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Research Article

On Hybrid numbers with Gaussian Mersenne Coefficients

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Abstract

In this paper, we consider hybrid numbers with Gaussian Mersenne coefficients and investigate their interesting properties such as the Generating function, Binet formula, Cassini, Catalan, Vajda, D'Ocagne and Honsberger identities. Moreover, we illustrate the results with some examples.

Keywords: generating function, binet formula, cassini identity, catalan identity, vajda identity.

Gauss Mersenne Katsayılı Hibrit Sayılar

Öz

Bu çalışmada, Gauss Mersenne katsayılı hibrit sayıların tanımı ve bazı karakteristik özellikleri incelendi. Bu bağlamda, Üreteç Fonksiyonu, Binet formülü, Cassini, Catalan, Vajda, D'Ocagne ve Honsberger eşitlikleri elde edildi.

Anahtar Kelimeler: üreteç fonksiyonu, binet formülü, cassini özelliği, catalan özelliği, vajda özelliği

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1. Introduction

Recently, there has been a huge amount of interest to hybrid numbers which can be considered as a generalization of the complex numbers and composed of a combination of the complex ($i^2 = -1$), hyperbolic ($h^2 = 1$) and dual numbers ($\varepsilon^2 = 0$). The set of hybrid numbers, for details see [1], are defined as below:

$$\mathbb{K} = \{a + bi + c\varepsilon + dh: a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}.$$

Here, we want to take your attention that the product of any two hybrid numbers is done by exploiting the following table, please see [1]:

Table 1. Multiplication rule

$\bullet$	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1-h$	$\varepsilon+i$
ε	ε	$h+1$	0	$-\varepsilon$
h	h	$-\varepsilon-i$	ε	1

The Gaussian Mersenne sequence, denoted by GM_n, is defined by the recurrence relation

$$GM_n = 3GM_{n-1} - 2GM_{n-2}, \qquad \text{for n} \geq 2$$

with the initial conditions $GM_0 = -i/2, GM_1 = 1$. Note that the recurrence relation of Gaussian Mersenne sequence [2] can be rewritten as follows:

$$GM_n = M_n + iM_{n-1}$$

where $M_0 = 0$ and $M_1 = 1$. Some values of the Gaussian Mersenne numbers are given in

Table 2. Gaussian Mersenne numbers

n	0	1	2	3	4	5	6	7
GM_n	$-i/2$	1	$3+i$	$7+3i$	$15+7i$	$31+15i$	$63+31i$	$127+63i$

In literature, many researchers investigate some remarkable properties of some well-known sequences. For example, in [3], the authors get some properties for the Mersenne-Lucas hybrid numbers. In [4], the authors scrutinize some identities for the Mersenne, Jacobsthal and Jacobsthal-Lucas hybrid numbers. The authors obtain some results for the generalized tetranacci hybrid numbers, in [5]. For some similar studies, please see the references [6, 7, 8, 9, 10, 11, 12] and the references therein.

In this paper, we consider the hybrid numbers with Gaussian Mersenne coefficients. Then we get some characteristic relations of them.

2. Preliminaries

At this section, we give the definition of hybrid numbers with Gaussian Mersenne coefficients, denoted by HGM_n.

Definition 2.1. : Let us define the hybrid numbers with Gaussian Mersenne coefficients as below:

$$HGM_n = GM_n + GM_{n+1}i + GM_{n+2}\varepsilon + GM_{n+3}h; \;\; n \geq 0,$$

where HGM_n denotes the nth hybrid numbers with Gaussian Mersenne coefficients.

