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On Fibonacci (k,p)-Numbers and Their Interpretations

Yasemin TAŞYURDU [*1], Berke CENGİZ[1]

Abstract

In this paper, we define new kinds of Fibonacci numbers, which generalize both Fibonacci, Jacobsthal, Narayana numbers and Fibonacci p-numbers in the distance sense, using the definition of a distance between numbers by a recurrence relation according to a new parameter k. Tiling and combinatorial interpretations of these numbers are presented, and explicit formulas that allow us to calculate the nth number are given. Also, their generating functions are obtained and sums formulas of these numbers with special subscripts are given by tiling interpretations that allow the derivation of their properties.

Keywords: Combinatorial identities, Fibonacci p-numbers, generalized Fibonacci numbers, tilings

1. INTRODUCTION

Fibonacci numbers are given by recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial terms $F_0 = 0$, $F_1 = 1$. These numbers have many generalizations, applications and interpretations presented in different ways [1]. Most of generalizations are obtained by changing the coefficient and distance between the added terms in the recurrence relation of the Fibonacci numbers and their initial terms. For instance, k-Fibonacci numbers are defined by recurrence relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$ with initial terms $F_{k,0} = 0$, $F_{k,1} = 1$ by generalizing Fibonacci numbers according to the value of a new parameter k for the coefficient of the $(n-1)$th term in recurrence relation [2]. Then, a new family of k-Fibonacci numbers is defined and its period and many properties of this family are presented [3-7].

On the other hand, taking into account the parameter p for distance between the added terms in the recurrence relation of the Fibonacci numbers, Fibonacci p-numbers are determined the by recurrence relation

$$F_p(n) = F_p(n-1) + F_p(n-p-1),\ n > p+1$$

with initial terms $F_p(1) = F_p(2) = \cdots = F_p(p+1) = 1$ for $p \geq 0$, and presented $(p \times 1) \times (p \times 1)$ companion matrix for these numbers [8, 9]. Some authors have presented fundamental identities of the Fibonacci p-numbers that are similar to well-known properties of the Fibonacci numbers and have provided various general formulas for these numbers. Using various properties of Pascal's triangle, the Fibonacci

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p-numbers can be derived and their matrix representations are given [10, 11].

The Fibonacci p-numbers have led to the discovery of an infinite amount of the number sequences presented by recurrence relations called generalized or distance Fibonacci numbers. For instance, generalized Fibonacci numbers are defined by recurrence relation $F(k,n) = F(k,n-1) + F(k,n-k)$ for $n \geq k+1$ and with initial terms $F(k,n) = n+1$ for $0 \leq n \leq k$, the integer $k \geq 1$ by considering different parameters for both the initial terms and the distance between the added terms in the recurrence relation of the Fibonacci numbers [12]. In [13], distance Fibonacci numbers are introduced by recurrence relation $Fd(k,n) = Fd(k,n-k+1) + Fd(n-k)$ for $n \geq k$ and with initial terms $Fd(k,n) = 1$ for $0 \leq n \leq k-1$, integers $k \geq 2$ and $n \geq 0$. Then $(2,k)$-distance Fibonacci numbers are given by $F_2(k,n) = F_2(k,n-2) + F_2(k,n-k)$ for $n \geq k$ and with initial terms $F_2(k,i) = 1$ for $i = 0,1,\dots k-1$, integers $k \geq 1$ and $n \geq 0$ as a new kind of the distance Fibonacci numbers [14].

Combinatorial and tiling interpretations are extensively used in researching generalized Fibonacci numbers and their properties. These numbers are interpreted as the number of tiling of a board of length n, a $1 \times n$ grid with cells labeled $1, 2, \dots, n$, using squares and dominoes of various lengths. For instance, the nth Fibonacci number counts the number of distinct ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes. Then, many well-known relationships among Fibonacci numbers are provided via combinatorial and tiling proofs [15-17]. In [18], the tiling representations of Fibonacci p-numbers are introduced.

The aim of this study is to generalize the well-known Fibonacci, the Fibonacci type and the distance Fibonacci numbers and define new kinds of the Fibonacci, the Jacobsthal, the Narayana numbers and the Fibonacci p-numbers generalized in the distance sense using many generalization criterion used to generalize recurrence sequences presented by recurrence relations, as in the studies cited above. It is to derive the general formulas, generating functions and some identities for these generalized Fibonacci numbers. It is also to express these numbers with set decomposition, combinatorial, tiling interpretations, and give their special cases and generalize all the results.

