2-Rainbow Domination Number of Some Graphs

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Abstract

A *2-rainbow domination function* of a graph G is a function *f* that assigns to each vertex a set of colors chosen from the set {1,2}, such that for any $v \in V(G), f(v) = \emptyset$ implies $\bigcup_{u \in N(v)} |f(v)| = \{1,2\}$. The *2-rainbow domination number* $\gamma_{r2}(G)$ of a graph G is minimum $w(f) = \sum_{v \in V(G)} |f(v)|$ over all such functions *f*. In this paper, we show that upper bounds of 2-rainbow domination numbers of several classes of graphs.

Keywords - Vulnerability, domination, 2-rainbow domination , $k-$th power of a graph

1 Introduction

Domination have been extensively studied concept of Graph Theory. It has lots of variations, rainbow domination is one of them. Rainbow domination introduced by Brešar, Henning and Rall in 2005 [1]. They started to study on *k*-rainbow domination of a graph G [1,2]. Brešar and Šumenjak gave exact values of 2- rainbow domination numbers of several classes of graphs and also they shown that sharp bounds of GP(n,k) [3]. In 2009, Chungling et. all. showed that exact values of $\gamma_{r2}(P(n,2))$ for some α and n values [5]. Xu, showed that $\gamma_{r2}(P(n,3)) \leq n-1$ for all $n \geq 13$ and $\gamma_{r2}(P(n,3)) \leq n - \left\lfloor \frac{n}{8} \right\rfloor + \beta$, where $\beta = 0 \; for \; n = 0,2,4,5,6,7,13,14,15 (mod\ 16)$ and $\beta = 1 \; for \; n = 1,3,8,9,10,11,12\ (mod\ 16)$ [6].

Throughout this paper, we consider finite, simple and undirected graphs. For standard Graph Theory terminology not given here we refer to [4].

Let $C = \{1,2,\dots,k\}$ be a set of k colors and f be a function that assigns to each vertex a set of colors chosen from C, that is $f : V(G) \to \mathrm{P}(\{1,2,3,\dots,k\})$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have

$$\bigcup_{u \in N(v)} |f(v)| = \{1,2,\dots,k\}$$

then f is called a *k*-rainbow domination function (*k*RDF) of G. The weight $w(f)$ of *f* is

$$w(f) = \sum_{v \in V(G)} |f(v)|.$$

The minimum weight of a *k*RDF of graph G is called *k*-rainbow domination number of G and shown as $\gamma_{rk}(G)$ [6].

In this paper, we show that upper bounds of $\gamma_{r2}(P_n^k)$, $\gamma_{r2}(C_n^k)$, $\gamma_{r2}(H_n)$, where (P_n^k) and (C_n^k) are power of path with n vertices, P_n and cycle with n vertices, C_n respectively and also H_n be the Helm graph.

2 2-Rainbow Domination Number of Some Graphs

In this part, we show upper bounds for 2-rainbow domination number of some graphs. We use procedure of [7] to prove the upper bounds of $\gamma_{r2}(P_n^k)$ and $\gamma_{r2}(C_n^k)$.

Definition 1. The k-th power of a graph G, denoted G^k, is a graph with the same vertex set as G, such that two vertices are adjacent in G^k if and only if their distance is at most k in G.

Theorem 1. Let P_n^k be k-th power of P_n where $k \geq 2$ and $n \geq 2k+2$;

$$\gamma_{r2}(P_n^k) \leq \begin{cases} 2\left\lceil \frac{n}{2k+2} \right\rceil - 1, & n \equiv 1 (\mathrm{mod}\ 2k+2); \\ 2\left\lceil \frac{n}{2k+2} \right\rceil, & n \equiv 2,3,\ldots,k+2\ (\mathrm{mod}\ 2k+2); \\ 2\left\lceil \frac{n}{2k+2} \right\rceil + 1, & n \equiv k+3,\ldots,2k,2k+1,0\ (\mathrm{mod}\ 2k+2). \end{cases}$$

Proof. Clearly, for the proof it suffices to construct a 2RDF of P_n^k of weight $2\left\lceil \frac{n}{2k+2} \right\rceil - 1$, $2\left\lceil \frac{n}{2k+2} \right\rceil$ or $2\left\lceil \frac{n}{2k+2} \right\rceil + 1$. We use lines to denote a 2RDF, where in the line there are values of vertices $v_1, v_2, \ldots, v_n$. We use 0,1,2 and 3 to denote subsets $\emptyset, \{1\}, \{2\}, \{1,2\}$, respectively. We distinguish the following cases:

