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A Note on a Special Metric Space with Triple Fixed Points

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ABSTRACT

A number satisfying the equation a = h(a) is called a fixed point of the function h since it doesn't change upon application of a map as geometrically. As an example, the function given by $a^2 = h(a)$ for all a has the two fixed points 0 and 1. Specifically, an element (a,b,c) is called a triple fixed point of h. H satisfies some conditions such as h(a,b,c) = a, h(b,a,b) = b, and h(c,b,a) = c. In this paper, we introduce a tripled fixed point theory in C*-algebra valued metric space and provide several results. We demonstrate existence and uniqueness of triple fixed point in a such space. In addition, we provide some examples to illustrate our results.

Üçlü Sabit Noktalı Özel Bir Metrik Uzay Üzerine Bir Not

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ÖZ

a = h(a) denklemini sağlayan a sayısına h fonksiyonunun sabit noktası denir, çünkü geometrik olarak bir dönüşümün uygulanmasıyla değişmez. Örnek olarak, tüm a değerleri için için $a^2 = h(a)$ ile verilen fonksiyon, 0 ve 1 olmak üzere iki sabit noktaya sahiptir. Spesifik olarak, h fonksiyonu, h(a,b,c) = a, h(b,a,b) = b ve h(c,b,a) = c gibi bazı koşulları sağlıyorsa, bir (a,b,c) elemanına h fonksiyonunun üçlü sabit noktası denir. Bu makalede, C* cebir değerli metrik uzayda üçlü sabit nokta teorisini tanıtıyoruz ve bunun için bir kaç sonuç elde ediyoruz. Böyle bir uzayda üçlü sabit noktanın varlığını ve tekliğini gösteriyoruz. Ek olarak, sonuçlarımızı göstermek için bazı örnekler sunuyoruz.

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1.Introduction

After working on concepts of cone or C*-algebra valued metric space, lots of consequences on the theory of fixed point have been considered/proved in such space as some references contained therein.

"Fixed point Theory and Application" (by Agarwal et al, 2001) is a fundamental book for fixed point theory in the literature. In this book, basic and useful results such as Banach's contraction theorem, different types of contraction mappings, applications of them in analysis, topology or applied mathematics are given along with plenty of examples.

Borcout (2012) worked on the problem as a triple coincidence of mappings and tripled common fixed points in the partially ordered metric space and established useful resultants in 2012.

Berinde and Borcout (2011) also extended some results on coupled fixed theorems by considering notations of tripled fixed points considering non-linear mappings in non-totally ordered complete metric spaces.

Ciric (1981) considered Edelstein's contractive mapping theorem and took steps in the direction to make a sweeping statement in the other way on complete and compact metric spaces.

Cosentine (2014) with his colleagues introduced a common fixed point on ordered or normal cone metric space and provided some examples for the theory of integral equations. Also, they gave numerical examples of integral equation theory in other works too.

Jha (2002) prepared the survey works on some applications of Banach contraction Principle such as the theory of the differentiable, the system of linear algebraic equations or O.D.E, theory of integral equations in 2002. His important discoveries about the connection between differential equations and fixed point theory led him to solve some initial value problems by using the Banach contraction mapping principle.

Lin (2001) prepared a book on the theory C*-algebras related to systems of dynamics, quantum mechanics, representation of operator theory, etc strongly… López (2017) explained some basic notations on metric spaces, the Banach contraction mapping and fixed point theory in his lecture notes.

Gholamian and his colleagues (2017) demonstrated a new type of improved metric spaces considering the concept of b_2 and C*-algebra valued metric spaces. They also obtained entity and unicity solutions for a kind of integral equations. Harandi (2013) also proved several important statements on the coupled fixed points using quasi-contraction type mapping for s metric space.

The book by Murphy (1990) has made a contribution worthy of note insights related to the fields of fundamental theorems of the theory for operators, Kaplanski's theorem of density, K-theory, a product of tensors, theory of representation with C*-algebras.

Wnag and Guo (2011) introduced c-distance on a cone metric space and also, obtained new results on common fixed point theorems.

The main idea of Srinuvasa et al. (2019) was to initiate bipolar metric space and found out the existence and uniqueness of tripled fixed point results for covariant mapping with some numerical examples in 2019.

Ma, Jiang and Sun (2014) studied the mappings related to C*-valued metric spaces and introduced many theorems on fixed points for such spaces. They laid out a new kind of metric space and proved several theorems for fixed points with self-maps of contractive or expansive conditions on such spaces.