2. FIBONACCI (k,p)-NUMBERS

In this section, new generalizations of the well-known Fibonacci, the Fibonacci type and the distance Fibonacci numbers, called Fibonacci (k,p)-numbers, are presented according to a new parameter k. The nth Fibonacci (k,p)-number is expressed by set decomposition, combinatorial and tiling interpretations that allow the derivation of its properties.

Definition 1. For integers $k,p \geq 1$ and $n \geq p$, the nth Fibonacci (k,p)-number is defined by recurrence relation

$$F_p^{(k)}(n) = F_p^{(k)}(n-1) + kF_p^{(k)}(n-p) \quad (1)$$

with initial terms $F_p^{(k)}(n) = 1$ for $n = 0,1,2,\dots,p-1$. The sequences of the Fibonacci (k,p)-numbers are denoted by $\left\{F_p^{(k)}(n)\right\}_{n\geq 0}$.

Sequences of the Fibonacci (k,p)-numbers for $p = 1,2,3,4,5,6$:

$$\left\{F_1^{(k)}(n)\right\}_{n\geq 0} = \{1, 1+k, 1+2k+k^2, 1+3k+3k^2+k^3, 1+4k+6k^2+4k^3+k^4, \dots\}$$

$$\left\{F_2^{(k)}(n)\right\}_{n\geq 0} = \{1,1,1+k,1+2k,1+3k+k^2, 1+4k+3k^2, 1+5k+6k^2+k^3, \dots\}$$

$$\left\{F_3^{(k)}(n)\right\}_{n\geq 0} = \{1, 1, 1, 1+k, 1+2k, 1+3k, 1+4k+k^2, 1+5k+3k^2, 1+6k+6k^2, \dots\}$$

$$\left\{F_4^{(k)}(n)\right\}_{n\geq 0} = \{1, 1, 1, 1, 1+k, 1+2k, 1+3k, 1+4k, 1+5k+k^2, 1+6k+3k^2, \dots\}$$

$$\left\{F_5^{(k)}(n)\right\}_{n\geq 0} = \{1, 1, 1, 1, 1, 1+k, 1+2k, 1+3k, 1+4k, 1+5k, 1+6k+k^2, 1+7k+3k^2, \dots\}$$

$$\left\{F_6^{(k)}(n)\right\}_{n\geq 0} = \{1, 1, 1, 1, 1,1, 1+k, 1+2k, 1+3k, 1+4k, 1+5k, 1+6k, 1+7k+k^2, \dots\}$$

Definition 1 is the general form of many generalized Fibonacci numbers defined by well-known recurrence relations and the distance sense. Special cases of the Fibonacci (k,p)-numbers obtained according to parameters k and p are given in the Table 1. Therefore, any result obtained throughout the study for the Fibonacci (k,p)-numbers is valid for all numbers mentioned in Table 1.

Table 1 Special cases of the Fibonacci (k,p)-numbers

$\boldsymbol{k}$	$\boldsymbol{p}$	***Symbol***	$\boldsymbol{nth}$ **Fibonacci** $(\boldsymbol{k},\boldsymbol{p})$**-number**
1	2	$F_2^{(1)}(n) = F_{n+1}$	F_n, nth Fibonacci number [1]
1	$p+1$	$F_{p+1}^{(1)}(n) = F_p(n+1)$	$F_p(n)$, nth Fibonacci p-number [8]
2	2	$F_2^{(2)}(n) = J_{n+1}$	J_n, nth Jacobsthal number [19]
1	3	$F_3^{(1)}(n) = N_{n+1}$	N_n, nth Narayana number [20]

For some values of k and p, the sequences of the Fibonacci (k,p)-numbers, $\left\{F_p^{(k)}(n)\right\}_{n\geq 0}$ indexed in The On-Line Encyclopedia of Integer Sequences [21], from now on OEIS, are:

