \frac{\mu^2 y}{A^2} - \frac{By}{A^2}\left(\eta + \frac{\delta}{z}\right) \\ &\leq -\frac{\mu^2}{A^2 x}\left(x - \frac{\mu}{A}\right)^2 - \frac{\mu^2}{A^2 z}\left(z - \frac{\mu}{B}\right)^2 \\ &\quad + \frac{y}{A^2}(\alpha A - \mu^2 - B\eta - \delta\mu). \end{aligned} \tag{12}$$

Note that $\alpha A - \mu^2 - B\eta - \delta\mu \leq 0$ if $R_0 \leq 1$. Therefore, (12) implies $V'(x, y, z) \leq 0$ and $V'(x, y, z) = 0$ holds only when $(x, y, z) = (A/\mu, 0, B/\mu)$ or $x = A/\mu, z = B/\mu$, and $R_0 = 1$, which means that $y = 0$. The maximum invariant set in $\{(x, y, z) \in \Omega : L' = 0\}$ is the Singleton P_0. By LaSalle's Invariant Principle [21], see also, for example, [22, 24, 23], $(x(t), y(t), z(t)) \to P_0$ as $t \to \infty$. This and the local stability of P_0 established in Theorem 3.1 imply that P_0 is globally asymptotically stable in Ω. □

Next, we will study the global asymptotic stability of $P^* = (x^*, y^*, z^*)$ in $\overset{\circ}{\Omega}$ when $R_0 > 1$. In $\overset{\circ}{\Omega}$, we have $x, y, z > 0$. Let

$$V_2(x, y, z) = x - x^* - x^* \ln\left(\frac{x}{x^*}\right) + y - y^* - y^* \ln\left(\frac{y}{y^*}\right) + z - z^* - z^* \ln\left(\frac{z}{z^*}\right). \tag{13}$$

The derivative of V_2 along the solution of System (2) can be calculated as

$$\begin{aligned} V_2'(x,y,z) &= \frac{A}{x}(x-x^*)+\mu\,(x^*-x)+\alpha(x^*y-xy^*) \\ &+\frac{\delta+\eta z}{z}\,(y^*z-yz^*)+\mu(y^*-y) \\ &+\frac{B}{z}\,(z-z^*)+\mu(z^*-z). \end{aligned}$$

Note that

$$x^*y-xy^*=x^*(y-y^*)+y^*(x^*-x), \quad y^*z-yz^*=y^*(z-z^*)+z^*(y^*-y).$$

The above $V_2'(x,y,z)$ can be written as

$$\begin{aligned} V_2'(x,y,z) &= (x-x^*)\left(\frac{A}{x}-\mu-\alpha y^*\right) \\ &+(y-y^*)\left(\alpha x^*-\mu-\frac{\delta+\eta z}{z}z^*\right) \\ &+(z-z^*)\left(\frac{B}{z}+\frac{\delta+\eta z}{z}y^*-\mu\right). \end{aligned}$$

Now, from the first equation of System (3), we have

$$\frac{A}{x^*}=\alpha y^*+\mu$$

and this results in

$$\frac{A}{x}-\mu-\alpha y^*=\frac{A}{x}-\frac{A}{x^*}=\frac{A}{xx^*}(x^*-x).$$

Using expressions for x^* and y^* in (4) and (5), we can get

$$\alpha x^*-\mu-\frac{\delta+\eta z}{z}z^*=\frac{\delta}{z}(z-z^*)$$

and

$$\frac{B}{z}+\frac{\delta+\eta z}{z}y^*-\mu=-\frac{B\eta+\delta\mu}{z(\delta+\eta z^*)}(z-z^*).$$

We finally obtain

$$V_2'(x,y,z)=-\frac{A}{xx^*}(x-x^*)^2+\frac{\delta}{z}(y-y^*)(z-z^*)-\frac{B\eta+\delta\mu}{z(\delta+\eta z^*)}(z-z^*)^2. \tag{14}$$

Theorem 4.2. *Assume that $R_0>1$. If $\delta=0$, then $P^*=(x^*,y^*,z^*)$ is globally asymptotically stable in $\mathring{\Omega}$.*