Later on, many other authors also studied the entity of fixed points for self-mappings with contractive-type conditions. Özer and Omran (2016- 2019) studied C*-algebra valued metric spaces and presented several kinds of fixed point theorems in varieties of metric spaces.

Tomar et al. (2021) presented three different results on fixed point theory with different scapes in 2021. In their first result, they established common fixed point results of two maps holdings Hardy-Roger type contraction in complete partial metric space with some conditions. Also, they considered solving a Cantilever Beam problem using their results. Their second result was prepared on the results of C* Algebra valued matric spaces to show why such metric space results cannot be brought into existence from generalized results in other spaces with illustrates. In their third paper (published in 2021) they introduced C*-algebra valued partial metric space with contractiveness and expansiveness to elicit the fixed point results in the most generalized environment and they generalized many existing results to solve integral or operator equations.

For a survey of tripled fixed point theorems and related topics, we refer the readers to all references.

In this work, the general concept of C*-Algebra valued metric space is presented. Using the results from the preliminaries section, a function is defined and proved that it has unique triple fixed points in such space. A new general example is prepared to support our main theorem. Also, some special outcomes are obtained from these results and extra two numerical examples (one satisfies our results but another doesn't) are given to discuss the obtained results in this work.

2.Preliminaries, Materials and Method

Definition 2.1. (Özer and Omran, 2016) Let V be a C*-algebra. It is defined that V is a Banach algebra over the set of complex numbers with a map $\alpha \to \alpha^*$ $(\alpha \in V)$ if it holds the followings:

1. $\alpha^{**} = \alpha,\ \forall\alpha \in V$ (Involution property)
2. $(\alpha + \beta) = \alpha^* + \beta^*$ and $(\alpha\beta) = \alpha^*\beta^*$ for all α, β in V.
3. For every complex number λ and every α in V, we have $(\lambda\alpha)^* = \lambda\alpha^*$.
4. For all α in V, we have $\|\alpha\alpha^*\|$=$\|\alpha\|\|\alpha^*\|$.

Also, V is defined as v^*- algebra if the above first three identities are satisfied and the last identity is equivalent to $\|\alpha\alpha^*\| = \|\alpha\|^2$ named as the C*-identity.

Definition 2.2. (Özer and Omran, 2017) Let V and W be C*-algebras. Then, the operator $H : V \to W$ is a C*-homomorphism if conditions are performed as follows:

i. $H(\alpha^*) = (H(\alpha))^*$

ii. $H(\alpha\beta) = H(\alpha)H(\beta)$ for all α, β in V.

C*-homomorphism is called as C*-identity if any homomorphism (defined from C*-algebra to C*-algebra) is bounded with the norm ≤ 1.
From the literature, we can give following basic and simple examples too:

Example 2.3.

(1) If $\mathrm{Ш}_s(\mathbb{C})$ is considered as the set of all square matrices $s \times s$ over the set of the complex numbers with the involution conjugate transpose, then $\mathrm{Ш}_s(\mathbb{C})$ is determined as a C*-algebra.

(2) Let Ʀ be a Hilbert space. If γ^* is the dual of the operator, then γ : Ʀ → Ʀ becomes also a C*-algebra for the set of all bounded operators Ƀ(Ʀ).

Definition 2.4. (Özer and Omran, 2016) Let V be a C*-algebra and α be self-adjoint element ($\alpha^* = \alpha$) for α in V. Thus, α is named as a positive element if $\sigma(\alpha)$ (spectrum of α) is a positive real number.

Additionally, if α is a positive element, then it is denoted as $\alpha \geq 0$ and it is supposed that $V_+ = \{\alpha \in V : \alpha \geq 0\}$ is named as a positive set of V.

Lemma 2.5. (Özer and Omran, 2019) Let V be a C*- Algebra. Then, the followings are correct;

1. Let $\alpha \in V$ be normal. Then, $\alpha\alpha^* \geq 0$.
2. Let $\alpha \in V$ be a self adjoint and $||\alpha|| \leq 1$. Then, $\alpha \geq 0$
3. Let $\alpha, \beta \in V_+$. Then, $\alpha + \beta \in V_+$.
4. V_+ is closed in V.