- $\left\{F_1^{(1)}(n)\right\}_{n\geq 0}$ in A000079

$\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \dots\}$

- $\left\{F_2^{(1)}(n)\right\}_{n\geq 0}$ in A000045

$\{1,1,2,3,5,8,13,21,34,55,89,144,233,377, \dots\}$

- $\left\{F_6^{(1)}(n)\right\}_{n\geq 0}$ in A005708

$\{1,1,1,1,1,1,2,3,4,5,6,7,9,12,16,21, \dots\}$

- $\left\{F_2^{(2)}(n)\right\}_{n\geq 0}$ in A001045

$\{1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, \dots\}$

- $\left\{F_2^{(4)}(n)\right\}_{n\geq 0}$ in A006131

$\{1,1,5,9,29,65,181,441,1165, \dots\}$

- $\left\{F_2^{(8)}(n)\right\}$ in A015443

$\{1,1,9,17,89,225,937,2737,10233, \dots\}$

More generally, for $p = 1,2$ the Table 2 and Table 3 are obtained:

Table 2 Fibonacci $(k, 1)$-numbers

n / $F_p^{(k)}(n)$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$F_1^{(1)}(n)$	1	2	4	8	16	32	64	128	256
$F_1^{(2)}(n)$	1	3	9	27	81	243	729	2187	6561
$F_1^{(3)}(n)$	1	4	16	64	256	1024	4096	16384	65536
$F_1^{(4)}(n)$	1	5	25	125	625	3125	15625	78125	390625
$F_1^{(5)}(n)$	1	6	36	216	1296	7776	46656	279936	1679616
$F_1^{(6)}(n)$	1	7	49	343	2401	16807	117649	823543	5764801
$F_1^{(7)}(n)$	1	8	64	512	4096	32768	262144	2097152	16777216
$F_1^{(8)}(n)$	1	9	81	729	6561	59049	531441	4782969	43046721
$F_1^{(9)}(n)$	1	10	100	1000	10000	100000	1000000	10000000	100000000
$F_1^{(10)}(n)$	1	11	121	1331	14641	161051	1771561	19487171	214358881

Table 3 Fibonacci $(k, 2)$-numbers

n / $F_p^{(k)}(n)$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$F_2^{(1)}(n)$	1	1	2	3	5	8	13	21	34
$F_2^{(2)}(n)$	1	1	3	5	11	21	43	85	171
$F_2^{(3)}(n)$	1	1	4	7	19	40	97	217	508
$F_2^{(4)}(n)$	1	1	5	9	29	65	181	441	1165
$F_2^{(5)}(n)$	1	1	6	11	41	96	301	781	2286
$F_2^{(6)}(n)$	1	1	7	13	55	133	463	1261	4039
$F_2^{(7)}(n)$	1	1	8	15	71	176	673	1905	6616
$F_2^{(8)}(n)$	1	1	9	17	89	225	937	2737	10233
$F_2^{(9)}(n)$	1	1	10	19	109	280	1261	3781	15130
$F_2^{(10)}(n)$	1	1	11	21	131	341	1651	5061	21571

For $k = 1,2$ the Table 4 and Table 5 are obtained:

Table 4 Fibonacci $(1, p)$-numbers

n / $F_p^{(k)}(n)$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$F_1^{(1)}(n)$	1	2	4	8	16	32	64	128	256
$F_2^{(1)}(n)$	1	1	2	3	5	8	13	21	34
$F_3^{(1)}(n)$	1	1	1	2	3	4	6	9	13
$F_4^{(1)}(n)$	1	1	1	1	2	3	4	5	7
$F_5^{(1)}(n)$	1	1	1	1	1	2	3	4	5
$F_6^{(1)}(n)$	1	1	1	1	1	1	2	3	4
$F_7^{(1)}(n)$	1	1	1	1	1	1	1	2	3
$F_8^{(1)}(n)$	1	1	1	1	1	1	1	1	2
$F_9^{(1)}(n)$	1	1	1	1	1	1	1	1	1
$F_{10}^{(1)}(n)$	1	1	1	1	1	1	1	1	1

Table 5 Fibonacci $(2, p)$-numbers

$\boldsymbol{F_p^{(k)}(n)}$ \ $\boldsymbol{n}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\boldsymbol{F_1^{(2)}(n)}$	1	3	9	27	81	243	729	2187	6561
$\boldsymbol{F_2^{(2)}(n)}$	1	1	3	5	11	21	43	85	171
$\boldsymbol{F_3^{(2)}(n)}$	1	1	1	3	5	7	13	23	37
$\boldsymbol{F_4^{(2)}(n)}$	1	1	1	1	3	5	7	9	15
$\boldsymbol{F_5^{(2)}(n)}$	1	1	1	1	1	3	5	7	9
$\boldsymbol{F_6^{(2)}(n)}$	1	1	1	1	1	1	3	5	7
$\boldsymbol{F_7^{(2)}(n)}$	1	1	1	1	1	1	1	3	5
$\boldsymbol{F_8^{(2)}(n)}$	1	1	1	1	1	1	1	1	3
$\boldsymbol{F_9^{(2)}(n)}$	1	1	1	1	1	1	1	1	1
$\boldsymbol{F_{10}^{(2)}(n)}$	1	1	1	1	1	1	1	1	1