Proof. Using the Lyapunov function given in (13) assuming $\delta=0$, by (14), we end up with

$$V_2'(x,y,z)=-\frac{A}{xx^*}(x-x^*)^2-\frac{B\eta+\delta\mu}{z(\delta+\eta z^*)}(z-z^*)^2.$$

Obviously, $V_2'(x,y,z)\le 0$ for all $(x,y,z)\in\mathring{\Omega}$ and $V_2'(x,y,z)=0$ holds only for the set

$$M=\{(x,y,z)\in\mathring{\Omega}\colon x=x^*,\ z=z^*\}.$$

The maximum invariant set in M is the unique equilibrium P^*. Again, by the asymptotic stability theorem [21, 22, 24, 23], all solutions starting inside $\mathring{\Omega}$ converge to P^*. This fact and the local stability of the equilibrium P^* established in Theorem 3.1 imply the global asymptotic stability of P^*. □

Remark 4.3. *We want to point out that we only proved that P^* is globally asymptotically stable in $\overset{\circ}{\Omega}$ when $\delta = 0$ in Theorem 4.2. But numerical simulations show that P^* is globally asymptotically stable in $\overset{\circ}{\Omega}$ as long as it exists, i.e., if $R_0 > 1$. See Figure 1 for the convergence of the solutions to P^* for different initial points when $\delta > 0$. Here we choose $A = 2, B = 0.1, \mu = .2, \alpha = 1, \beta = 1, \eta = 0.5, \delta = 0.4$. Calculation shows that $R_0 = 11.7647 > 1$ and $P^* = (3.7809, 0.3276, 6.3816)$. Solutions converge to P^* regardless where the solution starts.*

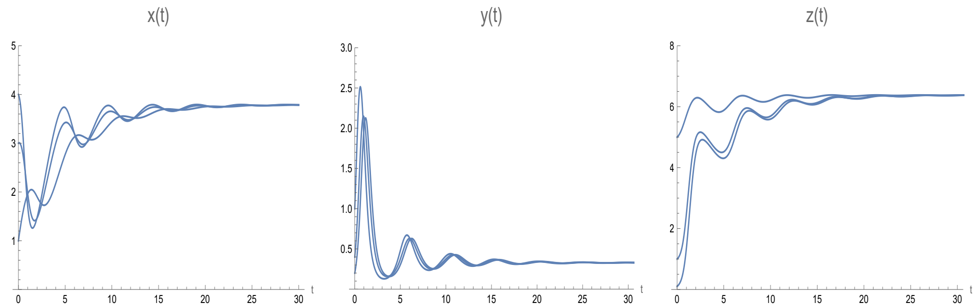


Figure 1: Solutions converge to P^*.

Remark 4.4. *From Sections 3 and 4, we can conclude that the dynamics of System (2) is completely determined by the basic reproduction R_0 when there is no delay. If $R_0 \leq 1$, P_0 is the only equilibrium and all solutions converge to P_0. That means that in this case, the number of active online network users will eventually approach to zero. On the other hand, while $R_0 > 1$, all solutions converge to P^*. The active online network users will settle at the level of $y^* > 0$ over time.*

5. Hopf bifurcation

As we see from Remark 4.2, the dynamics of System (2) is completely determined by the basic reproduction R_0 when the delay $\tau = 0$. We are interested in the question that if the delay τ could cause the stability switch of the UPE P^* as it increases. In this section, we study the occurrence of Hopf bifurcations using the delay τ as the bifurcation parameter. Note that when $R_0 > 1$ there is a unique UPE $P^* = (x^*, y^*, z^*)$. For this section, we will always assume that $R_0 > 1$.