Remark: Let $\alpha, \beta \in V_+$ then, it can be defined $\alpha \leq \beta$ for $\alpha - \beta \geq 0$ and obtained that $(V_+, \leq)$ is a partially ordered relation. If we consider that V is a C*-algebra with a unit element, then we get $||\alpha|| \leq ||\beta||$ under the restriction $0 \leq \alpha \leq \beta$ for all $\alpha, \beta \in V$. Besides, the positive element in a C*-algebra is called as normal.

C*-algebra valued metric space started to be considered by Ma who also worked on the self-mapping theorem of fixed points. In this part, after introducing several other notifications, the C*-algebra valued metric space is discussed. Moreover, the triple fixed point theory is presented.

Definition 2.6. (Ma et al. 2014) Let X be a set ($X \neq \emptyset$), V be a C*- algebra and d be a function defined from $X \times X$ to V_+. If the following given circumstances are provided, then triple (X, V, d) is named as C*- algebra valued metric space.

1. $d(\alpha_1, \alpha_2) = 0$ if and only if $\alpha_1 = \alpha_2$.
2. $d(\alpha_1, \alpha_2) = d(\alpha_2, \alpha_1)$.
3. $d(\alpha_1, \alpha_2) \leq d(\alpha_1, \alpha_3) + d(\alpha_3, \alpha_2)$, for any $\alpha_1, \alpha_2, \alpha_3$ in X.

Definition 2.7. (Ma et al. 2014) Let (X, V, d) be a C*-algebra valued metric space and also α_n be a sequence in X. α_n is converged to α in X for delivered $\varepsilon > 0$, if there exists a natural number Ƞ in the set of natural numbers such that $\|d(\alpha_n, \alpha)\| \leq \epsilon,\ \forall n, m > Ƞ$.

Definition 2.8. (Ma et al. 2014) Let (X, V, d) be a C*-algebra valued metric space and α_n be a sequence in X. α_n is a called as Cauchy sequence in space X for given $\epsilon > 0$,if there is an existence Ƞ in IN, such that $\|d(\alpha_n, \alpha_m)\| \leq \epsilon,\ \forall n, m > Ƞ$.

Definition 2.9. (Ma et al. 2014) The tripled C*-algebra valued metric space (X, V, d) is completed if every Cauchy sequence converges in X.

Example 2.10. Let X be a Banach space. The triple C*-algebra valued metric space (X, V, d) will be completed with metric d defined as $d(\alpha_1, \alpha_2) = \|\alpha_1 - \alpha_2\|h$, for all α_1, α_2 in X and h in V_+.

Definition 2.11. (Ma et al. 2014) Let (X, V, d) be a C*-algebra valued metric space. The operator $\delta: X \to X$ is called to be contractive on X if there comes into being $\alpha \in V$, $\|\alpha\| \leq 1$ and holds the following inequality:

$$d\big(\delta(\mu_1), \delta(\mu_2)\big) \leq \alpha^* d(\mu_1, \mu_2)\alpha, \quad \text{for all} \quad \mu_1, \mu_2 \text{ in } X.$$

Example 2.12. Let $X = [-1,1]$ and $V = \mathbb{R}^3$ be given with $\|(\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3)\| = (\,|\mathfrak{Y}_1|,\ |\mathfrak{Y}_2|,\ |\mathfrak{Y}_3|\,)$. Putting an order on V as follows:

$$(\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3) \leq (\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$$
$$\text{if and only if}$$
$$\mathfrak{Y}_1 \leq \mathfrak{T}_1,\ \mathfrak{Y}_2 \leq \mathfrak{T}_2 \ \text{ and } \ \mathfrak{Y}_3 \leq \mathfrak{T}_3.$$

We have that $(V_+, \leq)$ is partially order relation since $V_+ = V = \mathbb{R}^3$ (since V is finite dimensional space). Now assume $d: X \times X \to V_+$ is defined as: $d(\mathfrak{Y}_1, \mathfrak{Y}_2) = (|\mathfrak{Y}_1 - \mathfrak{Y}_2|, 0,0)$.

Thus, (X, V, d) is obtained as C*- algebra valued metric space.