2.1. Generating Functions for $\boldsymbol{F_p^{(k)}(n)}$

In this section, generating functions are constructed for the sequences of the Fibonacci (k, p)-numbers, $\left\{F_p^{(k)}(n)\right\}$ that generalize the well-known Fibonacci, the Fibonacci-like and the distance Fibonacci sequences.

We define $G_p^{(k)}(x)$ the generating functions of the sequences $\left\{F_p^{(k)}(n)\right\}$ such that

$$G_p^{(k)}(x) = \sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^n \tag{2}$$

where integers $k, p \geq 1$. Then the generating functions of the sequences $\left\{F_p^{(k)}(n)\right\}$ are given in the following theorem.

Theorem 2. Let $k, p \geq 1$ be integers. Generating functions for the sequences of the Fibonacci (k, p)-numbers, $\left\{F_p^{(k)}(n)\right\}$ are

$$G_p^{(k)}(x) = \frac{1}{1 - x - kx^p}.$$

Proof. Using equations (1) and (2), we have

$$G_p^{(k)}(x) = \sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^n$$

$$= F_p^{(k)}(0) + F_p^{(k)}(1)x^1 + F_p^{(k)}(2)x^2$$

$$+ \cdots + F_p^{(k)}(n)x^n + \cdots$$

and

$$G_p^{(k)}(x) - xG_p^{(k)}(x) - kx^p G_p^{(k)}(x)$$

$$= \sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^n - \sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^{n+1}$$

$$-k \sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^{n+p}$$

$$= \sum_{n=0}^{p-1} F_p^{(k)}(n)\, x^n + \sum_{n=p}^{\infty} F_p^{(k)}(n)\, x^n$$

$$-\sum_{n=0}^{p-2} F_p^{(k)}(n)\, x^{n+1} - \sum_{n=p-1}^{\infty} F_p^{(k)}(n)\, x^{n+1}$$

$$-k \sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^{n+p}$$

$$= \sum_{n=0}^{p-1} F_p^{(k)}(n)\, x^n - \sum_{n=0}^{p-2} F_p^{(k)}(n)\, x^{n+1}$$

$$+\Bigg(\sum_{n=0}^{\infty} F_p^{(k)}(n+p)\, x^{n+p}$$

$$-\sum_{n=0}^{\infty} F_p^{(k)}(n+p-1)\, x^{n+p}$$

$$\left.-k\sum_{n=0}^{\infty} F_p^{(k)}(n)\, x^{n+p}\right)$$

$$=\sum_{n=0}^{p-1} x^n - \sum_{n=0}^{p-2} x^{n+1}$$

$$+\sum_{n=0}^{\infty}\Big(F_p^{(k)}(n+p) - F_p^{(k)}(n+p-1)$$

$$-kF_p^{(k)}(n)\Big)x^{n+p}$$

$$=1$$

with $F_p^{(k)}(n) = 1$ for $n = 0,1,2,\dots,p-1$ and the generating functions of the sequences $\left\{F_p^{(k)}(n)\right\}$ are

$$G_p^{(k)}(x) = \frac{1}{1-x-kx^p}.$$

Thus, the proof is completed.

Using parameters k and p given in Table 1 in Theorem 2, the generating functions for the special cases of the sequences of the Fibonacci (k,p)-numbers are given in the following corollary.

Corollary 3. The generating functions of the sequences given in Table 1 are:

- $G_2^{(1)}(x) = \frac{1}{1-x-x^2}$ for the Fibonacci sequences, $\{F_{n+1}\}$ with $k=1$, $p=2$, [22]
- $G_{p+1}^{(1)}(x) = \frac{1}{1-x-x^{p+1}}$ for sequences of Fibonacci p-numbers, $\left\{F_p(n+1)\right\}$ with $k=1, p=p+1$, [10]
- $G_2^{(2)}(x) = \frac{1}{1-x-2x^2}$ for the Jacobsthal sequences, $\{J_{n+1}\}$ with $k=2, p=2$, [22]
- $G_3^{(1)}(x) = \frac{1}{1-x-x^3}$ for Narayana sequences, $\{N_{n+1}\}$ with $k=1$, $p=3$.

