Journal of New Results in Science 11(3) (2022) 190-198

Journal of New Results in Science

https://dergipark.org.tr/en/pub/jnrs

Research Article

Open Access

E-ISSN: 1304-7981 https://doi.org/10.54187/jnrs.1177925

Different computational approach for Fourier transforms by using variational iteration method

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Keywords

VIM,

Fourier transform,

Dirac delta function

Abstract — In this paper, we present another method for computing Fourier transforms of functions considering the Variational Iteration Method (VIM). Through our procedure, the Fourier transforms of functions can be calculated precisely and without reference to complex integration.

Subject Classification (2020): 42A38, 35A15.

1. Introduction

The variational iteration technique is one of the evaluation techniques that can be employed to many linear and nonlinear problems and reduces the computational effort. Numerous authors have worked for years to develop the method of variation iteration [1–3]. Starting from the Inokuti-Sekine-Mura technique as a starting point [4], it has developed into a full-fledged theory thanks to the work of several academics, including Wazwaz [5] and Moghimi [6]. The variational iteration technique [1–3, 5] is suitable for the treatment of a large class of linear or nonlinear differential problems. It is a suitable computational method for applications in the sciences [7–10]. Gubes [11] performed the variational iteration method (VIM) to obtain Laplace and Sumudu transforms. Xu and Lee [12] implemented the (VIM) for solving the boundary layer equations of magnetohydrodynamic flow over a nonlinear stretched sheet. Wazwaz [13] also applied this method to solve the linear and nonlinear ODE with variable coefficients. This method does not require much time when applied on the computer. In recent years, Düz et al. [14] have implemented the differential transformation technique to obtain Fourier transform of functions.

This paper is about a new calculation of Fourier transforms of functions using the variational iteration method (VIM) with first order linear IVP.

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Article History: Received: 20 Sep 2022 - Accepted: 04 Dec 2022 - Published: 31 Dec 2022

2. The Basic Definitions and Theorems

2.1. Variational Iteration Method

To provide the core concept of the technique, consider the following general nonlinear differential equation:

$$L[\beta(u)] + N[\beta(u)] = \psi(u) \tag{2.1}$$

where N, L are the non-linear and linear operator respectively, and $\psi(u)$ is a given continuous function.

The main advantage of this technique is that it creates a correction function for Equation (2.1), which can be represented as follows.

$$\beta_{J+1}(u) = \beta_J(u) + \int_0^u \lambda(u,s)\left[L\beta_J(s) + N\hat{\beta}_J(s) - \psi(s)\right]ds \tag{2.2}$$

Here, β_J is the J^{th} approximation solution, λ is a General lagrange multiplier that may be ideally found using variational theory and $\hat{\beta}_J$ is restricted variation.

The choice of the first approximation function $\beta_0(u)$ can affect the approximation positively or negatively.

As a result, the exact solution to Equation (2.1) can be obtained by:

$$\beta = \lim_{J\to\infty} \beta_J$$

2.2. Fourier Transform and Dirac Delta Distribution

To begin our discussion of the new computation of Fourier transforms of functions, we represent the Fourier transform of $h(u)$ as [14, 15]

$$\begin{aligned} \mathscr{F}[h(u)] &= \hat{h}(w) \\ &= \int_{-\infty}^{\infty} h(u)e^{-iwu}du \end{aligned} \tag{2.3}$$

and inverse Fourier transform is:

$$\begin{aligned} h(u) &= \mathscr{F}^{-1}[\hat{h}(w)] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(w)e^{iwu}dw \end{aligned}$$

The integral in Equation (2.3) is convergent to the function of the w if the following condition is satisfied

$$\int_{-\infty}^{\infty} |h(u)|du < \infty$$

The Fourier transform of exponential, polynomial and trigonometric functions are defined with the Dirac delta function, we will now introduce you to some properties of the Dirac delta function, which play a big role in Fourier transform.

Some properties of the dirac delta functions are as following: [14, 16, 17]

i. $\delta(w) = \begin{cases} 0, & w \neq 0 \\ \infty, & w = 0 \end{cases}$

ii. $\int_{-\infty}^{\infty} \delta(w)dw = 1$

iii. $\delta(bw) = \frac{1}{|b|}\delta(w), b \neq 0$

iv. $\delta(w^2 - b^2) = \frac{1}{2|b|}\left(\delta(w-b) + \delta(w+b)\right), b \neq 0$

v. $h(0) = \int_{-\infty}^{\infty} h(u)\delta(w)du$

There are a number of suitable functions that can estimate the delta function in the limiting process. One of them is the family of Gaussian curves, which are

$$\phi(w,b) = \frac{1}{\sqrt{2\pi b}} e^{\frac{-w^2}{2b}}, b > 0 \tag{2.4}$$

As $b \to 0^+$ family of $\phi(w,b)$ functions satisfies exact the same properties of delta function. Thus,

$$\delta(w) = \lim_{b \to 0^+} \phi(w,b) \tag{2.5}$$

The Fourier Transform of delta function [1] is

$$\mathscr{F}[\delta(w)] = \int_{-\infty}^{\infty} \delta(w)e^{-iwu}dw = 1$$

and it's inverse is

$$\mathscr{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1.e^{iwu}du = \delta(w)$$