3. Results and Discussion

Theorem 3.1. Assume that $(\text{Ҳ},\ \mathbb{V}, d)$ is a C*-algebra valued metric space and the mapping $\mathcal{S}$ is defined from Ҳ × Ҳ × Ҳ up to Ҳ and satisfying (1) as follows:

$$d\big(\mathcal{S}(\mu, \eta, \zeta), \mathcal{S}(\theta, \phi, s)\big) \leq \alpha d(\mu, \theta)\alpha^* + \beta d(\eta, \phi)\beta^* + \gamma d(\zeta, s)\gamma^* \tag{1}$$

for all $\mu, \eta, \zeta, \theta, \phi, s \in$ Ҳ and $\alpha, \beta, \gamma \in \mathbb{V}_+$ such that

$$\|\alpha\| \leq \frac{1}{3}, \|\beta\| \leq \frac{1}{3}, \|\gamma\| \leq \frac{1}{3}.$$

Then, $\mathcal{S}$ has a unicity triple fixed point.

Proof. Choose $\mu_o,\ \eta_o,\ \zeta_o \in \mathfrak{X}$ as follows:

$$\mu_1 = \mathcal{S}(\mu_o,\eta_o,\zeta_o),\ \eta_1 = \mathcal{S}(\eta_o,\zeta_o,\mu_o),\ \zeta_1 = \mathcal{S}(\zeta_o,\mu_o,\eta_o). \tag{2}$$

$$\mu_2 = \mathcal{S}(\mu_1,\eta_1,\zeta_1),\ \eta_2 = \mathcal{S}(\eta_1,\zeta_1,\mu_1),\ \zeta_2 = \mathcal{S}(\zeta_1,\mu_1,\eta_1). \tag{3}$$

$$\mu_m = \mathcal{S}(\mu_{m-1},\ \eta_{m-1},\ \zeta_{m-1}), \qquad \eta_m = \mathcal{S}(\eta_{m-1},\ \zeta_{m-1},\mu_{m-1}), \qquad \zeta_m = \mathcal{S}(\zeta_{m-1},\mu_{m-1},\eta_{m-1}) \tag{4}$$

$$\mu_{m+1} = \mathcal{S}(\mu_m,\eta_m,\zeta_m), \quad \eta_{m+1} = \mathcal{S}(\eta_m,\zeta_m,\mu_m), \quad \zeta_{m+1} = \mathcal{S}(\zeta_m,\mu_m,\eta_m) \tag{5}$$

From (1) and above equations, we have

$$\begin{aligned} d(\mu_{m+1},\mu_m) &= d\big(\mathcal{S}(\mu_m,\eta_m,\zeta_m),\mathcal{S}(\mu_{m-1},\ \eta_{m-1},\ \zeta_{m-1})\big) \\ &\le \alpha d(\mu_m,\mu_{m-1})\alpha^* + \beta d(\eta_m,\eta_{m-1})\beta^* + \gamma d(\zeta_m,\zeta_{m-1})\gamma^* \end{aligned}$$

In the same way, these equations are given as follows:

$$\begin{aligned} d(\eta_{m+1},\eta_m) &= d\big(\mathcal{S}(\eta_m,\zeta_m,\mu_m),\mathcal{S}(\eta_{m-1},\ \zeta_{m-1},\mu_{m-1})\big) \\ &\le \alpha d(\eta_m,\eta_{m-1})\alpha^* + \beta d(\zeta_m,\zeta_{m-1})\beta^* + \gamma d(\mu_m,\mu_{m-1})\gamma^* \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}(\zeta_{m+1},\zeta_m) &= d\big(\mathcal{S}(\zeta_m,\mu_m,\eta_m),\mathcal{S}(\zeta_{m-1},\mu_{m-1},\eta_{m-1})\big) \\ &\le a d(\zeta_m,\zeta_{m-1})\alpha^* + \beta d(\mu_m,\mu_{m-1})\beta^* + \gamma d(\eta_m,\eta_{m-1})\gamma^* \end{aligned}$$

Hence, following inequality is obtained.

$$\begin{aligned} d_{mm} \le\ & a d(\mu_m,\mu_{m-1})\alpha^* + \beta d(\eta_m,\eta_{m-1})\beta^* + \gamma d(\zeta_m,\zeta_{m-1})\gamma^* + a d(\eta_m,\eta_{m-1})\alpha^* \\ & + \beta d(\zeta_m,\zeta_{m-1})\beta^* + \gamma d(\mu_m,\mu_{m-1})\gamma^* + a d(\zeta_m,\zeta_{m-1})\alpha^* + \beta d(\mu_m,\mu_{m-1})\beta^* \\ & + \gamma d(\eta_m,\eta_{m-1})\gamma^* \end{aligned}$$

where $d_{mm} = d(\mu_{m+1},\mu_m) + d(\eta_{m+1},\eta_m) + d(\zeta_{m+1},\zeta_m)$ for all $m \ge 0$.