By using the definition of the HGM_n, we can write:

$$\begin{aligned} HGM_n &= GM_n + GM_{n+1}i + GM_{n+2}\varepsilon + GM_{n+3}h \\ &= 3GM_{n-1} - 2GM_{n-2} \\ &\quad +(3GM_n - 2GM_{n-1})i \\ &\quad +(3GM_{n+1} - 2GM_n)\varepsilon \\ &\quad +(3GM_{n+2} - 2GM_{n+1})h \end{aligned}$$

and

$$HGM_{n-1} = GM_{n-1} + GM_n i + GM_{n+1}\varepsilon + GM_{n+2}h$$

$$HGM_{n-2} = GM_{n-2} + GM_{n-1}i + GM_n\varepsilon + GM_{n+1}h$$

In other words, for $n \geq 2,$ the hybrid numbers with Gaussian Mersenne coefficients can be rewritten by following recurrence

$$HGM_n = 3HGM_{n-1} - 2HGM_{n-2},$$

with initial conditions $HGM_0 = 1 + \frac{7}{2}i + 6\varepsilon + 6h$ and $HGM_1 = 3 + 10i + 14\varepsilon + 12h$.

Table 3. Some Gaussian Mersenne hybrid numbers

n	HGM_n
0	$1+\frac{7}{2}i+6\varepsilon+6h$
1	$3+10i+14\varepsilon+12h$
2	$7+23i+30\varepsilon+24h$
3	$15+49i+62\varepsilon+48h$
4	$31+101i+126\varepsilon+96h$
5	$63+205i+254\varepsilon+192h$
6	$127+413i+510\varepsilon+384h$
7	$255+829i+1022\varepsilon+768h$
8	$511+1661i+2046\varepsilon+1536h$

In order to find the generating function for the hybrid numbers with Gaussian Mersenne coefficients, we have to write the sequence as a power series where each term of the sequence corresponds to the coefficients of the series. For more details, please see [13].

Lemma 2.1. :

$$HGM_n = 2HGM_{n-1} + (1+3i+2\varepsilon)$$

Proof. For $n \geq 0$, by exploiting $GM_{n+1} = 2GM_n + 1 + i$, please see [2];

$$HGM_n = GM_n + GM_{n+1}i + GM_{n+2}\varepsilon + GM_{n+3}h$$

$$= 2HGM_{n-1} + (1+3i+2\varepsilon).$$

3. Main Theorem and Proof

Theorem 3.1. : (Generating Function) The generating function for the hybrid numbers with Gaussian Mersenne coefficients is given by

$$g(x) = \sum_{n=0}^{\infty} (HGM_n)x^n = \frac{\left(1+\frac{7}{2}i+6\varepsilon+6h\right)+\left(\frac{-i}{2}-4\varepsilon-6h\right)x}{1-3x+2x^2}.$$

Proof. We first write

$$g(x) = HGM_0 + HGM_1 x + HGM_2 x^2 + \ldots + HGM_n x^n + \cdots$$

and compute

$$\begin{aligned} -3xg(x) &= -3HGM_0 x - HGM_1 x^2 - 3HGM_2 x^3 - \ldots - 3HGM_{n-1} x^n - \ldots \\ 2x^2 g(x) &= 2HGM_0 x^2 + 2HGM_1 x^3 + 2HGM_2 x^4 + \ldots + 2HGM_{n-2} x^n + \ldots \cdot \end{aligned}$$

From the above it follows that

$$(1 - 3x + 2x^2) g(x) = HGM_0 + (HGM_1 - 3HGM_0)x$$

$$+(HGM_2 - 3HGM_1 + 2HGM_0)x^2$$

$$+(HGM_3 - 3HGM_2 + 2HGM_1)x^3$$

$$\vdots$$

$$+(HGM_n - 3HGM_{n-1} + 2HGM_{n-2})x^n + \cdots .$$

As a result, we find

$$\begin{aligned} g(x) &= \frac{HGM_0 + (HGM_1 - 3HGM_0)x}{1 - 3x + 2x^2} \\ &= \frac{\left(1 + \frac{7}{2}i + 6\varepsilon + 6h\right) + \left(\frac{-i}{2} - 4\varepsilon - 6h\right)x}{1 - 3x + 2x^2}. \end{aligned}$$

So, the proof is completed.

The Binet formula is obtained by the following result.

