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Spinor Representations of Positional Adapted Frame in the Euclidean 3-Space*

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The main goal of this paper is to study together the spinors, which have a major place in several disciplines from mathematics to physics, and Positional Adapted Frame (PAF) which is a new frame that attracts the attention of many researchers. In accordance with this purpose, we introduce the spinor representations for the trajectories endowed with PAF in the Euclidean 3-space $\mathbb{E}^3$, and construct the spinor equations of PAF vectors. Then, we find the relations between spinor representations of PAF and Serret-Frenet frame. Also we give some results and present some geometric interpretations with respect to this relationship. Moreover, we present an illustrative numerical example in order to support the given theorems and results.

Keywords: *Kinematics of a particle, positional adapted frame, spinors.*
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1. Introduction

Despite its long history, the curve theory is still an issue of interest in different dimensions and spaces. The concept of moving frames is an important part of the curve theory. The discovery of the moving frame called today as Serret-Frenet frame was a turning point for this theory. The reason why this frame is called the Serret-Frenet frame is that Serret and Frenet discussed this issue at different times unaware of each other in 19th century.

From the discovery of Serret-Frenet frame until today, by considering the curve theory, a lot of interesting studies have been performed with the aid of Serret-Frenet frame. The readers are referred to [4, 18, 19, 41] for some of these studies. Also, the researchers have introduced many other moving frames similar to this frame. The first ones come to mind are Darboux frame [6] and Bishop frame [2]. In addition to these, the readers, which are interested, can find some of the other frames in [7, 8, 22, 26, 30, 34, 43]. Recently, one of the newest of them is given in the study [30] by Özen and Tosun. For the trajectories having non-vanishing angular momentum in Euclidean 3-space, the authors introduced the Positional Adapted Frame (PAF) in the aforementioned study. This frame enables the researchers to handle the kinematics of the particle, the differential geometry of the surface, and the differential geometry of the trajectory. On the other hand, PAF is a useful tool to study the motion of an electron (with electrical charge $-e$, and mass m) under a constant magnetic field along an axis. Certainly, this is so special motion for electrons. Spin is a more general motion than the aforementioned motion for electrons. Even it can be said that it is the most important motion for electrons. The practical tools to study this spin motion are spinors.

Spinors are physical concepts which are used in various fields of theoretical and applied sciences. Quantum mechanics, theory of relativity and physics can be given as some examples to these fields. The word "spinor"

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is a term used for the first time by Paul Ehrenfest in quantum physics in 1920s [39]. Spinors are considered as multi linear transformations and by this property, spinors are mathematical structure. According to the mathematicians spinors are a vectorial structure and this multi-linear property does not matter [11, 13, 14]. The first mathematician who studied the spinors in geometrical sense is Élie Cartan [3]. When examining the representation of groups, Cartan found the mathematical forms of spinors in 1913. Cartan indicated that spinors satisfy a linear representation of the groups of rotations of a space of any dimension. Therefore, spinors are directly related to geometry in addition to their relationship with physics [17]. Cartan [3] obtained the geometric construction of spinor representations. This study is quite fundamental study because of the fact that it presents some fundamental definitions and concepts with respect to the geometry of spinor representations. In accordance with this basic study, in the vector space $\mathbb{C}^3$, the set of isotropic vectors constructs a two-dimensional surface in the space $\mathbb{C}^2$. Conversely, these vectors in $\mathbb{C}^2$ represent the same isotropic vectors. Cartan stated that these vectors are complex as two-dimensional in the space $\mathbb{C}^2$. Moreover, Cartan [3] expressed the spinors comprising of two complex components in terms of vectors in Euclidean 3-space and specified that spinors supply a linear representation of the groups of rotations of a space of any dimension [11, 13]. The triads of unit vectors which are orthogonal to each other were stated in terms of a single vector having two complex components, that is called a spinor [3, 9, 10]. On the other hand, spin matrices which express the spin in quantum theory were presented by W. Pauli in [32]. Today these matrices are known as Pauli matrices, as well. By utilizing the Pauli matrices, spin can be represented in the quantum theory [11, 39]. Pauli expressed that the wave function of an electron can be represented by a vector (with two complex components) which is called spinor in 1927 [17]. The other study with respect to the spinors in geometric meaning is performed by Vivarelli [40]. Afterwards, Vivarelli obtained some relations between the spinors and quaternions, and provided the spinor representation of rotations in E^3 by utilizing the relations between the rotations in E^3 and quaternions [13, 40]. In addition to these, hyperbolic spinors were studied [1, 12, 23, 24] in Minkowski 3-space. Also, spinors have an substantial and basic role in Clifford algebra, which is called geometric algebra by W. K. Clifford, as well [5,27,28,39]. Furthermore, the Fibonacci spinors were investigated in [13].