The characteristic equation of System (2) at the unique equilibrium P^* when $\tau \geq 0$ is the determinant of the matrix

$$J^* = \begin{pmatrix} \lambda + \alpha y^* + \mu & \alpha x^* & 0 \\ -\alpha y^* & \lambda + \mu - \alpha x^* + \eta z^* + \delta e^{-\lambda\tau} & \eta y^* \\ 0 & -\eta z^* - \delta e^{-\lambda\tau} & \lambda - \eta y^* + \mu \end{pmatrix},$$

which is

$$(\lambda + \mu)(\lambda^2 + a_1\lambda + b_1 + (\delta\lambda + c_1)e^{-\lambda\tau}) = 0, \tag{15}$$

where

$$\begin{aligned} a_1 &= 2\mu + \alpha y^* + \eta z^* - \alpha x - \eta y^*, \\ b_1 &= \mu^2 + (\alpha y^* + \eta z^* - \alpha x^* - \eta y^*)\mu + \alpha\eta y^*(x^* + z^* - y^*), \\ c_1 &= \delta(\mu + \alpha y^*). \end{aligned} \tag{16}$$

One root of Equation (15) is $\lambda = -\mu < 0$. The other roots are determined by the transcendental equation

$$\lambda^2 + a_1\lambda + b_1 + (\delta\lambda + c_1)e^{-\lambda\tau} = 0. \tag{17}$$

From the discussion in Section 2, we know that if $R_0 > 1$ and $\tau = 0$, P^* is locally asymptotically stable. Our interest is to see whether or not the delay τ will cause the stability of P^* to switch as τ increases while R_0 remains larger than the unity. Due to Lemma 1.1, we need to investigate if a zero of Equation (17) will appear on or cross the imaginary axis. Keep in mind that when $R_0 > 1$, $z^* > B/\mu$.

From (16 and use expressions given in (4 and (5), we can get

$$b_1 + c_1 = \frac{(\mu z^* - B)\left(\alpha\left(\eta\left(B + \eta(z^*)^2\right) + \delta^2 + \delta(\mu + 2\eta z^*)\right) + \eta(\delta + \eta z^*)^2\right)}{(\delta + \eta z^*)^2} > 0$$

since $z^* > B/\mu$. Therefore, $\lambda = 0$ is not a root of (17). Now, let $\lambda = \omega i$ ($\omega > 0$) be a root to Equation (17). Plug it into (17), we get

$$-\omega^2 + a_1\omega i + b_1 + (\delta\omega i + c_1)(\cos(\omega\tau) - i\sin(\omega\tau)) = 0.$$

Separating the real and imaginary parts gives

$$c_1\cos(\omega\tau) + \delta\omega\sin(\omega\tau) = \omega^2 - b_1, \tag{18}$$

$$-c_1\sin(\omega\tau) + \delta\omega\cos(\omega\tau) = -a_1\omega. \tag{19}$$

Squaring both sides and adding them together yields

$$\omega^4 + (a_1^2 - \delta^2 - 2b_1)\omega^2 + b_1^2 - c_1^2 = 0.$$

Let $p = w^2$ and denote $a_2 = a_1^2 - \delta^2 - 2b_1$ and $b_2 = b_1^2 - c_1^2$. Then the above equation can be rewritten as

$$p^2 + a_2 p + b_2 = 0. \tag{20}$$

The following result is well known.

Lemma 5.1. *For Equation (20), we have*

(a) *If $b_2 < 0$ or if $b_2 = 0$ and $a_2 < 0$, then it has a unique positive root.*
(b) *If $a_2 \geq 0$ and $b_2 \geq 0$, then it has no positive roots.*
(c) *If $a_2 < 0$ and $b_2 > 0$, then it has no positive roots if $a_2^2 - 4b_2 < 0$; one positive root if $a_2^2 - 4b_2 = 0$; and two positive roots if $a_2^2 - 4b_2 > 0$.*