Case 1: $n \equiv 0\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k-1}\ \ 1$

Case 2: $n \equiv 1\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1$

Case 3: $n \equiv 2\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 13$

Case 4: $n \equiv 3\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 102$

Case 5: $n \equiv 4\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1002$

⋮

Case k+3 : $n \equiv k+2\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1\underbrace{00\ldots0}_{k}\ \ 2$

Case k+4 : $n \equiv k+3\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1\underbrace{00\ldots0}_{k}$
23

Case k+5 : $n \equiv k+4\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1\underbrace{00\ldots0}_{k}$
201

Case k+6 : $n \equiv k+5\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ \ldots\ \ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1\underbrace{00\ldots0}_{k}$
2001

⋮

Case 2k+2 : $n \equiv 2k+1\ mod(2k+2)$

$1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ldots\ 1\underbrace{00\ldots0}_{k}\ \ 2\underbrace{00\ldots0}_{k}\ \ 1\underbrace{00\ldots0}_{k}$
$2\underbrace{00\ldots0}_{k-2}\ \ 1$

In each case, one can check that the function above is a 2RDF and is of weight

$$2\left\lceil \frac{n}{2k+2} \right\rceil - 1$$

for $n \equiv 1\ mod(2k+2)$, weight

$$2\left\lceil \frac{n}{2k+2} \right\rceil$$

for $n \equiv 2,3,\ldots,k+2\ mod(2k+2)$ and weight

$$2\left\lceil \frac{n}{2k+2} \right\rceil + 1$$

for $n \equiv k+3,\ldots,2k,2k+1,0\ mod(2k+2)$.
Therefore,

$$\gamma_{r2}(P_n^k) \leq \begin{cases} 2\left\lceil \frac{n}{2k+2} \right\rceil - 1, & n \equiv 1 (\mathrm{mod}\ 2k+2); \\ 2\left\lceil \frac{n}{2k+2} \right\rceil, & n \equiv 2,3,\ldots,k+2\ (\mathrm{mod}\ 2k+2); \\ 2\left\lceil \frac{n}{2k+2} \right\rceil + 1, & n \equiv k+3,\ldots,2k,2k+1,0\ (\mathrm{mod}\ 2k+2). \end{cases}$$

□

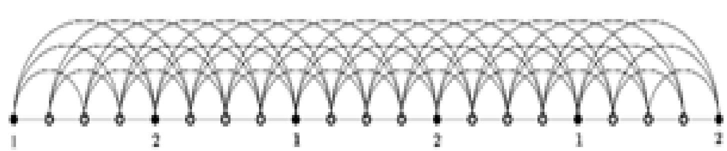

Figure 1. A 2RDF of weight 6 of P_{21}^3

Theorem 2. Let C_n^k be k-th of C_n where $k \geq 2$ and $n \geq 2k+2$;

$$\gamma_{r2}(C_n^k) \leq \begin{cases} 2\left\lceil \frac{n}{2k+2} \right\rceil - 1, & n \equiv 1 \pmod{2k+2}; \\ 2\left\lceil \frac{n}{2k+2} \right\rceil, & n \not\equiv 1 \pmod{2k+2}. \end{cases}$$

Proof: Clearly, for the proof it suffices to construct a 2RDF of C_n^k of weight $2\left\lceil \frac{n}{2k+2} \right\rceil - 1$ or $2\left\lceil \frac{n}{2k+2} \right\rceil$. We use lines to denote a 2RDF, where in the line there are values of vertices $v_1, v_2, ..., v_n$. We use 0,1,2 and 3 to denote subsets $\emptyset, \{1\}, \{2\}, \{1,2\}$, respectively.

Case 1: $n \equiv 0 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k}$

Case 2: $n \equiv 1 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1$

Case 3: $n \equiv 2 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 13$

Case 4: $n \equiv 3 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 102$

Case 5: $n \equiv 4 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1002$

Case 6: $n \equiv 5 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 10002$

⋮

Case k+3: $n \equiv k+2 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1\underbrace{00...0}_{k}$

2

Case k+4: $n \equiv k+3 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1\underbrace{00...0}_{k}$

20

Case k+5: $n \equiv k+4 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1\underbrace{00...0}_{k}$

200

⋮

Case 2k+1: $n \equiv 2k \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1\underbrace{00...0}_{k}$

$2\underbrace{00...0}_{k-2}$

Case 2k+2: $n \equiv 2k+1 \; mod(2k+2)$

$1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; \ldots \; 1\underbrace{00...0}_{k} \; 2\underbrace{00...0}_{k} \; 1\underbrace{00...0}_{k}$

$2\underbrace{00...0}_{k-1}$

In each case, one can check that the function above is a 2RDF and is of weight

$$2\left\lceil \frac{n}{2k+2} \right\rceil - 1$$

for $n \equiv 1 \; (mod\; 2k+2)$ and weight

$$2\left\lceil \frac{n}{2k+2} \right\rceil$$

For $\mathbf{n \not\equiv 1 (mod\; 2k+2)}$.