On the other hand, del Castillo and Barrales introduced the spinor equations of the Serret-Frenet frame in the study [10], and it became a turning point for several researchers. Studying together the spinors and frames attracted several researchers. In the existing literature, some studies have been done and ongoing with respect to this interesting topic. Researchers investigated several types of frames and some special curves in the Euclidean 3-space and Minkowski 3-space by using the term spinors. Ünal et al. [38] studied on the spinor representations of Bishop frame, and Kişi and Tosun examined the spinor formulas of Darboux frame [25]. Also, Şenyurt and Çalışkan investigated the spinor equations of Sabban frame [36]. Spinor representations of Bertrand [11], involute-evolute [14] and successor curves [15] were constructed. Also, spinor representation of framed Mannheim and framed Bertrand curves were investigated by Yazıcı et al. [42] and İşbilir et al. [21], respectively. In the Minkowski-3 space, by using the concept of the hyperbolic spinors of curves in Minkowski space [23], hyperbolic spinor representations of Serret-Frenet frame [24], Darboux frame [1], and Bishop frame [12] are constructed.

At the beginning of this study, we started with the question "What kind of results and interpretations do we get if we combine spinors and PAF?" and the results aroused our curiosity. To answer this question, we intend to examine this issue. The primary purpose of this paper is to study together the spinors, which have a great deal of interests by researchers and several disciplines from mathematics to physics, and Positional Adapted Frame, which is a new type special and interesting frame. We absolutely believe that this study contributes to the literature and leads the future studies.

This article is organized as follows. In Section 2, we give the necessary information with respect to the spinors, the fundamental concepts of differential geometry and Positional Adapted Frame to be understood the ensuing sections. In Section 3, the spinor representations are introduced for Positional Adapted Frame. Then, we indicate that PAF derivative formulas can be stated with a single equation for a vector having two complex components. Also, the relations between spinor representations of Positional Adapted Frame and Serret-Frenet frame are found. In Section 4, an example including an illustrative figure is presented. Then, the conclusions are given in Section 5.

2. Preliminaries

In this section, we remind some required notions and often used notations with respect to the spinors, PAF and Serret-Frenet frame.

2.1. Spinors

A spinor is represented as

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

where the vectors $u, v, w \in \mathbb{R}^3$ satisfy the following equations

$$\begin{aligned} u + iv &= \eta^t \sigma \eta, \\ w &= -\widehat{\eta}^t \sigma \eta. \end{aligned} \tag{2.1}$$

Here $u + iv$ is an isotropic vector and w is a real vector. Also, "t" denotes the transposition, $\widehat{\eta}$ denotes the conjugation [3] (or mate [9]) of η, and $\overline{\eta}$ denotes the complex conjugation of η. Additionally, the following equation with respect to the spinors can be written [10]:

$$\widehat{\eta} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\eta} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\eta}_1 \\ \overline{\eta}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\eta}_2 \\ \overline{\eta}_1 \end{pmatrix}.$$

Moreover, the following 2×2 matrices, which are Cartesian components of the vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, are derived from Pauli matrices and are given as [10]:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Let $\rho = (\rho_1, \rho_2, \rho_3) \in \mathbb{C}^3$ be an isotropic vector ($\langle \rho, \rho \rangle = 0$) where $\mathbb{C}^3$ represents the three-dimensional complex vector space. The isotropic vectors in $\mathbb{C}^3$ generate a two-dimensional surface in $\mathbb{C}^2$. If the surface is parametrized by using η_1 and η_2, then $\rho_1 = \eta_1^2 - \eta_2^2, \rho_2 = i(\eta_1^2 + \eta_2^2), \rho_3 = -2\eta_1\eta_2$ can be written. Also,