Plug a_1, b_1, c_1 given in (16) and x^* and y^* given in (4) and (5) into a_2 and b_2, a tedious and long calculation gives

$$a_2 = a_1^2 - \delta^2 - 2b_1 = \frac{1}{(\delta + \eta z^*)^2} P_1(z^*), \tag{21}$$

$$b_2 = b_1^2 - c_1^2 = \frac{(\mu z^* - B)}{(\delta + \eta z^*)^4} P_2(z^*) P_3(z^*), \tag{22}$$

where

$$\begin{aligned} P_1(z) = {} & -2\eta^2\mu(\alpha + \eta)z^3 + (2B\eta^2(\alpha + \eta) + \mu(\alpha^2\mu - 4\alpha\delta\eta - 2\alpha\delta\eta^2))z^2 \\ & + (B(4\alpha\delta\eta + 2\delta\eta^2 - 2\alpha\mu) - 2\alpha\delta^2\mu)z \\ & + B^2(\eta^2 + \alpha^2) + \delta^2\mu^2 + 2B\delta(\alpha\delta + \eta\mu), \end{aligned} \tag{23}$$

$$\begin{aligned} P_2(z) = {} & (\alpha\eta^2 + \eta^3)z^2 + (2\alpha\delta\eta + 2\delta\eta^2)z + \alpha\delta^2 + \alpha\delta\mu + \alpha B\eta + \delta^2\eta, \\ P_3(z) = {} & (\alpha\eta^2\mu + \eta^3\mu)z^3 - (\alpha B\eta^2 + B\eta^3)z^2 \\ & + (-\alpha\delta^2\mu + \alpha\delta\mu^2 + \alpha B\eta\mu - 2B\delta\eta^2 - 3\delta^2\eta\mu)z \\ & - \alpha B^2\eta + \alpha B\delta^2 - \alpha B\delta\mu - B\delta^2\eta - 2\delta^3\mu. \end{aligned} \tag{24}$$

Note that $P_1(z)$ is a degree three polynomial with $P_1(B/\mu) = (B\eta + \delta\mu)^2 > 0$ and $P_1(z) \to -\infty$ as $z \to \infty$. By Descartes' Rule of Signs, P_1 has one or three positive roots. Let $z_1 > B/\mu$ be the first value of z such that $P_1(z_1) = 0$. Obviously, $P_2(z^*) > 0$, and $\mu z^* - B > 0$ as $z^* > B/\mu$ if $R_0 > 1$. $P_3(z)$ is also a degree three polynomial of z such that

$$P_3(B/\mu) = -4B\delta^2\eta - 2\delta^3\mu - \frac{2B^2\delta\eta^2}{\mu} < 0$$

and $P_3(z) \to \infty$ as $z \to \infty$. Thus, P_3 has one or three positive roots. Let $z_2 > B/\mu$ be the first value of z such that $P_3(z_2) = 0$. If we assume that both P_1 and P_3 have one positive root, i.e., z_1 and z_2 are the only positive numbers such that $P_1(z_1) = 0$ and $P_3(z_2) = 0$. We then have the following cases in terms of z_1, z_2 and z^* along with the signs of a_2 and b_2 :

(I) If $z_1 = z_2$, then
  (1) $z^* < z_1 \implies a_2 > 0, b_2 < 0$;
  (2) $z^* = z_1 \implies a_2 = 0, b_2 = 0$;
  (3) $z^* > z_1 \implies a_2 < 0, b_2 > 0$.

(II) If $z_1 < z_2$, then
  (1) $z^* \leq z_1 \implies a_2 \geq 0, b_2 < 0$;
  (2) $z_1 < z^* \leq z_2 \implies a_2 < 0, b_2 \leq 0$;
  (3) $z^* > z_2 \implies a_2 < 0, b_2 > 0$.

(III) If $z_1 > z_2$, then
  (1) $z^* < z_2 \implies a_2 > 0, b_2 < 0$;
  (2) $z_2 \leq z^* \leq z_1 \implies a_2 \geq 0, b_2 \geq 0$;
  (3) $z^* > z_1 \implies a_2 < 0, b_2 > 0$.

By Lemma 5.1, we have the following results based on these cases.

Theorem 5.2. *Let $R_0 > 1$, and let a_2, b_2, P_1 and P_3 be defined by (21), (22), (23), and (24). Assume that P_1 and P_3 have a unique positive root z_1 and z_2, respectively. We then have:*

(I) *When any of the following conditions is satisfied, Equation (20) has no positive roots.*
  (1) $z_1 = z_2$ *and* $z^* = z_1$;
  (2) $z_1 > z_2$ *and* $z_2 \leq z^* \leq z_1$;
  (3) $z^* > \max\{z_1, z_2\}$ *and* $a_2^2 - 4b_2 < 0$.