3. INTERPRETATIONS OF THE FIBONACCI (k,p)-NUMBERS

In this section, the Fibonacci (k,p)-numbers are expressed with different interpretations such as set decomposition, combinatorial and tiling interpretations. Using these interpretations, general formulas and identities for the Fibonacci (k,p)-numbers are obtained.

3.1. Set Decomposition for $F_p^{(k)}(n)$

We now represent numbers $F_p^{(k)}(n)$ according to special decompositions of the set of n integers. Assume that $k \geq 1$ and $p > 1$ are integers and $S_n = \{1,2,\dots,n\}$ is the set of n integers. Let $\mathcal{A} = \{A_i : i \in I\}$ be the family of subsets of the set S_n such that each subset A_i contains consecutive integers and satisfies the following conditions

- **i.** $|A_i| = 1$ or $|A_i| = p$ for $i \in I$,
- **ii.** If $|A_i| = p$, it may be colored with one of k different colors,
- **iii.** $A_i \cap A_j = \emptyset$ for $i \neq j$, $i,j \in I$,
- **iv.** $|\bigcup_{i\in I} A_i| = n$

for $n \geq 1$.

Each the family $\mathcal{A}$ is a color decomposition of the set of n integers related to k different colors and is called as a (k,p)-decomposition of the set S_n.

Theorem 4. Let $k,n \geq 1$ and $p > 1$ be integers. Then the number of all (k,p)-decompositions of the set S_n is equal to $F_p^{(k)}(n)$.

Proof. By induction. Let $k, n \geq 1$ and $p > 1$ be integers and $S_n = \{1,2, \dots, n\}$. Indicate by $s(k,p)_n$ the number of all (k,p)-decompositions of the set S_n. To complete the proof, we show that $s(k,p)_n = F_p^{(k)}(n)$. For $n < p$, $\mathcal{A}$ the family can be obtained in exactly one way with $|A_i| = 1, i \in I$ for each subset, so S_n has 1 such family and $s(k,p)_n = 1 = F_p^{(k)}(n)$. For $n \geq p$, either $\{1\} \in \mathcal{A}$ or $\{1,2, \dots, p\} \in \mathcal{A}$, and suppose that $s(k,p)_n = F_p^{(k)}(n)$ holds for n. We show that it is true for $n+1$ which means $s(k,p)_{n+1} = F_p^{(k)}(n+1)$. Let $s_1(k,p)_{n+1}$ be the number of all (k,p)-decompositions of the set S_{n+1} such that $\{1\} \in \mathcal{A}$ and let $s_p(k,p)_{n+1}$ be the number of all (k,p)-decompositions of the set S_{n+1} such that $\{1,2, \dots, p\} \in \mathcal{A}$. Since these two cases are mutually exclusive, by the addition principle $s(k,p)_{n+1} = s_1(k,p)_{n+1} + s_p(k,p)_{n+1}$. If $\{1\} \in \mathcal{A}$, then the number of all (k,p)-decompositions of the set $S_{n+1-1} = \{2,3, \dots, n+1\}$ is $s(k,p)_{n+1-1}$. Add $\{1\}$ to each of the families of subsets of the set S_{n+1-1}. Then we have $s(k,p)_{n+1-1} = s_1(k,p)_{n+1}$ the families. If $\{1,2, \dots, p\} \in \mathcal{A}$, then the number of all (k,p)-decompositions of the set $S_{n+1-p} = \{p+1, p+2, \dots, n+1\}$ is $s(k,p)_{n+1-p}$. Add $\{1,2, \dots, p\}$ to each of the families of subsets of the set S_{n+1-p} with a choice of k different colors. Then we $ks(k,p)_{n+1-p} = s_p(k,p)_{n+1}$ the families. On the other hand, using the induction's hypothesis and the recurrence relation (1), we obtain

$$\begin{aligned} s(k,p)_{n+1} &= s_1(k,p)_{n+1} + s_p(k,p)_{n+1} \\ &= s(k,p)_n + ks(k,p)_{n+1-p} \\ &= F_p^{(k)}(n) + kF_p^{(k)}(n+1-p) \\ &= F_p^{(k)}(n+1) \end{aligned}$$

and the theorem is proved.

This interpretation of set decomposition allows a tiling approach for the Fibonacci (k,p)-numbers.