$$\eta_1 = \pm\sqrt{\frac{\rho_1 - i\rho_2}{2}} \quad \text{and} \quad \eta_2 = \pm\sqrt{\frac{-\rho_1 - i\rho_2}{2}} \tag{2.3}$$

can be given [3]. According to the equalities (2.1) and (2.2), we can write the followings:

$$\rho_1 = \eta^t \sigma_1 \eta = \eta_1^2 - \eta_2^2, \quad \rho_2 = \eta^t \sigma_2 \eta = i(\eta_1^2 + \eta_2^2), \quad \rho_3 = \eta^t \sigma_3 \eta = -2\eta_1\eta_2,$$

and

$$\begin{aligned} u + iv &= \left(\eta_1^2 - \eta_2^2, i(\eta_1^2 + \eta_2^2), -2\eta_1\eta_2\right), \\ w &= \left(\eta_1\overline{\eta}_2 + \overline{\eta}_1\eta_2, i(\eta_1\overline{\eta}_2 - \overline{\eta}_1\eta_2), |\eta_1|^2 - |\eta_2|^2\right). \end{aligned} \tag{2.4}$$

Since the vector $u + iv \in \mathbb{C}^3$ is an isotropic vector, u, v, w are mutually orthogonal, and then $|u| = |v| = |w| = \overline{\eta}^t \eta$ and $\langle u \wedge v, w \rangle = \det(u, v, w) > 0$. On the other hand, if the vectors $u, v, w \in \mathbb{R}^3$ are mutually orthogonal vectors of the same magnitude $(\det(u, v, w) > 0)$, then there exists a spinor which is defined as in the equation (2.1) [3, 9–11, 14, 25, 36, 38].

Proposition 2.1. *Let η and ψ be two spinors. Then the followings are satisfied:*

$$\eta^t \sigma \psi = \psi^t \sigma \eta, \tag{2.5}$$

$$\overline{\eta^t \sigma \psi} = -\widehat{\eta}^t \sigma \widehat{\psi}, \tag{2.6}$$

$$\widehat{(\varrho_1 \eta + \varrho_2 \psi)} = \overline{\varrho}_1 \widehat{\eta} + \overline{\varrho}_2 \widehat{\psi}, \tag{2.7}$$

$$\widehat{\widehat{\eta}} = -\eta, \tag{2.8}$$

where $\varrho_1, \varrho_2 \in \mathbb{C}$ [10].

Since the matrices $\sigma_1, \sigma_2, \sigma_3$ which are given in the equation (2.2) are symmetric, the equation (2.5) holds. Also, since the spinors η and $-\eta$ match up with the same ordered orthogonal basis $\{u, v, w\}$ with $|u| = |v| = |w|$ and $\det(u, v, w) > 0$, the correspondence between the spinors and orthogonal bases which are written in the equation (2.1) is two-to-one. Note that, the ordered triads $\{u, v, w\}, \{v, w, u\}, \{w, u, v\}$ correspond to different spinors. Furthermore, if $\eta \neq 0$, then the set $\{\eta, \overline{\eta}\}$ is linearly independent [3,9–11,14,25,36,38]. For more detailed information about spinors, we can also refer to the studies [3,9,10,27,28,37,40].

2.2. Positional Adapted Frame

In this subsection, at first, we remind Serret-Frenet frame briefly and basically since the PAF is related to Serret-Frenet frame.

Let P be a moving point particle of constant mass m in the Euclidean 3-space E^3 given with the standard inner product. Suppose that the unit speed parametrization of the trajectory of P is shown with $\alpha = \alpha(s)$. It means that s is the arc-length parameter. In that case, the vectors $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$ comprise the Serret-Frenet frame of $\alpha = \alpha(s)$. The base vectors $\mathbf{T}(s), \mathbf{N}(s)$ and $\mathbf{B}(s)$ of Serret-Frenet frame are said to be the unit tangent, unit principal normal and unit binormal vectors, respectively. Note that we will use a prime to denote the differentiation with respect to the arc-length parameter s in most cases, just like we did above. On the other hand, the derivative formulas for Serret-Frenet frame are given as:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{2.9}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ is the torsion function [33].