Theorem 3.2. (**Binet Formula**): For $n \geq 0$, the Binet formula for the hybrid numbers with Gaussian Mersenne coefficients is given by

$$HGM_n = A2^n + B,$$

where $A = 2 + \frac{13}{2}i + 8\varepsilon + 6h$ and $B = -1 - 3i - 2\varepsilon$. (2.1)

Proof. By exploiting the generating function and the definition of the hybrid numbers with Gaussian Mersenne coefficients, we get

$$\begin{aligned} g(x) &= \frac{HGM_0 + (HGM_1 - 3HGM_0)x}{1 - 3x + 2x^2} \\ &= \frac{\left(1 + \frac{7}{2}i + 6\varepsilon + 6h\right) + \left(\frac{-i}{2} - 4\varepsilon - 6h\right)x}{1 - 3x + 2x^2} \\ &= \frac{A}{(1 - 2x)} + \frac{B}{(1 - x)}, \end{aligned}$$

where

$$A = \left(2 + \frac{13}{2}i + 8\varepsilon + 6h\right) \text{ and } B = (-1 - 3i - 2\varepsilon).$$

It can be rewritten

$$\begin{aligned} \frac{\left(1 + \frac{7}{2}i + 6\varepsilon + 6h\right) + \left(\frac{-i}{2} - 4\varepsilon - 6h\right)x}{1 - 3x + 2x^2} &= \frac{\left(2 + \frac{13}{2}i + 8\varepsilon + 6h\right)}{1 - 2x} + \frac{(-1 - 3i - 2\varepsilon)}{1 - x} \\ &= \left(\sum_{n=0}^{\infty} A2^n x^n\right) + \left(\sum_{n=0}^{\infty} Bx^n\right) \\ &= \sum_{n=0}^{\infty} (A2^n + B)x^n \end{aligned}$$

where:

$$g(x) = \sum_{n=0}^{\infty} (A2^n + B)x^n.$$

i.e.;

$$HGM_n = A2^n + B.$$

So, the proof is completed.

Example 3.1. : For $n = 3$, the hybrid numbers with Gaussian Mersenne coefficients HGM_3, with the Binet formula is

$$\begin{aligned} HGM_3 &= A2^3 + B \\ &= \left(2 + \frac{13}{2}i + 8\varepsilon + 6h\right)2^3 - 1 - 3i - 2\varepsilon \\ &= 15 + 49i + 62\varepsilon + 48h. \end{aligned}$$

Theorem 3.3. (**Cassini Identity**): For $n > 0$, the Cassini identity for the hybrid numbers with Gaussian Mersenne coefficients is given by

$$HGM_{n-1}HGM_{n+1} - HGM_n^2 = 2^n BA - 2^{n-1}AB,$$

where A and B are given in the equation (2.1).

Proof. From the Binet formula, we get;

$$\begin{aligned} HGM_{n-1}HGM_{n+1} - HGM_n^2 &= (A2^{n-1} + B)(A2^{n+1} + B) - (A2^n + B)(A2^n + B) \\ &= 2^{2n}A^2 + 2^{n-1}AB + 2^{n+1}BA + B^2 \\ &\quad -2^{2n}A^2 - 2^nAB - 2^nBA - B^2 \\ &= 2^{n-1}AB + 2^{n+1}BA - 2^nAB - 2^nBA \\ &= AB(2^{n-1} - 2^n) + BA(2^{n+1} - 2^n) \\ &= 2^nBA - 2^{n-1}AB. \end{aligned}$$

So, the proof is completed.

Example 3.2. : For $n = 3$, the Cassini identity is

$$\begin{aligned} HGM_2HGM_4 - HGM_3^2 &= 2^3BA - 2^2AB \\ &= 2^3\left(-\frac{39}{2} - \frac{61}{2}i - 18\varepsilon + 5h\right) \\ &\quad - 2^2\left(-\frac{39}{2} + \frac{11}{2}i - 6\varepsilon - 17h\right) \\ &= -78 - 266i - 120\varepsilon + 108h. \end{aligned}$$

Theorem 3.4. (**Catalan Identity**): For $n, r \geq 0$, the Catalan identity for the hybrid numbers with Gaussian Mersenne coefficients is given by

$$HGM_{n-r}HGM_{n+r} - HGM_n^2 = 2^{n-r}(1 - 2^r)[AB - 2^rBA],$$

where A and B are given in the equation (2.1).