3.2. A Tiling Approach to $F_p^{(k)}(n)$

We now give the tiling interpretations for the sequences of the Fibonacci (k,p)-numbers and show that the nth Fibonacci (k,p)-number counts the number of distinct ways to tile a board of length n, called a $1 \times n$ board, using boards of lengths 1 and different colored p.

Assume that $k \geq 1$ and $p > 1$ are integers. Let us represent the numbers $F_p^{(k)}(n)$ as the number of distinct ways to tile a $1 \times n$ board using 1×1 boards (squares) and colored $1 \times p$ boards (p-ominoes), where there are k different colors for p-ominoes. Suppose we begin from the leftmost when placing cells and a $1 \times n$ board is splinted as follows:

$1 \times c_1$	$1 \times c_2$	$\cdots$	$1 \times c_i$

Where each $1 \times c_i$ board satisfies the following conditions

- **i.** $c_i \in \{1, p\}$ for $i \in I$,
- **ii.** If $c_i = p$, it may be colored with one of k different colors,
- **iii.** $\sum_{i \in I} c_i = n$.

Theorem 5. For integers $k, n \geq 1$ and $p > 1$, $F_p^{(k)}(n)$ counts the number of distinct ways to tile a $1 \times n$ board with 1×1 squares and colored $1 \times p$, p-ominoes, where there are k different colors for p-ominoes.

Proof. By induction. Let $k, n \geq 1, p > 1$ be integers. Indicate by $(k,p)_n$ the number of distinct ways to tile a $1 \times n$ board using 1×1 squares and colored $1 \times p$, p-ominoes, where there are k different colors for p-ominoes. To complete the proof, we show that $(k,p)_n = F_p^{(k)}(n)$. For $n < p$, all cells in tilings are 1×1 square and it can be obtained in exactly one way, so $(k,p)_n = 1 = F_p^{(k)}(n)$. For $n \geq p$, the first cell in the tilings is either a 1×1 square or a $1 \times p$, p-omino, and suppose that $(k,p)_n = F_p^{(k)}(n)$

holds for n. We show that it is true for $n+1$ which means $(k,p)_{n+1} = F_p^{(k)}(n+1)$. From now on, according to two options of the first cell, there are $(k,p)_{n+1-1}$ distinct ways to tile a $1 \times n$ board whose first cell is a 1×1 square, and there are $k(k,p)_{n+1-p}$ distinct ways to tile a $1 \times n$ board whose first cell is a $1 \times p$, p-omino with a choice of k different colors. Since these two cases are mutually exclusive, by the addition principle $(k,p)_{n+1} = (k,p)_n + k(k,p)_{n+1-p}$. On the other hand, using the induction's hypothesis we have $(k,p)_n = F_p^{(k)}(n)$ and $(k,p)_{n+1-p} = F_p^{(k)}(n+1-p)$. By the recurrence relation (1), we have

$$\begin{aligned}(k,p)_{n+1} &= (k,p)_n + k(k,p)_{n+1-p}\\ &= F_p^{(k)}(n) + kF_p^{(k)}(n+1-p)\\ &= F_p^{(k)}(n+1)\end{aligned}$$

and the theorem is proved.

We now give another general formula that directly presents the nth Fibonacci (k,p)-number.

3.3. Combinatorial Representation of $F_p^{(k)}(n)$

We now present combinatorial formulas for the Fibonacci (k,p)-numbers. An explicit formula that allows us to calculate the nth Fibonacci (k,p)-number, $F_p^{(k)}(n)$ are given in the following theorem.

Theorem 6. Let $k,p \geq 1$ and $n \geq 0$ be integers. Then

$$F_p^{(k)}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \binom{n-i(p-1)}{i} k^i$$

Proof. By induction. Let $k,p \geq 1$ and $n \geq 0$ be integers. If $n < p$, then $\left\lfloor \frac{n}{p} \right\rfloor = 0$. Using Definition 1, we have

$$F_p^{(k)}(n) = \sum_{i=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \binom{n-i(p-1)}{i} k^i = 1$$

for $n < p$. Assume that $n \geq p$. Let us suppose this formula is true until n. Then