In general, we get

$$\begin{aligned} d_{mm} &\le (\alpha+\beta+\gamma)d_{(m-1)(m-1)}(\alpha^*+\beta^*+\gamma^*) \\ &\le (\alpha+\beta+\gamma)^2 d_{(m-2)(m-2)}(\alpha^*+\beta^*+\gamma^*)^2 \\ &\le \cdots \\ &\le (\alpha+\beta+\gamma)^m d_{oo}(\alpha^*+\beta^*+\gamma^*)^m \end{aligned}$$

where $d_{oo} = d(\mu_1,\mu_0) + d(\eta_1,\eta_0) + d(\zeta_1,\zeta_0) = \mathcal{H}$.

If we put $\mathfrak{S} = (\alpha+\beta+\gamma)$ in the upper inequality, then we obtain following result,

$$d_{mm} \leq \mathfrak{S}^m \mathcal{H} (\mathfrak{S}^*)^m = \mathfrak{S}^m \mathcal{H}^{\frac{1}{2}} \mathcal{H}^{\frac{1}{2}} (\mathfrak{S}^*)^m = \left(\mathfrak{S}^m \mathcal{H}^{\frac{1}{2}}\right)\left(\mathcal{H}^{\frac{1}{2}} (\mathfrak{S}^m)\right)^*$$

For $m+1 > n$, we also obtain

$$d_{mn} = d(\mu_{m+1}, \mu_n) + d(\eta_{m+1}, \eta_n) + d(\zeta_{m+1}, \zeta_n)$$

$$\leq d(\mu_{m+1}, \mu_n) + d(\mu_m, \mu_n) + d(\eta_{m+1}, \eta_n) + d(\eta_m, \eta_n) + d(\zeta_{m+1}, \zeta_n) + d(\zeta_m, \zeta_n)$$

$$\leq \left(\mathfrak{S}^m \mathcal{H}^{\frac{1}{2}}\right)\left(\mathcal{H}^{\frac{1}{2}} (\mathfrak{S}^m)\right)^* + (\mathfrak{S}^{m-1} \mathcal{H}^{\frac{1}{2}})\left(\mathcal{H}^{\frac{1}{2}} (\mathfrak{S}^{m-1})\right)^* + \cdots + (\mathfrak{S}^n \mathcal{H}^{\frac{1}{2}})\left(\mathcal{H}^{\frac{1}{2}} (\mathfrak{S}^n)\right)^*$$

$$= \sum_{k=n}^{m} \left|\mathfrak{S}^k \mathcal{H}^{\frac{1}{2}}\right|^2.$$

Hence, we have

$$\|d_{mn}\| \leq \left\|\sum_{k=n}^{m} \left|\mathfrak{S}^k \mathcal{H}^{\frac{1}{2}}\right|^2\right\| \leq \sum_{k=n}^{m} \|\mathfrak{S}^{k}\|^2 \|\mathcal{H}^{\frac{1}{2}}\|^2$$

$$= \|\mathcal{H}^{\frac{1}{2}}\|^2 \sum_{k=n}^{m} \|\mathfrak{S}\|^{2k} = \|\mathcal{H}^{\frac{1}{2}}\|^2 \frac{\|\mathfrak{S}\|^{2n}}{1 - \|\mathfrak{S}\|}$$

(by Geometry series)

So, the following equivalents are satisfied.

$$\|d(\mu_{m+1}, \mu_n) + d(\eta_{m+1}, \eta_n) + d(\zeta_{m+1}, \zeta_n)\| \leq \| \mathcal{H}^{\frac{1}{2}} \|^2 \frac{\|\mathfrak{S}\|^{2n}}{1 - \|\mathfrak{S}\|}$$

and

$$\|\mathcal{H}^{\frac{1}{2}}\|^2 \frac{\|\mathfrak{S}\|^{2n}}{1 - \|\mathfrak{S}\|} = 0_{\mathcal{H}}$$

since $\|\mathfrak{S}\| \leq 1$.

These imply that μ_m, η_m, ζ_m are Cauchy sequences in Ӿ with respect to $\mathbb{V}$. Since (Ӿ, $\mathbb{V}, d$) is completed, then following equations are hold;

$$\lim_{m\to\infty} \mu_m = \mu, \qquad \lim_{m\to\infty} \eta_m = \eta, \qquad \lim_{m\to\infty} \zeta_m = \zeta,$$

where $\mu, \eta, \zeta \in$ Ӿ.