Hence,

$$\gamma_{r2}(C_n^k) \leq \begin{cases} 2\left\lceil \frac{n}{2k+2} \right\rceil - 1, & n \equiv 1 \pmod{2k+2}; \\ 2\left\lceil \frac{n}{2k+2} \right\rceil, & n \not\equiv 1 \pmod{2k+2}. \end{cases}$$

□

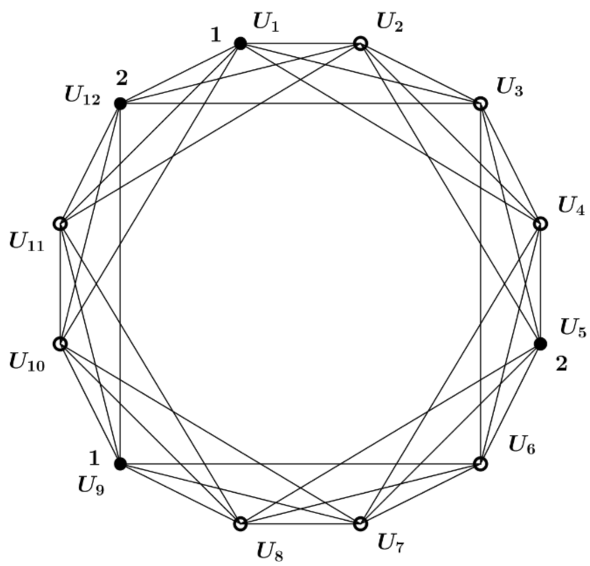

Figure 2. A 2RDF of weight 4 of C_{12}^{3}

Definition 2. *The Helm graph* H_n is the graph obtained from on *(n+1)*-wheel graph by joining a pendant edge at each vertex of the cycle.

W_{n+1} is a graph that contains a cycle of vertex n and for which every graph in the cycle is connected to one other graph vertex (which is known as the Hub).

Let the hub vertex be v_{n+1}, vertices on the cycle be $v_1, v_2, \ldots, v_n$ and the end vertices of the graph be $u_1, u_2, \ldots, u_n$.

Theorem 3. Let H_n be the Helm graph. Upper bound of *2-Rainbow domination number* of the H_n for $n \geq 3$ is

$$\boldsymbol{\gamma_{r2}}\left(H_n\right) \leq \mathbf{n+1}.$$

Proof. The 2RDFs values are used {1},{2},∅ which we denote by 1,2,0, respectively.

$f : V(H_n) \rightarrow \mathrm{P}$ ({1,2}) be defined as follows,

$$\boldsymbol{f}(\mathbf{v_i}) = \mathbf{0}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}.$$
$$\boldsymbol{f}(\boldsymbol{v_{n+1}}) = \mathbf{2},$$
$$\boldsymbol{f}(\boldsymbol{u_i}) = \mathbf{1}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}.$$

Then f is a 2RDF of H_n. Therefore,

$\boldsymbol{\gamma_{r2}}\left(H_n\right) \leq w\ (\boldsymbol{f}) = \boldsymbol{n} + \mathbf{1}.$ □

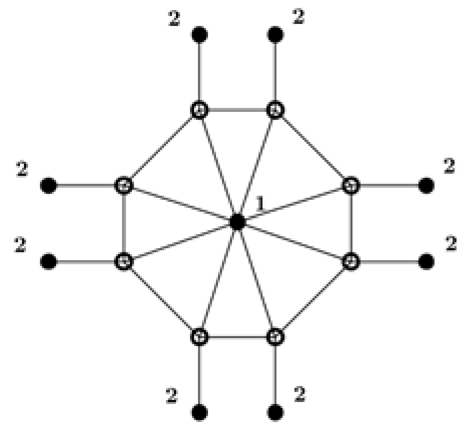

Figure 3. A 2RDF of H_8

5 References

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