Proof. By considering the Binet formula, we have;

$$\begin{aligned} HGM_{n-r}HGM_{n+r} - HGM_n^2 &= (A2^{n-r} + B)(A2^{n+r} + B) - (A2^n + B)(A2^n + B) \\ &= 2^{2n}A^2 + 2^{n-r}AB + 2^{n+r}BA + B^2 \\ &\quad -2^{2n}A^2 - 2^nAB - 2^nBA - B^2 \\ &= 2^{n-r}AB + 2^{n+r}BA - 2^nAB - 2^nBA \\ &= AB(2^{n-r}(1 - 2^r)) + BA(2^n(2^r - 1)) \end{aligned}$$

$$= 2^{n-r}(1-2^r)[AB - 2^r BA].$$

So, the proof is completed.

Example 3.3. : For $n = 3, r = 1$, the Catalan identity is

$$HGM_2 HGM_4 - HGM_3^2 = 2^2(1-2)[AB - 2BA]$$
$$= -2^2 AB + 2^3 BA$$
$$= 2^3 BA - 2^2 AB$$
$$= 2^3\left(-\frac{39}{2} - \frac{61}{2}i - 18\varepsilon + 5h\right)$$
$$-2^2\left(-\frac{39}{2} + \frac{11}{2}i - 6\varepsilon - 17h\right)$$
$$= -78 - 266i - 120\varepsilon + 108h.$$

Theorem 3.5. (**Vajda Identity**): For $n, m, r \geq 0$, the Vajda identity for the hybrid numbers with Gaussian Mersenne coefficients is given by

$$HGM_{n+r}HGM_{n+k} - HGM_n HGM_{n+r+k} = 2^n(2^r - 1)[AB - 2^k BA],$$

where A and B are given in the equation (2.1).

Proof. We conclude from the Binet formula that

$$HGM_{n+r}HGM_{n+k} - HGM_n HGM_{n+r+k} = (A2^{n+r} + B)(A2^{n+k} + B) - (A2^n + B)(A2^{n+r+k} + B)$$
$$= 2^{2n+r+k}A^2 + 2^{n+r}AB + 2^{n+k}BA + B^2$$
$$-2^{2n+r+k}A^2 - 2^n AB - 2^{n+r+k}BA - B^2$$
$$= 2^{n+r}AB + 2^{n+k}BA - 2^n AB - 2^{n+r+k}BA$$
$$= 2^n(2^r - 1)AB - 2^{n+k}(2^r - 1)BA$$
$$= 2^n(2^r - 1)[AB - 2^k BA].$$

So, the proof is completed.

Example 3.4. : For $n = 2, r = 1, k = 1$, the Vajda identity is

$$HGM_3 HGM_3 - HGM_2 HGM_4 = 2^2(2-1)[AB - 2BA]$$

$$= 2^2 AB - 2^3 BA$$

$$= -(2^3 BA - 2^2 AB)$$

$$= -\left[2^3\left(-\frac{39}{2} - \frac{61}{2}i - 18\varepsilon + 5h\right)\right.$$

$$\left. -2^2\left(-\frac{39}{2} + \frac{11}{2}i - 6\varepsilon - 17h\right)\right]$$

$$= -(-78 - 266i - 120\varepsilon + 108h)$$

$$= 78 + 266i + 120\varepsilon - 108h \ .$$

Theorem 3.6. (**D'Ocagne Identity**): For $n, m \geq 0$, the D'Ocagne identity for the Gaussian Mersenne hybrid numbers is given by

$$HGM_m HGM_{n+1} - HGM_n HGM_{m+1} = (2^n - 2^m)[2BA - AB],$$

where A and B are given in the equation (2.1).