Now, by using inequality (1), we get

$$d(\mathcal{S}(\mu, \eta, \zeta), \mu) \leq d(\mathcal{S}(\mu, \eta, \zeta), \mu_{n+1}) + d(\mu_{m+1}, \mu)$$

$$= d(\mathcal{S}(\mu, \eta, \zeta), \mathcal{S}(\mu_m, \eta_m, \zeta_m)) + d(\mu_{m+1}, \mu)$$

$$\leq \alpha d(\mu, \mu_m)\alpha^* + \beta d(\eta, \eta_m)\beta^* + \gamma d(\zeta, \zeta_m)\gamma^* + d(\mu_{m+1}, \mu)$$

Therefore,

$$\lim_{m\to\infty} d(\mu,\mu_m)=0, \qquad \lim_{m\to\infty} d(\eta,\eta_m)=0 \quad \text{and} \quad \lim_{m\to\infty} d(\zeta,\zeta_m)=0$$

It refers that

$$\lim_{m\to\infty} d(\mathcal{S}(\mu,\eta,\zeta),\mu)=0.$$

In the same way,

$$lim_{m\to\infty} \; d(\mathcal{S}(\eta,\zeta,\mu),\eta)=0 \quad \text{and} \quad \lim_{m\to\infty} d(\mathcal{S}(\zeta,\mu,\eta),\zeta)=0.$$

Thus, (μ,η,ζ) is tripled fixed point for $\mathcal{S}$.

Now, let (μ',η',ζ') be another tripled fixed point of $\mathcal{S}$. Then some of the equalities will be given as follows:

$$d(\mu',\mu)=d\big(\mathcal{S}(\mu',\eta',\zeta'),\mathcal{S}(\mu,\eta,\zeta)\big)\le \alpha d(\mu',\mu)\alpha^*+\beta d(\eta',\eta)\beta^*+\gamma d(\zeta',\zeta)\gamma^*$$

$$d(\zeta',\zeta)\le \alpha d(\zeta',\zeta)\alpha^*+\beta d(\mu',\mu)\beta^*+\gamma d(\eta',\eta)\gamma^*$$

$$d(\eta',\eta)\le \alpha d(\eta',\eta)\alpha^*+\beta d(\zeta',\zeta)\beta^*+\gamma d(\mu',\mu)\gamma^*$$

Hence, it is:

$$d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)=$$

$$(\alpha+\beta+\gamma)[d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)](\alpha+\beta+\gamma)^*=$$

$$\mathfrak{S}[d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)]\mathfrak{S}^*.$$

Therefore, we obtain following equality by using $\|\mathfrak{S}\|\le 1$;

$$\|d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)\|=$$

$$\|\mathfrak{S}\|\|d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)\|\|\mathfrak{S}^*\|\le$$

$$\|d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)\|\|\mathfrak{S}^*\|$$

This implies a contradiction. Hence we obtain following equation,

$$d(\mu',\mu)+d(\eta',\eta)+d(\zeta',\zeta)=0_{\mathcal{H}}$$

Also, it is obtained;

$$d(\mu',\mu)=d(\eta',\eta)=d(\zeta',\zeta)=0$$

since $d(\mu',\mu), d(\eta',\eta), d(\zeta',\zeta)$ are positive numbers. So, it can be easily observed that

$$\mu'=\mu, \qquad \eta'=\eta \quad \text{and} \quad \zeta'=\zeta.$$

Therefore, $\mathcal{S}$ has a oneness tripled Fixed Point.

Corollary 3.2. Suppose that $(\text{Ꭓ}, \mathbb{V}, d)$ is a C^*- algebra valued metric space and the mapping Ꮥ is defined from $\text{Ꭓ} \times \text{Ꭓ} \times \text{Ꭓ}$ up to Ꭓ and satisfying;

$$d\big(\text{Ꮥ}(\mu,\eta,\zeta), \text{Ꮥ}(\theta,\phi,s)\big) \leq \aleph[d(\mu,\theta) + d(\eta,\phi) + d(\zeta,s)]\aleph^*$$

for all $\mu,\eta,\zeta,\theta,\phi,s \in \text{Ꭓ}$ and $\aleph \in \mathbb{V}_+$ such that $\|\aleph\| \leq \frac{1}{3}$. Then Ꮥ has an individuality tripled fixed point.