$$\begin{aligned}F_p^{(k)}(n+1) &= F_p^{(k)}(n) + kF_p^{(k)}(n+1-p)\\
&= \sum_{i=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \binom{n-i(p-1)}{i} k^i\\
&\quad + \sum_{i=0}^{\left\lfloor \frac{n+1-p}{p} \right\rfloor} \binom{n+1-p-i(p-1)}{i} k^{i+1}\\
&= \binom{n}{0} + \sum_{i=1}^{\left\lfloor \frac{n}{p} \right\rfloor} \binom{n-i(p-1)}{i} k^i\\
&\quad + \sum_{i=0}^{\left\lfloor \frac{n+1}{p} \right\rfloor - 1} \binom{n+1-p-i(p-1)}{i} k^{i+1}\\
&= 1 + \sum_{i=0}^{\left\lfloor \frac{n}{p} \right\rfloor - 1} \binom{n-(i+1)(p-1)}{i+1} k^{i+1}\\
&\quad + \sum_{i=0}^{\left\lfloor \frac{n+1}{p} \right\rfloor - 1} \binom{n-(i+1)(p-1)}{i} k^{i+1}\\
&= 1\\
&\quad + \sum_{i=0}^{\left\lfloor \frac{n+1}{p} \right\rfloor - 1} \binom{n-(i+1)(p-1)+1}{i+1} k^{i+1}\\
&= 1 + \sum_{i=1}^{\left\lfloor \frac{n+1}{p} \right\rfloor} \binom{n+1-i(p-1)}{i} k^i\\
&= \sum_{i=0}^{\left\lfloor \frac{n+1}{p} \right\rfloor} \binom{n+1-i(p-1)}{i} k^i\end{aligned}$$

which completes the proof.

Note that in the proof of Theorem 6, $a < b \Rightarrow \binom{a}{b} = 0$, $a < 0 \Rightarrow \binom{a}{b} = 0$ and $\binom{a}{0} = 1$ for $\forall\ a$ are considered.

Using parameters k and p given in Table 1 in Theorem 6, well-known combinatorial formulas that allow us to calculate the nth terms of the special cases of the sequences of the Fibonacci (k,p)-numbers are given in the following corollary.

Corollary 7. The combinatorial formulas of the sequences given in Table 1 are:

- $F_{n+1} = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i}$ for the Fibonacci numbers, $F_2^{(1)}(n) = F_{n+1}$ with $k = 1, p = 2$, [22]
- $F_p(n+1) = \sum_{i=0}^{\left\lfloor \frac{n}{p+1} \right\rfloor} \binom{n-ip}{i}$ for the Fibonacci p-numbers, $F_{p+1}^{(1)}(n) = F_p(n+1)$ with $k = 1$, $p = p+1$, [11]
- $J_{n+1} = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} 2^i$ for the Jacobsthal numbers, $F_2^{(2)}(n) = J_{n+1}$ with $k = 2, p = 2$,
- $N_{n+1} = \sum_{i=0}^{\left\lfloor \frac{n}{3} \right\rfloor} \binom{n-2i}{i}$ for the Narayana numbers, $F_3^{(1)}(n) = N_{n+1}$ with $k = 1, p = 3$.

3.4. Identities for $F_p^{(k)}(n)$

We now give the identities and the sums formulas of the Fibonacci (k,p)-numbers with special subscripts via tiling proof in the following theorems.

Theorem 8. Let $k, p \geq 1$ and $n \geq 2p-1$ be integers. Then

$$F_p^{(k)}(n) = (k-1)F_p^{(k)}(n-p) + F_p^{(k)}(n-1) + F_p^{(k)}(n-p+1) - kF_p^{(k)}(n-2p+1).$$

Proof. This result is obtained directly from the equation (1).