Now, let us discuss the Positional Adapted Frame (PAF). Assume that the angular momentum vector of P about the origin is always non-zero, then the Positional Adapted Frame is well defined during the motion of P (and so during the trajectory $\alpha = \alpha(s)$). The first base vector of PAF is $\mathbf{T}(s)$ as in the Serret-Frenet frame. The second and third base vectors are given by

$$\mathbf{M}(s) = \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{B}(s)$$

and

$$\mathbf{Y}(s) = \frac{\langle -\alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{B}(s).$$

The relation between the Serret-Frenet frame and Positional Adapted Frame is as follows:

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(s) & -\sin\phi(s) \\ 0 & \sin\phi(s) & \cos\phi(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} \tag{2.10}$$

where $\phi(s)$ is the angle between the vectors $\mathbf{N}(s)$ and $\mathbf{M}(s)$ which is positively oriented from $\mathbf{N}(s)$ to $\mathbf{M}(s)$. In addition to these, the derivative formulas of PAF are expressed as

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{M}'(s) \\ \mathbf{Y}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix} \tag{2.11}$$

where

$$\begin{cases} k_1(s) = \kappa(s)\cos\phi(s), \\ k_2(s) = \kappa(s)\sin\phi(s), \\ k_3(s) = \tau(s) - \phi'(s). \end{cases} \tag{2.12}$$

The rotation angle $\phi(s)$, mentioned above, is calculated as in the following:

$$\phi(s)=\begin{cases}\arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right) \; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle>0,\\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right)+\pi \; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle<0,\\ -\frac{\pi}{2} \; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle=0\;, \;\; \langle\alpha(s),\mathbf{N}(s)\rangle>0,\\ \frac{\pi}{2} \; if \;\; \langle\alpha(s),\mathbf{B}(s)\rangle=0\;, \;\; \langle\alpha(s),\mathbf{N}(s)\rangle<0.\end{cases}$$

The elements of the system $\{\mathbf{T}(s),\mathbf{M}(s),\mathbf{Y}(s),k_1(s),k_2(s),k_3(s)\}$ are called as PAF apparatuses of $\alpha=\alpha(s)$ [30]. Throughout this study, we consider the trajectories which are parametrized by arc-length parameter and whose PAFs are well-defined. One can find more details on PAF in the studies [16,20,29–31,35].

Another thing that can be of importance is the spinor representations of the Serret-Frenet equations given in (2.9). This topic is investigated in the quite fundamental study [10] by del Castillo and Barrales. They found spinor equivalent of the Serret-Frenet equations in [10]. This study has been a milestone for many researchers. The authors obtained a spinor ψ satisfying

$$\mathbf{N}+i\mathbf{B}=\psi^t\sigma\psi, \tag{2.13}$$

$$\mathbf{T}=-\widehat{\psi}^t\sigma\psi, \tag{2.14}$$

where $\overline{\psi}^t\psi=1$ satisfies and the spinor ψ symbolizes the triad $\{\mathbf{N},\mathbf{B},\mathbf{T}\}$. In this case, the Serret-Frenet equations are equivalent to the following single spinor equation [10]:

$$\frac{d\psi}{ds}=\frac{1}{2}\left(-i\tau\psi+\kappa\widehat{\psi}\right) \tag{2.15}$$

where κ and τ show the curvature and torsion functions of the curve.

3. The spinor representations of Positional Adapted Frame

The goal of this section is to determine the spinor representation of Positional Adapted Frame in $\mathbb{E}^3$. We find the spinor equations of PAF vectors and components of them. Also, we obtain the relations between the spinor representations of the Positional Adapted Frame and the Serret-Frenet frame. Then, we give some geometric properties and results with respect to them.

Definition 3.1. Suppose that P is the moving point particle in the Euclidean 3-space and $\alpha=\alpha(s)$ endowed with PAF is the unit speed parametrization of the trajectory of P. Let η match up with the triad $\{\mathbf{M},\mathbf{Y},\mathbf{T}\}$. Then, the spinor representations of PAF of the trajectory α are obtained as follows:

$$\mathbf{M}+i\mathbf{Y}=\eta^t\sigma\eta, \tag{3.1}$$

$$\mathbf{T}=-\widehat{\eta}^t\sigma\eta, \tag{3.2}$$

where $\overline{\eta}^t\eta=1$.