(II) *When any of the following conditions is satisfied, Equation (20) has a unique positive root.*
  (1) $z_1 < z_2$ *and* $z^* \leq z_2$;
  (2) $z_1 \geq z_2$ *and* $z^* < z_2$;
  (3) $z^* > \max\{z_1, z_2\}$ *and* $a_2^2 - 4b_2 = 0$.

(III) *Equation (20) has two positive roots if* $z^* > \max\{z_1, z_2\}$ *and* $a_2^2 - 4b_2 > 0$.

Now assume that $R_0 > 1$ and Equation (20) has at lease one positive root. We study Hopf bifurcations of System (2) at P^*. Solving p from Eq.(20) for the positive roots gives

$$p^{\pm} = \frac{1}{2}\Big[- (a_1^2 - \delta^2 - 2b_1) \pm \sqrt{(a_1 - \delta^2 - 2b_1)^2 - 4(b_1^2 - c_1^2)}\Big].$$

Note that if Equation (20) has a unique positive root, then it is p^+. Let $\omega^\pm = \sqrt{p^\pm}$. Solving for $\sin(\omega\tau)$ and $\cos(\omega\tau)$ from (5.2) and (5.3), we get

$$\cos(\omega\tau) = \frac{c_1\omega^2 - a_1\delta\omega^2 - b_1c_1}{c_1^2 + \delta^2\omega^2} = d_1(\omega)$$

and

$$\sin(\omega\tau) = \frac{\omega(a_1c_1 - b_1\delta + \delta\omega^2)}{c_1^2 + \delta^2\omega^2} = d_2(\omega).$$

Define $\tau_n^\pm, n = 0, 1, 2, \cdots$, as

$$\tau_n^\pm = \begin{cases} \frac{1}{\omega^\pm}(\arccos d_1(\omega^\pm) + 2n\pi) & \text{if } d_2(\omega^\pm) > 0, \\ \frac{1}{\omega^\pm}(2\pi - \arccos d_1(\omega^\pm) + 2n\pi) & \text{if } d_2(\omega^\pm) \le 0. \end{cases} \tag{25}$$

Hence, $\tau_n^\pm > 0$ and Equation (17) has a pair of purely imaginary roots $\pm i\omega^\pm$ when $\tau = \tau_n^\pm$ for $n = 0, 1, 2, \cdots$. Next, we try to establish the transversality condition for Hopf bifurcations, see [25, 26] for details about Hopf bifurcations. For $\tau \ge 0$, let

$$\lambda(\tau) = \alpha(\tau) + iw(\tau) \tag{26}$$

be the root of Equation (17) satisfying

$$\alpha(\tau_n^\pm) = 0, \quad w(\tau_n^\pm) = w^\pm.$$

Differentiating both sides of Equation (17) with respect to τ gives

$$\frac{d\lambda}{d\tau}[2\lambda + a_1 + (\delta - c_1\tau - \delta\tau\lambda)e^{-\lambda\tau}] = \lambda(c_1 + \delta\lambda)e^{-\lambda\tau}.$$

Notice that

$$\lambda^2 + a_1\lambda + b_1 = -(\delta\lambda + c_1)e^{-\lambda\tau},$$

it follows

$$\begin{aligned} \left(\frac{d\lambda}{d\tau}\right)^{-1} &= -\frac{2\lambda + a_1}{\lambda(\lambda^2 + a_1\lambda + b_1)} + \frac{\delta}{\lambda(\delta\lambda + c_1)} - \frac{\tau}{\lambda} \\ &= -\frac{1}{\lambda}\left[\tau + \frac{2\lambda + a_1}{\lambda^2 + a_1\lambda + b_1} - \frac{\delta}{\delta\lambda + c_1}\right]. \end{aligned}$$