The delta function is an even function: $\delta(-w) = \delta(w)$ and instead of the above integral expression of the delta function it is more usual to express :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwu}du = \delta(w) \tag{2.6}$$

From the Fourier Integral Theorem :

$$\begin{aligned} h(u) &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty} \hat{h}(w)e^{iwu}dw \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} h(v)e^{-iwv}dv)e^{iwu}dw \\ &= \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{\infty} h(v)(\int_{-\infty}^{\infty} e^{-i(v-u)w}dw)dv \end{aligned}$$

From the previous equation and Equation (2.6), $h(u)$ can be written as follows

$$h(u) = \int_{-\infty}^{\infty} h(v)\delta(v-u)dv$$

3. Results Using VIM

In this study, we tried to find some elementary functions Fourier transform using VIM, this kind of work was done using VIM and DTM for Laplace, Sumudu transforms and we were inspired by this work.

Theorem 3.1. Let $w \in \mathbb{C}$, $h(u)$ is an analytic function, and consider the linear IVP given as:

$$\beta' - iw\beta = ih(u), \beta(0) = 0 \tag{3.1}$$

then the Fourier Transforms of $h(u)$ is

$$\mathscr{F}[h(u)] = \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\frac{e^{-iwu}}{i}\lim_{J\to\infty}\beta_J\right]_{u=B}^{u=A}$$

Proof.

$\beta' - iw\beta = ih(u)$ this ODE can be represented as follows

$$(\beta e^{-iwu})' = ih(u)e^{-iwu}$$

by integrate both sides concerning u from $-\infty$ to ∞, we get the relationship between the Fourier Transform and previous equation as follows

$$\lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\beta e^{-iwu}\right]_{u=B}^{u=A} = i\int_{-\infty}^{\infty} h(u)e^{-iwu}du$$

that means :

$$\mathscr{F}[h(u)] = \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\frac{e^{-iwu}}{i}\beta\right]_{u=B}^{u=A} \tag{3.2}$$

Besides, by creating the VIM formula of Equation (3.1) at $\lambda = -1$, we will find the Fourier transform of $h(u)$ as:

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta_J'(s) - iw\beta_J(s) - ih(s)\right] ds \tag{3.3}$$

$$\beta = \lim_{J\to\infty}\beta_J \tag{3.4}$$

We substitute β into Equation (3.2) to get

$$\mathscr{F}[h(u)] = \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\frac{e^{-iwu}}{i}\lim_{J\to\infty}\beta_J\right]_{u=B}^{u=A} \tag{3.5}$$

Now, we provide some examples to find Fourier Transforms of some functions using the VIM

Example 3.2. Suppose that $h(u) = 1$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta_J'(s) - iw\beta_J(s) - i\right] ds$$

$\beta_0(u) = 0$.

We will now find some of $\beta_J(u)$

$$\begin{aligned} \beta_1(u) &= iu \\ \beta_2(u) &= iu + \tfrac{i^2wu^2}{2!}, \\ \beta_3(u) &= iu + \tfrac{i^2wu^2}{2!} + \tfrac{i^3w^2u^3}{3!}, \cdots \end{aligned} \tag{3.6}$$

from Equations (3.4) and (3.6), we get

$$\beta = \lim_{J\to\infty}\beta_J = \frac{e^{iwu}-1}{w} \tag{3.7}$$

We substitute β into Equation (3.2) to get the Fourier Transform of 1

$$\begin{aligned}\mathscr{F}[1] &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{e^{-iwu}}{i}\left(\tfrac{e^{iwu}-1}{w}\right)\right]_{u=B}^{u=A}\\ &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{1-e^{-iwu}}{iw}\right]_{u=B}^{u=A}\\ &= \textstyle\int_{-\infty}^{\infty}e^{-iwu}du\end{aligned}$$

by Equation (2.6), $\mathscr{F}[1]=2\pi\delta(w)$.