Theorem 3.1. *Let the unit speed trajectory $\alpha=\alpha(s)$ be endowed with PAF and the spinor η match up with the triad $\{\mathbf{M},\mathbf{Y},\mathbf{T}\}$. Then, the PAF equations are equivalent to the following single spinor equation:*

$$\frac{d\eta}{ds}=\frac{1}{2}\left[-ik_3\eta+(k_1+ik_2)\,\widehat{\eta}\right]. \tag{3.3}$$

Proof. Differentiating the equation (3.1) according to the parameter s, we get:

$$\frac{d\mathbf{M}}{ds}+i\frac{d\mathbf{Y}}{ds}=\left(\frac{d\eta}{ds}\right)^t\sigma\eta+\eta^t\sigma\frac{d\eta}{ds}. \tag{3.4}$$

Since $\{\eta, \widehat{\eta}\}$ is a basis for spinors, we can write:

$$\frac{d\eta}{ds} = \chi\eta + \mu\widehat{\eta} \tag{3.5}$$

where χ and μ are complex valued functions. By means of the equations (2.11), (3.4), (3.5) and (2.5), we have:

$$\begin{aligned} -k_1\mathbf{T} + k_3\mathbf{Y} + i\left(-k_2\mathbf{T} - k_3\mathbf{M}\right) &= \left(\chi\eta + \mu\widehat{\eta}\right)^t \sigma\eta + \eta^t\sigma\left(\chi\eta + \mu\widehat{\eta}\right) \\ &= 2\chi\eta^t\sigma\eta + 2\mu\widehat{\eta}^t\sigma\eta. \end{aligned}$$

With the help of the equations (3.1) and (3.2), we can write:

$$-ik_3\left(\mathbf{M} + i\mathbf{Y}\right) - \left(k_1 + ik_2\right)\mathbf{T} = 2\chi\left(\mathbf{M} + i\mathbf{Y}\right) - 2\mu\mathbf{T}.$$

Therefore, we get:

$$\chi = -\frac{ik_3}{2} \quad \text{and} \quad \mu = \frac{k_1 + ik_2}{2}. \tag{3.6}$$

Substituting the equation (3.6) in the equation (3.5) gives us the equation (3.3). □

Theorem 3.2. *Let the unit speed trajectory $\alpha = \alpha(s)$ be endowed with PAF and the spinor η match up with the triad $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$. Then, the spinor equations for the PAF vectors are written as follows:*

$$\mathbf{T} = -\widehat{\eta}^t\sigma\eta, \tag{3.7}$$

$$\mathbf{M} = \frac{1}{2}\left(\eta^t\sigma\eta - \widehat{\eta}^t\sigma\widehat{\eta}\right), \tag{3.8}$$

$$\mathbf{Y} = -\frac{i}{2}\left(\eta^t\sigma\eta + \widehat{\eta}^t\sigma\widehat{\eta}\right). \tag{3.9}$$

Proof. Let the spinor η match up with the triad $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ of the unit speed trajectory α. According to the equation (3.1), we can write $\mathbf{M} = Re(\eta^t\sigma\eta)$ and $\mathbf{Y} = Im(\eta^t\sigma\eta)$. We know already that $\mathbf{T} = -\widehat{\eta}^t\sigma\eta$ from the equation (3.2). By using the well-known properties of complex numbers such as $Re(\varrho) = \frac{1}{2}\left(\varrho + \overline{\varrho}\right)$ and $iIm(\varrho) = \frac{1}{2}\left(\varrho - \overline{\varrho}\right)$ for every $\varrho \in \mathbb{C}$, we have:

$$\mathbf{M} = \frac{1}{2}\left(\eta^t\sigma\eta + \overline{\eta^t\sigma\eta}\right),$$

$$\mathbf{Y} = -\frac{i}{2}\left(\eta^t\sigma\eta - \overline{\eta^t\sigma\eta}\right).$$

Based on the last two equations, we get the equations (3.8) and (3.9) with the help of the equation (2.6). □

Considering the equations (3.7)-(3.9) which are given in the Theorem 3.2, we can say the followings:
For spinor equation $\eta^t\sigma\eta$, we obtain:

$$\begin{cases} \eta^t\sigma_1\eta = \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = \eta_1^2 - \eta_2^2, \\ \eta^t\sigma_2\eta = \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = i(\eta_1^2 + \eta_2^2), \\ \eta^t\sigma_3\eta = \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = -2\eta_1\eta_2. \end{cases} \tag{3.10}$$