Substituting $\tau = \tau_n^\pm$ and $\lambda = i\omega^\pm$ into the equality above, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^\pm} = \frac{-1}{iw^\pm}\left[\tau_n^\pm + \frac{2iw^\pm + a_1}{-(w^\pm)^2 + a_1iw^\pm + b_1} - \frac{\delta}{\delta iw^\pm + c_1}\right].$$

It follows that

$$\begin{aligned} \mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^\pm} &= \frac{a_1^2 - 2b_1 + 2(\omega^\pm)^2}{a_1^2(\omega^\pm)^2 + (b_1 - (\omega^\pm)^2)^2} - \frac{\delta^2}{c_1^2 + \delta^2(\omega^\pm)^2} \\ &= \frac{a_1^2 - 2b_1 + 2(\omega^\pm)^2 - \delta^2}{a_1^2(\omega^\pm)^2 + (b_1 - (\omega^\pm)^2)^2} \\ &= \frac{\pm\sqrt{(a_1 - \delta^2 - 2b_1)^2 - 4(b_1^2 - c_1^2)}}{a_1^2(\omega^\pm)^2 + (b_1 - (\omega^\pm)^2)^2} \\ &= \frac{\pm\sqrt{a_2^2 - 4b_2}}{c_1^2 + \delta^2(\omega^\pm)^2}. \end{aligned} \tag{27}$$

We thus obtained that $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^+} > 0$ and $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^-} < 0$. The discussion above establishes the following stability and Hopf bifurcation results.

Theorem 5.3. *Assume that $R_0 > 1$ and let $a_2, b_2, P_1, P_3, z^*, \omega^+, \tau_0^+$ be defined above. Assume that P_1 and P_3 have a unique positive root z_1 and z_2, respectively. We then have the following results.*

(I) *All roots of Equation (17) have negative real parts for all delay $\tau \geq 0$, if*

   (1) $z_1 = z_2$ *and* $z^* = z_1$, *or*

   (2) $z_1 > z_2$ *and* $z_2 \leq z^* \leq z_1$, *or*

   (3) $z^* > \max\{z_1, z_2\}$ *and* $a_2^2 - 4b_2 < 0$.

   Therefore, P^* is locally asymptotically stable for all $\tau \geq 0$.

(II) *There is a $\tau_0^+ > 0$ such that all roots of Equation (17) have negative real parts for all $\tau \in [0, \tau_0^+)$, and it has a pair of purely imaginary roots $\pm\omega^+ i$ and all other roots have negative real parts when $\tau = \tau_0^+$, if*

   (1) $z_1 < z_2$ *and* $z^* \leq z_2$, *or*

   (2) $z_1 \geq z_2$ *and* $z^* < z_2$, *or*

   (3) $z^* > \max\{z_1, z_2\}$ *and* $a_2^2 - 4b_2 > 0$.

   Therefore, P^* is locally asymptotically stable for all $\tau < \tau_0^+$. Hopf bifurcation occurs as τ passes cross $\tau = \tau_0^+$.

Remark 5.4. *Under conditions (II)(1) and (II)(2), Equation (20) has a unique positive root. By the local stability of P^* when $R_0 > 1$ and $\tau = 0$, the transversality condition (27), and Lemma 1.1, we can get that all roots of Equation (17) have negative parts when $\tau < \tau_0^+$. It has a pair of pure imaginary roots $\pm i\omega^+$ when $\tau = \tau_n^+$, and it has $2(n+1)$ roots with positive real parts when $\tau \in (\tau_n^+, \tau_{n+1}^+)$, $n = 0, 1, 2, \cdots$. P^* is unstable when $\tau > \tau_0^+$.*

Remark 5.5. *Under the condition $z^* > \max\{z_1, z_2\}$ and $a_2^2 - 4b_2 = 0$, Equation (20) has a unique positive root. We can get that all roots of Equation (17) have negative parts when $\tau < \tau_0^+$. It has a pair of pure imaginary roots $\pm i\omega^+$ when $\tau = \tau_n^+, n = 0, 1, 2, \cdots$, but $Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^+} = 0$ from (27). The Hopf bifurcation transversality condition fails.*