Theorem 9. Let $k, p \geq 1$ and $n \geq 0$ be integers. Then

$$F_p^{(k)}(n+p) = \sum_{i=0}^{n} kF_p^{(k)}(i) + 1.$$

Proof. From Theorem 5, the number of distinct ways to tile a $1 \times (n+p)$ board using 1×1 squares and k different colored $1 \times p$, p-ominoes is equal to $F_p^{(k)}(n+p)$. Thus, the left-hand side of this identity is counted. If we show that the right-hand side of the identity gives the same count, the proof is completed. Suppose that we have a $1 \times (n+p)$ board and place the cells beginning from the leftmost cell. There is exactly a tiling where all the cells are 1×1 square. In the other cases there is at least a $1 \times p$, p-omino. Tilings containing only a $1 \times p$, p-omino can be partitioned according to the location of the first $1 \times p$, p-omino, counting from the left. There are $kF_p^{(k)}(n)$ distinct ways to tile $1 \times (n+p-p)$ board if $1 \times p$, p-omino appears for the first time in the first cell, and $kF_p^{(k)}(n-1)$ distinct ways to tile $1 \times (n+p-p-1)$ board if $1 \times p$, p-omino appears for the first time in the second cell where there are k different colors for p-ominoes. Similarly, if $1 \times p$, p-omino appears for the first time in the last cell, there are n times 1×1 square before the location of $1 \times p$, p-omino and we get $kF_p^{(k)}(0)$ distinct ways to tile $1 \times (n+p-p-n)$ board where there are k different colors for p-ominoes. The desired result is the sum of all the tilings and $\sum_{i=0}^{n} kF_p^{(k)}(i) + 1 = F_p^{(k)}(n+p)$ is obtained. So, the proof is completed.

Theorem 10. Let $k, p \geq 1$ and $n \geq 0$ be integers. For $0 \leq r \leq p-1$, we have

$$F_p^{(k)}(np+r+1)$$

$$=\begin{cases} k^{n+1}+\sum_{i=0}^{n} k^{n-i}F_p^{(k)}(ip+r), & r=p-1 \\ \sum_{i=0}^{n} k^{n-i}F_p^{(k)}(ip+r), & r<p-1 \end{cases}$$

Proof. From Theorem 5, the number of distinct ways to tile a $1\times(np+r+1)$ board using 1×1 squares and k different colored $1\times p$, p-ominoes is equal to $F_p^{(k)}(np+r+1)$. Thus, the left-hand side of this identity is counted. If we show that the right-hand side of the identity gives the same count, the proof is completed. Suppose that we have a $1\times(np+r+1)$ board and place the cells beginning from the leftmost cell. If $r<p-1$, then all tilings have at least a 1×1 square. Tilings containing only a 1×1 square can be partitioned according to the location of the first 1×1 square, counting from the left. There are $F_p^{(k)}(np+r)$ distinct ways to tile $1\times(np+r+1-1)$ board if 1×1 square appears for the first time in the first cell, and $kF_p^{(k)}((n-1)p+r)$ distinct ways to tile $1\times(np+r+1-1-p)$ board if 1×1 square appears for the first time in the second cell where there are k different colors for p-ominoes. Similarly, if 1×1 square appears for the first time in the last cell, there are i times $1\times p$, p-omino before the location of 1×1 square and we get $k^iF_p^{(k)}(p(n-i)+r)$ distinct ways to tile $1\times(np+r+1-1-ip)$ board where there are k different colors for p-ominoes. The desired result is the sum of all the tilings and $\sum_{i=0}^{n} k^{n-i}F_p^{(k)}(ip+r)=F_p^{(k)}(np+r+1)$ is obtained. Otherwise, if $r=p-1$, there is exactly one more tiling where all tilings are $1\times p$, p-omino and the number of such tilings is k^{n+1} with a choice of k different colors. So $\sum_{i=0}^{n} k^{n-i}F_p^{(k)}(ip+r)+k^{n+1}=F_p^{(k)}(np+r+1)$ is obtained. So, the proof is completed.

4. CONCLUSION AND SUGGESTION

The generalizations and applications of the Fibonacci and the Fibonacci-type numbers have been presented in many ways. Most of generalizations are related to arbitrary parameters for initial terms, the coefficient and distance between the added terms in the recurrence relation of the Fibonacci numbers. In this paper, we define new generalizations of the well-known Fibonacci, the Fibonacci-type and the distance Fibonacci numbers, called Fibonacci (k,p)-numbers, according to a new parameter k. Thus, we have generalized the well-known Fibonacci, the Jacobsthal, the Narayana numbers and the Fibonacci p-numbers in the distance sense given in [1, 8, 19, 20]. Then the Fibonacci (k,p)-numbers are expressed by set decomposition, combinatorial and tiling interpretations which allow to derive properties of them via set decomposition, combinatorial and tiling proofs. Also, explicit formulas that allow us to calculate the nth terms of the sequences of the Fibonacci (k,p)-numbers are given. Finally, generating functions, sums formulas and identities for these numbers are obtained.

It would be interesting to study these numbers in matrix theory. More general formulas that allow us to calculate the nth Fibonacci (k,p)-number and relations like the known relations between the well-known Fibonacci and Fibonacci type numbers can be explored.

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The Declaration of Research and Publication Ethics

The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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