Example 3.3. Suppose that $h(u)=e^{bu}$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u)=\beta_J(u)-\int_0^u\left[\beta_J'(s)-iw\beta_J(s)-ie^{bs}\right]ds$$

$\beta_0(u)=0.$

We will now find some of $\beta_J(u)$

$$\begin{aligned}\beta_1(u) &= \tfrac{i}{b}\left(e^{bu}-1\right)\\ \beta_2(u) &= \tfrac{i}{b}\left(e^{bu}-1\right)+\tfrac{i^2w}{b}\left(\tfrac{e^{bu}-1}{b}-u\right)\\ \beta_3(u) &= \tfrac{i}{b}\left(e^{bu}-1\right)+\tfrac{i^2w}{b}\left(\tfrac{e^{bu}-1}{b}-u\right)+\tfrac{i^3w^2}{b}\left(\tfrac{e^{bu}-1}{b^2}-\tfrac{u}{b}-\tfrac{u^2}{2!}\right),\cdots\end{aligned}\tag{3.8}$$

from Equations (3.4) and (3.8), we get

$$\beta=\lim_{J\to\infty}\beta_J=\frac{i\left(e^{bu}-e^{iwu}\right)}{b-iw}\tag{3.9}$$

We substitute β into Equation (3.2) to get the Fourier Transform of e^{bu}

$$\begin{aligned}\mathscr{F}[e^{bu}] &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{e^{-iwu}}{i}\left(\tfrac{i\left(e^{bu}-e^{iwu}\right)}{b-iw}\right)\right]_{u=B}^{u=A}\\ &= \lim_{A\longrightarrow\infty}\lim_{B\longrightarrow-\infty}\left[\tfrac{-\left(e^{-i(ib+w)u}-1\right)}{i(ib+w)}\right]_{u=B}^{u=A}\\ &= \textstyle\int_{-\infty}^{\infty}e^{-i(ib+w)u}du\end{aligned}\tag{3.10}$$

by Equation (2.6) $\mathscr{F}[e^{bu}]=2\pi\delta(ib+w)$.

Example 3.4. Suppose that $h(u)=u^J$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u)=\beta_J(u)-\int_0^u\left[\beta_J'(s)-iw\beta_J(s)-is^J\right]ds$$

$\beta_0(u)=0.$

We will now find some of $\beta_J(u)$

$$\begin{aligned}
\beta_1(u) &= \tfrac{i}{(J+1)} u^{J+1} \\
\beta_2(u) &= \tfrac{i}{(J+1)} u^{J+1} + \tfrac{i^2 w}{(J+1)(J+2)} u^{J+2} \\
\beta_3(u) &= \tfrac{i}{(J+1)} u^{J+1} + \tfrac{i^2 w}{(J+1)(J+2)} u^{J+2} + \tfrac{i^3 w^2}{(J+1)(J+2)(J+3)} u^{J+3} \\
&= \tfrac{iJ!}{(J+1)!} u^{J+1} + \tfrac{i^2 wJ!}{(J+2)!} u^{J+2} + \tfrac{i^3 w^2 J!}{(J+3)!} u^{J+3}, \cdots
\end{aligned} \tag{3.11}$$

from Equations (3.4) and (3.11), we get

$$\begin{aligned}
\beta &= \lim_{J\to\infty} \beta_J \\
&= \tfrac{iJ!}{(iw)^{J+1}} \left(e^{iwx} - 1 - \tfrac{iwu}{1} - \tfrac{(iwu)^2}{2!} - \cdots - \tfrac{(iwu)^J}{J!} \right)
\end{aligned} \tag{3.12}$$

We substitute β into Equation (3.2) to get the Fourier Transform of u^J

$$\begin{aligned}
\mathscr{F}[u^J] &= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\tfrac{e^{-iwu}}{i} \left(\tfrac{iJ!}{(iw)^{J+1}} \left(e^{iwu} - 1 - \tfrac{iwu}{1} - \tfrac{(iwu)^2}{2!} - \cdots - \tfrac{(iwu)^J}{J!} \right) \right) \right]_{u=B}^{u=A} \\
&= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\tfrac{2\pi}{(-i)^J} \left(\tfrac{(-i)^J}{2\pi} (-1) e^{-iwu} \left(\tfrac{u^J}{iw} + \tfrac{Jx^{J-1}}{(iw)^2} + \cdots + \tfrac{J!u}{(iw)^J} + \tfrac{J!}{(iw)^{J+1}} \right) \right) \right]_{u=B}^{u=A} \\
&= 2\pi i^J \delta^{(J)}(w)
\end{aligned}$$

Example 3.5. Suppose that $h(u) = rect(u) = \begin{cases} \frac{1}{a}, & -\frac{a}{2} \le u \le \frac{a}{2} \\ 0, & otherwise \end{cases}$ and by Theorem 3.1, where $rect(u)$ is Rectangular Function, then we get