Remark 5.6. *Under the condition $z^* > \max\{z_1, z_2\}$ and $a_2^2 - 4b_2 > 0$, Equation (20) has two positive roots. It's easy to show that $\tau_0^+ < \tau_0^-$ and there exists an integer $i \geq 0$ such that*

$$\tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- < \cdots < \tau_i^+ < \tau_{i+1}^+.$$

We can conclude that P^* is asymptotically stable when τ is in the following intervals

$$[0, \tau_0^+), (\tau_0^-, \tau_1^+), \cdots, (\tau_{i-1}^-, \tau_i^+)$$

and unstable when τ is in the following intervals

$$(\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \cdots, (\tau_{i-1}^+, \tau_{i-1}^-).$$

6. Numerical simulations

In this section, we will conduct some numerical simulations to exhibit the theoretical results we obtained from the previous sections.

If we choose $A = 6, B = .2, \alpha = 1, \beta = 1, \eta = 0.5, \mu = 2, \delta = 0.4$. Then we have $P^* = (2.5, 0.4, 0.2)$, $R_0 = 1.22449 > 1$, and

$$a_1 = 1.8, \quad b_1 = 0.06, \quad c_1 = 0.96.$$

Consequently, we have

$$P_1(z) = 0.914 - 1.24z + 2.15z^2 - 1.5z^3$$

and

$$P_3(z) = -0.42 + 0.96z - 0.075z^2 + 0.75z^3.$$

They both have a unique positive root such that

$$z_1 = 1.17165, \quad z_2 = 0.4.$$

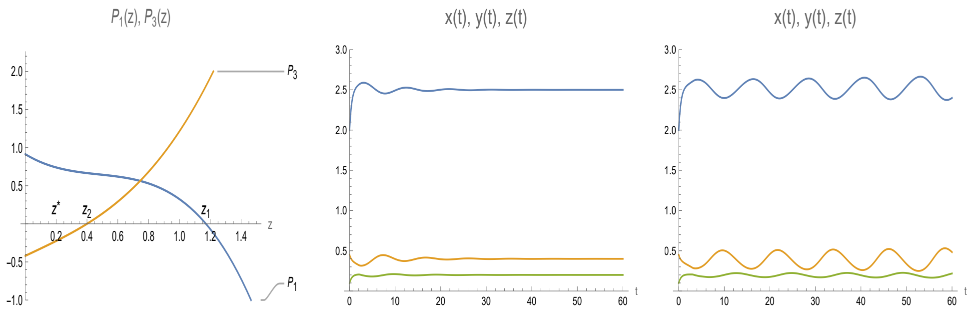
So we have

$$z^* = 0.2 < z_2 < z_1,$$

which fits the case (II)(2) from Theorem 5.2. Calculations give

$$a_2 = 2.96, \quad b_2 = -0.918.$$

In this case, Equation (20) has a unique positive root. We can find $\omega^+ = 0.5320039$ and $\tau_0^+ = 2.93232$, that means by Theorem 5.2 that P^* is locally asymptotically stable for all $\tau < 2.93232$, and when $\tau = 2.93232$, Equation (17) has a pair of purely imaginary roots $\pm 0.5320039i$ and all other roots have negative real parts. Hopf bifurcation occurs as τ passes cross $\tau = 2.93232$. See Figure 2(a) for the graphs of P_1 and P_2, 2(b) for solutions to converge to P^* for $\tau = 2$, and 2(c) for Hopf bifurcations to occur and periodic solutions to appear when $\tau = 2.93232$.



(a) Graphs of P_1 and P_3. (b) $\tau = 2$. Solutions converge to P^*. (c) $\tau = 2.93232$. Periodic solutions appear.

Figure 2: The case: $z^* < z_2 < z_1$.