Also, for the spinor equation $\widehat{\eta}^t\sigma\widehat{\eta}$, we have:

$$\begin{cases} \widehat{\eta}^t\sigma_1\widehat{\eta} = \begin{pmatrix}-\overline{\eta}_2 & \overline{\eta}_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\overline{\eta}_2\\ \overline{\eta}_1\end{pmatrix} = \overline{\eta}_2^2 - \overline{\eta}_1^2, \\ \widehat{\eta}^t\sigma_2\widehat{\eta} = \begin{pmatrix}-\overline{\eta}_2 & \overline{\eta}_1\end{pmatrix}\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}-\overline{\eta}_2\\ \overline{\eta}_1\end{pmatrix} = i\left(\overline{\eta}_2^2 + \overline{\eta}_1^2\right), \\ \widehat{\eta}^t\sigma_3\widehat{\eta} = \begin{pmatrix}-\overline{\eta}_2 & \overline{\eta}_1\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-\overline{\eta}_2\\ \overline{\eta}_1\end{pmatrix} = 2\overline{\eta}_1\,\overline{\eta}_2. \end{cases} \tag{3.11}$$

Finally, for the spinor equation $\widehat{\eta}^t\sigma\eta$, we get:

$$\begin{cases} \widehat{\eta}^t\sigma_1\eta = \begin{pmatrix} -\overline{\eta}_2 & \overline{\eta}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = -\overline{\eta}_2\eta_1 - \overline{\eta}_1\eta_2, \\ \widehat{\eta}^t\sigma_2\eta = \begin{pmatrix} -\overline{\eta}_2 & \overline{\eta}_1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = i\left(-\overline{\eta}_2\eta_1 + \overline{\eta}_1\eta_2\right), \\ \widehat{\eta}^t\sigma_3\eta = \begin{pmatrix} -\overline{\eta}_2 & \overline{\eta}_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = -|\overline{\eta}_1|^2 + |\overline{\eta}_2|^2. \end{cases} \tag{3.12}$$

Corollary 3.1. *Let the unit speed trajectory $\alpha = \alpha(s)$ be endowed with PAF and the spinor η match up with the triad $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$. In that case, the spinor components of PAF vectors are given as follows:*

$$\begin{aligned} \mathbf{T} &= \left(\eta_1\overline{\eta}_2 + \overline{\eta}_1\eta_2, i\left(\eta_1\overline{\eta}_2 - \overline{\eta}_1\eta_2\right), |\eta_1|^2 - |\eta_2|^2\right), \\ \mathbf{M} &= \frac{1}{2}\left(\eta_1^2 - \eta_2^2 + \overline{\eta}_1^2 - \overline{\eta}_2^2, i\left(\eta_1^2 + \eta_2^2 - \overline{\eta}_1^2 - \overline{\eta}_2^2\right), -2\left(\eta_1\eta_2 + \overline{\eta}_1\overline{\eta}_2\right)\right), \\ \mathbf{Y} &= -\frac{i}{2}\left(\eta_1^2 - \eta_2^2 - \overline{\eta}_1^2 + \overline{\eta}_2^2, i\left(\eta_1^2 + \eta_2^2 + \overline{\eta}_1^2 + \overline{\eta}_2^2\right), 2\left(\overline{\eta}_1\overline{\eta}_2 - \eta_1\eta_2\right)\right). \end{aligned}$$

Proof. Let the unit speed trajectory $\alpha = \alpha(s)$ be endowed with PAF and let the spinor η match up with the triad $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$. If we substitute the equations (3.10), (3.11) and (3.12) in the equations (3.7), (3.8) and (3.9), we complete the proof easily. □

In addition to these, let us give the spinor relations between the Positional Adapted Frame and Serret-Frenet frame.

Theorem 3.3. *Suppose that the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. Then, the relations between η and ψ can be given as:*

$$\eta^t\sigma\eta = e^{i\Omega}(\psi^t\sigma\psi), \tag{3.13}$$
$$\mathbf{T} = \mathbf{T},$$

where the angle Ω is the Euclidean angle between the vectors $\mathbf{B}$ and $\mathbf{Y}$.