Example 3.3. Assume that Ꭓ is Banach space and $d: \text{Ꭓ} \times \text{Ꭓ} \rightarrow \mathbb{V}$ is defined by $d(x,y) = \|x-y\|.h$ where h is self adjoint and positive operator in $\mathbb{V}$. Now, it is defined that

$$d\big(\text{Ꮥ}(\mu,\eta,\zeta), \text{Ꮥ}(\theta,\phi,s)\big) \leq \frac{\|\mu-\theta\|}{3}.I + \frac{\|\eta-\phi\|}{3}.I + \frac{\|\zeta-s\|}{3}.I \qquad (*)$$

for all $\mu,\eta,\zeta,\theta,\phi,s \in \text{Ꭓ}$ where I is unit in $\mathbb{V}$ and $\alpha,\beta,\gamma \in \mathbb{V}_+$. This condition satisfies the Theorem 3.1 with $\alpha = \beta = \gamma = \frac{I}{\sqrt{3}}$.

Following examples are given too.

Example 3.4. Let $\text{Ꭓ} = [-1,1]$ and $\mathbb{V} = \mathbb{R}^3$ are given with $\|(\beth_1,\beth_2,\beth_3)\| = (|\beth_1|, |\beth_2|, |\beth_3|)$. If we put order on $\mathbb{V}$ as follows;

$$(\beth_1,\beth_2,\beth_3) \leq (\eta_1,\eta_2,\eta_3) \quad \text{if and only if} \quad \beth_1 \leq \eta_1,\ \beth_2 \leq \eta_2,\ \beth_3 \leq \eta_3$$

then it is obtained that $(\mathbb{V}_+, \leq)$ is partially order relation since $\mathbb{V}_+ = \mathbb{V} = \mathbb{R}^3$ (since V is finite dimensional space).

Now, assume that $d: \text{Ꭓ} \times \text{Ꭓ} \rightarrow \mathbb{V}$ is defined as $d(\beth_1,\beth_2) = (|\beth_1 - \beth_2|, 0, 0)$ and Ꮥ is given as

$$\text{Ꮥ}(\beth,\eta,\zeta) = \begin{cases} 1 & , \beth\,\eta\,\zeta \neq 0 \\ 0 & , \beth\,\eta\,\zeta = 0 \end{cases}$$

Then, Ꮥ has two triple fixed points. Furthermore, these triple fixed points are just defined as (1,1,1) and (0,0,0). Thus, the condition in Theorem 3.1 is not satisfied.

Example 3.5. Considering that $\text{Ꭓ} = [0,1]$ and $\mathbb{V} = \text{Ш}_2(\mathbb{C})$ are determined by $\|\mathrm{D}\| = Max_i\, |d_i|$. If we put order on $\mathbb{V}$ as follows;

$$\mathrm{D} \leq \mathbb{F} \quad \text{if and only if} \quad d_i \leq f_i$$

for all i . Then it is obtained that $(\mathbb{V}_+, \leq)$ is a partial order relation since $\mathbb{V}_+ = \mathbb{V} = \text{Ш}_2(\mathbb{C})$ (i.e. since $\mathbb{V}$ is a finite dimensional space and $\text{Ш}_2(\mathbb{C})$ is defined in (1) of the Example 2.3 for s=2).

Now, assume $d: \text{Ꭓ} \times \text{Ꭓ} \rightarrow \mathbb{V}$ is defined as $d(\text{ꝥ}_1, \text{ꝥ}_2) = \begin{bmatrix} Max\,\{\text{ꝥ}, \text{ẟ}, \zeta\} & 0 \\ 0 & 0 \end{bmatrix}$ and Ꮥ is also given as $\text{Ꮥ}(\text{ꝥ},\eta,\zeta) = Max\,\{\text{ꝥ}, \text{ẟ}, \zeta\}$.

Then, S has a unique triple fixed point since Theorem 3.1 is satisfying.

4.Conclusions

The fixed point theory is a significant tool in nonlinear analysis (optimization, dynamic systems, ODE and PDE, game theory, mathematical modeling, etc.) to solve some problems with entity and unicity. That is why a kind of fixed points (triple fixed points) in C*-algebra valued metric spaces are constructed in this work. Also, numerical examples are given to support the results. It is hoped that our work can be useful for readers and take a place in the literature.

Declaration

This study does not require ethics committee approval.

Conflict of Interest

There is no conflict of authors in this work.

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