Proof. From the Binet formula, we see that

$$HGM_m HGM_{n+1} - HGM_n HGM_{m+1} = (A2^m + B)(A2^{n+1} + B)$$

$$-(A2^n + B)(A2^{m+1} + B)$$

$$= 2^{n+m+1}A^2 + 2^m AB + 2^{n+1} BA + B^2$$

$$-2^{n+m+1}A^2 - 2^n AB - 2^{m+1} BA - B^2$$

$$= 2^m AB + 2^{n+1} BA - 2^n AB - 2^{m+1} BA$$

$$= (2^m - 2^n)AB + (2^{n+1} - 2^{m+1})BA$$

$$= (2^n - 2^m)[2BA - AB].$$

So, the proof is completed.

Example 3.5. : For $n = 3$, $m = 2$, the D'Ocagne identity is

$$HGM_2 HGM_4 - HGM_3 HGM_3 = (2^3 - 2^2)[2BA - AB]$$

$$= (2^2)[2BA - AB]$$

$$= 2^3 BA - 2^2 AB$$

$$= 2^3\left(-\frac{39}{2}-\frac{61}{2}i-18\varepsilon+5h\right)$$

$$-2^2\left(-\frac{39}{2}+\frac{11}{2}i-6\varepsilon-17h\right)$$

$$= -78-266i-120\varepsilon+108h.$$

Theorem 3.7. (**Honsberger Identity**): For $n, m \geq 0$, the Honsberger identity for the hybrid numbers with Gaussian Mersenne coefficients is given by

$$HGM_n HGM_m + HGM_{n+1} HGM_{m+1} = 2^{n+m}(5A^2) + 2^n(3AB) + 2^m(3BA) + 2B^2,$$

where A and B are given in the equation (2.1).

Proof. According to the Binet formula, we see that

$$HGM_n HGM_m + HGM_{n+1} HGM_{m+1} = (A2^n + B)(A2^m + B)$$

$$+(A2^{n+1} + B)(A2^{m+1} + B)$$

$$= 2^{n+m}A^2 + 2^n AB + 2^m BA + B^2$$

$$+2^{n+m+2}A^2 + 2^{n+1}AB + 2^{m+1}BA + B^2$$

$$= 2^{n+m}(5A^2) + 2^n(3AB) + 2^m(3BA) + 2B^2.$$

So, the proof is completed.

Theorem 3.8. (**Summation Formula**):

$$\sum_{k=0}^{n} HGM_k = HGM_{n+1} - (n+1)(1+3i+2\varepsilon) - \left(1+\frac{7}{2}i+6\varepsilon+6h\right).$$

Proof. By using the Lemma 2.1. ;

$$HGM_1 = 2HGM_0 + (1+3i+2\varepsilon)$$

$$HGM_2 = 2HGM_1 + (1+3i+2\varepsilon)$$

$$HGM_3 = 2HGM_2 + (1+3i+2\varepsilon)$$

$$\vdots$$

$$HGM_{n+1} = 2HGM_n + (1+3i+2\varepsilon)$$

$$2\sum_{k=0}^{n} HGM_k = \sum_{k=1}^{n+1} HGM_k - (n+1)(1+3i+2\varepsilon)$$

$$= \sum_{k=0}^{n} HGM_k - (n+1)(1+3i+2\varepsilon) - HGM_0 + HGM_{n+1}$$

i.e.;

$$\sum_{k=0}^{n} HGM_k = HGM_{n+1} - (n+1)(1+3i+2\varepsilon) - \left(1+\frac{7}{2}i+6\varepsilon+6h\right).$$

So, the proof is completed.

4. Conclusion

In this study, we initially present the hybrid numbers with Gaussian Mersenne coefficients. Then, we investigate some interesting properties for them. At this content, we obtain the Binet formula by exploiting the generating function. Moreover, we get the Generating function, Cassini identity, Catalan identity, Vajda identity, D'Ocagne identity and Honsberger identity for the hybrid numbers with Gaussian Mersenne coefficients. Also, we illustrate the obtained results with some examples.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally.

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