Proof. According to the equation (2.10), we can write:

$$\begin{aligned} \mathbf{M} + i\mathbf{Y} &= \cos\Omega\mathbf{N} - \sin\Omega\mathbf{B} + i\left(\sin\Omega\mathbf{N} + \cos\Omega\mathbf{B}\right) \\ &= (\mathbf{N} + i\mathbf{B})(\cos\Omega + i\sin\Omega) \\ &= (\mathbf{N} + i\mathbf{B})e^{i\Omega}. \end{aligned} \tag{3.14}$$

From the equations (2.10), (2.13) and (2.14), we have $\eta^t\sigma\eta = e^{i\Omega}\left(\psi^t\sigma\psi\right)$ and $\mathbf{T} = \mathbf{T}$. □

Theorem 3.4. *Assume that the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. Then the following equation holds:*

$$\eta = \pm e^{\frac{i\Omega}{2}}\psi. \tag{3.15}$$

Proof. From the equation (3.13) and (3.14), we can say that the angle between the vectors $\mathbf{N} + i\mathbf{B}$ and $\mathbf{M} + i\mathbf{Y}$ is Ω. According to the equation (3.13) and (2.4), we can write $\eta^t\sigma\eta = \left(\eta_1^2 - \eta_2^2, i\left(\eta_1^2 + \eta_2^2\right), -2\eta_1\eta_2\right)$, and $\psi^t\sigma\psi = \left(\psi_1^2 - \psi_2^2, i\left(\psi_1^2 + \psi_2^2\right), -2\psi_1\psi_2\right)$. Then we have $\eta_1^2 = e^{i\Omega}\psi_1^2$ and $\eta_2^2 = e^{i\Omega}\psi_2^2$. Thus, $\eta_1 = \mp e^{i\frac{\Omega}{2}}\psi_1$ and $\eta_2 = \mp e^{i\frac{\Omega}{2}}\psi_2$ can be written. The spinor η and $-\eta$ correspond to the vector $\mathbf{M} + i\mathbf{Y}$, and the spinor ψ and $-\psi$ correspond to the vector $\mathbf{N} + i\mathbf{B}$, since the spinors η and $-\eta$ symbolize the same ordered orthonormal basis. Hence, we can write $\eta = \pm e^{\frac{i\Omega}{2}}\psi$. □

Corollary 3.2. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. In that case, the angle between the spinors η and ψ is $\Omega/2$.*

Theorem 3.5. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. Then the following relation between the spinors η and ψ is satisfied:*

$$\widehat{\eta} = \pm e^{-i\frac{\Omega}{2}}\widehat{\psi}. \tag{3.16}$$

Proof. Taking the conjugate of both sides of the equation (3.15), we get the equation $\widehat{\eta} = \pm\widehat{e^{i\frac{\Omega}{2}}\psi}$. By means of (2.7), we have the equation (3.16). □

Corollary 3.3. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. While the spinor η makes a rotation angle as $\Omega/2$ with the spinor ψ, the spinor $\widehat{\eta}$ makes the same rotation angle in the opposite direction with the spinor $\widehat{\psi}$.*

Theorem 3.6. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. The derivative of the spinor η can be expressed by using the Serret-Frenet curvatures κ and τ as follows:*

$$\frac{d\eta}{ds} = \frac{1}{2}\left[-i\left(\tau - \Omega'\right)\eta + \kappa e^{i\Omega}\widehat{\eta}\right]. \tag{3.17}$$

Proof. Using the equations (3.3) and (2.12) gives us the following:

$$\begin{aligned}\frac{d\eta}{ds} &= \frac{1}{2}[-ik_3\eta + (k_1 + ik_2)\widehat{\eta}] \\ &= \frac{1}{2}[-i(\tau - \Omega')\eta + \kappa(\cos\Omega + i\sin\Omega)\widehat{\eta}] \\ &= \frac{1}{2}\left[-i(\tau - \Omega')\eta + \kappa e^{i\Omega}\widehat{\eta}\right].\end{aligned}$$

□

Corollary 3.4. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. The angle between $\dfrac{d\eta}{ds}$ and $\widehat{\eta}$ is Ω on condition that $\Omega' = \tau$.*