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta_J'(s) - iw\beta_J(s) - irect(s) \right] ds$$

$\beta_0(u) = 0$. We will now find some of $\beta_J(u)$

$$\begin{aligned}
\beta_1(u) &= \tfrac{iu}{a} \\
\beta_2(u) &= \tfrac{iu}{a} + \tfrac{i^2 wu^2}{a2!}, \beta_3(u) = \tfrac{iu}{a} + \tfrac{i^2 wu^2}{a2!} + \tfrac{i^3 w^2 u^3}{a3!} \cdots
\end{aligned} \tag{3.13}$$

from Equations (3.4) and (3.13), we get

$$\beta = \lim_{J\to\infty} \beta_J = \frac{e^{iwu} - 1}{aw} \tag{3.14}$$

We substitute β into Equation (3.2) to get the Fourier Transform of $rect(u)$

$$\begin{aligned}
\mathscr{F}[rect(u)] &= \left[\tfrac{e^{-iwu}}{i} \left(\tfrac{e^{iwu}-1}{aw} \right) \right]_{u=-\frac{a}{2}}^{u=\frac{a}{2}} \\
&= \left[\tfrac{1-e^{-iwu}}{iaw} \right]_{u=-\frac{a}{2}}^{u=\frac{a}{2}} = \tfrac{\sin(\frac{aw}{2})}{\frac{aw}{2}} \\
&= \mathrm{sinc}(\tfrac{aw}{2\pi})
\end{aligned}$$

The definition of sinc function in [18].

Example 3.6. Suppose that $h(u) = \cos(au)$ and by Theorem 3.1, then we get

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta'_m(s) - iw\beta_J(s) - i\cos(bs)\right] ds$$

The previous formula can be represented as:

$$\beta_{J+1}(u) = \beta_J(u) - \int_0^u \left[\beta'_J(s) - iw\beta_J(s) - \frac{i}{2}\left(e^{ibs} + e^{-ibs}\right)\right] ds$$

$\beta_0(u) = 0$. We will now find some of $\beta_J(u)$

$$\begin{aligned}
\beta_1(u) &= \frac{1}{2b}\left(e^{ibu} - e^{-ibu}\right) \\
\beta_2(u) &= \frac{1}{2b}\left(e^{ibu} - e^{-ibu}\right) + \frac{w}{2b^2}\left(e^{ibu} + e^{-ibu}\right) \\
\beta_3(u) &= \left(\frac{1}{2b} + \frac{w^2}{2b^3}\right)\left(e^{ibu} - e^{-ibu}\right) + \frac{w}{2b^2}\left(e^{ibu} + e^{-ibu}\right) \\
\beta_4(u) &= \left(\frac{1}{2b} + \frac{w^2}{2b^3}\right)\left(e^{ibu} - e^{-ibu}\right) + \left(\frac{w}{2b^2} + \frac{w^3}{2b^4}\right)\left(e^{ibu} + e^{-ibu}\right), \ldots
\end{aligned} \tag{3.15}$$

from Equations (3.4) and (3.15) we get

$$\begin{aligned}
\beta &= \lim_{J\to\infty} \beta_J \\
&= \frac{e^{ibu}}{2(b-w)} - \frac{e^{-ibu}}{2(b+w)}
\end{aligned} \tag{3.16}$$

We substitute β into Equation (3.2) to get the Fourier Transform of $cos(bu)$

$$\begin{aligned}
\mathscr{F}[\cos(bu)] &= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\frac{e^{-iwu}}{i}\left(\frac{e^{ibu}}{2(b-w)} - \frac{e^{-ibu}}{2(b+w)}\right)\right]_{u=B}^{u=A} \\
&= \lim_{A\longrightarrow\infty} \lim_{B\longrightarrow-\infty} \left[\frac{e^{iu(b-w)}}{2i(b-w)} - \frac{e^{-iu(b+w)}}{2i(b+w)}\right]_{u=B}^{u=A} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left[e^{-i(w-b)u} + e^{-i(w+b)u}\right] dx
\end{aligned} \tag{3.17}$$

by Equation (2.6), $\mathscr{F}[\cos(bu)] = \pi[\delta(w-b) + \delta(w+b)]$.

The main advantage of VIM when calculating Fourier transforms of functions is that it can expand the convergence region for iterative solutions and reduce the size of the calculations. Besides, the solutions obtained using VIM do not need much effort and time compared to DTM [14], as explained in the above examples .

4. Conclusion

We have implemented the variational technique (VIM) to compute the Fourier transforms of functions differently. Furthermore, the results have shown that the method we have presented is a comprehensive and accurate scientific method that does not require much effort in determining Fourier transforms.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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