Next, if we choose $A = 20, B = .2, \alpha = 0.3, \beta = 1, \eta = 0.5, \mu = 2, \delta = 0.4$. Then we have $P^* = (8.67745, 1.01608, 0.406468)$, $R_0 = 1.22449 > 1$, and

$$a_1 = 1.39678, \quad b_1 = 0.0232058, \quad c_1 = 0.92193.$$

Consequently, we have

$$P_1(z) = 0.8328 - 0.176z - 0.44z^2 - 0.8z^3$$

and

$$P_3(z) = -0.3164 - 0.076z - 0.04z^2 + 0.4z^3.$$

They both have a unique positive root such that

$$z_1 = 0.800331, \quad z_2 = 1.03002.$$

So we have

$$z_1 < z_2,$$

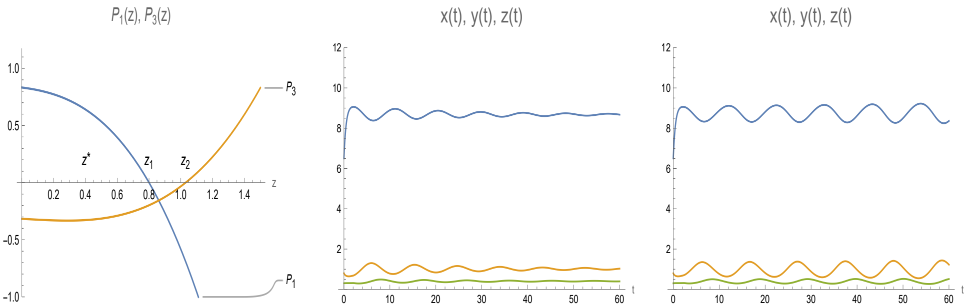
and

$$z^* = 0.406468 < z_2,$$

which fits the case (II)(1) from Theorem 5.2. Calculations give

$$a_2 = 1074459, \quad b_2 = -0.849417.$$

In this case, again Equation (20) has a unique positive root. We can find $\omega^+ = 0.629831$ and $\tau_0^+ = 2.28009$, that means by Theorem 5.2 that P^* is locally asymptotically stable for all $\tau < 2.28009$, and when $\tau = 2.28009$, Equation (17) has a pair of purely imaginary roots $\pm 0.629831i$ and all other roots have negative real parts. Hopf bifurcation occurs as τ passes cross $\tau = 2.28009$. See Figure 3(a) for the graphs of P_1 and P_2, 3(b) for solutions to converge to P^* for $\tau = 2$, and 3(c) for Hopf bifurcations to occur and periodic solutions to appear when $\tau = 2.28009$.



(a) Graphs of P_1 and P_3. (b) $\tau = 2$. Solutions converge to P^*. (c) $\tau = 2.28009$. Periodic solutions appear.

Figure 3: The case: $z^* < z_1 < z_2$.

7. Discussion

In this paper, we propose and study an online social network mathematical model. We divide the population into three major sub-classes: potential network users, active users, and people who are opposed to networks. Unlike the studies in the literature, in our model we assume that some people coming into the community that will never be interested in using online networks and active online social network users may lose their interest after a period of time (a time delay for abandoning networks). After a basic reproduction number R_0 is identified, we show that if $R_0 \leq 1$, the system has a unique user-free equilibrium P_0 and it exists for all parameter values, whereas if $R_0 > 1$, the system has two equilibria: P_0 and a unique user-prevailing equilibrium P^*. Local and global stability analysis for both P_0 and P^* is carried out when no delay is presented. In this case, the dynamics of the system is completely determined by the reproduction number R_0. If $R_0 \leq 1$, P_0 is globally asymptotically stable meaning that the number of active online social network users will converge to zero over time. Therefore, the social network will die out, and there will be fewer and fewer people using network. On the other hand, if $R_0 > 1$, i.e., each active user will generate more than one new network users, then P^* is globally asymptotically stable meaning that the number of active network users will settle at a certain number over time. The network users will be persistent. Assuming that $R_0 > 1$, we investigated whether or not the time delay for active users to abandon the network may cause the stability of P^* to switch as the delay τ is presented and increases. Conditions are established that guarantee that P^* will stay asymptotically stable for all delay $\tau \geq 0$, so people in all three sub-classes will settle at their corresponding numbers over time. We also found conditions and critical numbers such that under these conditions Hopf bifurcation occurs as the delay increases and crosses the critical numbers. Therefore, periodic solutions appear and the numbers of three sub-classes oscillate over time.

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