Theorem 3.7. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. In that case, the following equation is satisfied:*

$$\frac{d\eta}{ds} = \pm\frac{1}{2}e^{i\frac{\Omega}{2}}\left[-i(\tau - \Omega')\psi + \kappa\widehat{\psi}\right].$$

Proof. Using the equations (3.15), (3.16) and (3.17) yields the following:

$$\begin{aligned}\frac{d\eta}{ds} &= \frac{1}{2}\left[-i(\tau - \Omega')\eta + \kappa e^{i\Omega}\widehat{\eta}\right] \\ &= \frac{1}{2}\left[-i(\tau - \Omega')\left(\pm e^{i\frac{\Omega}{2}}\psi\right) + \kappa e^{i\Omega}\left(\pm e^{-i\frac{\Omega}{2}}\widehat{\psi}\right)\right] \\ &= \pm\frac{1}{2}e^{i\frac{\Omega}{2}}\left[-i(\tau - \Omega')\psi + \kappa\widehat{\psi}\right].\end{aligned}$$

□

Corollary 3.5. *Let the spinors η and ψ match up with the triads $\{\mathbf{M}, \mathbf{Y}, \mathbf{T}\}$ and $\{\mathbf{N}, \mathbf{B}, \mathbf{T}\}$ (related to α), respectively. Then we can say that the angle between $\dfrac{d\eta}{ds}$ and $\widehat{\psi}$ is $\Omega/2$ if $\Omega' = \tau$.*

4. Application

In this section, we present an example in order to understand the notions of the spinor representations of PAF in the Euclidean 3-space.

Example 4.1. In the Euclidean 3-space, suppose that a point particle P of constant mass m moves on the trajectory

$$\begin{aligned}\alpha &: (0, \pi) \mapsto E^3 \\ s &\mapsto \alpha(s) = \left(\frac{8}{17}\cos s, \frac{9}{17} - \sin s, -\frac{15}{17}\cos s\right)\end{aligned} \tag{4.1}$$

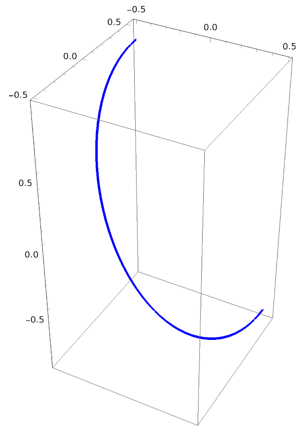

Figure 1. The trajectory of the moving point particle P given in (4.1)

which is a unit speed curve. In the Figure 1, the trajectory $\alpha = \alpha(s)$ can be seen.

It should be noted that the Figure 1 is drawn by utilizing the website Wolfram Mathematica (Wolfram Cloud). By straightforward calculations, we get the following Serret-Frenet apparatus:

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{8}{17}\sin s, -\cos s, \frac{15}{17}\sin s\right), \\ \mathbf{N}(s) = \left(-\frac{8}{17}\cos s, \sin s, \frac{15}{17}\cos s\right), \\ \mathbf{B}(s) = \left(-\frac{15}{17}, 0, -\frac{8}{17}\right), \end{cases} \quad \text{and} \quad \begin{cases} \kappa(s) = 1, \\ \tau(s) = 0. \end{cases}$$

By means of the equation (2.3), we have:

$$\psi_1 = \pm\sqrt{-\frac{4}{17}\cos s - i\left(\frac{15}{34} + \frac{\sin s}{2}\right)},$$
$$\psi_2 = \pm\sqrt{\frac{4}{17}\cos s + i\left(\frac{15}{34} - \frac{\sin s}{2}\right)}.$$

Via the equation (2.15), we obtain:

$$\frac{d\psi}{ds} = \frac{i}{2}\widehat{\psi}.$$

Since $\langle \alpha(s), \mathbf{B}(s)\rangle = 0$ and $\langle \alpha(s), \mathbf{N}(s)\rangle < 0$, we find $\Omega = \frac{\pi}{2}$. Then, the PAF apparatus can be given as:

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{8}{17}\sin s, -\cos s, \frac{15}{17}\sin s\right), \\ \mathbf{M}(s) = \left(\frac{15}{17}, 0, \frac{8}{17}\right), \\ \mathbf{Y}(s) = \left(-\frac{8}{17}\cos s, \sin s, \frac{15}{17}\cos s\right), \end{cases} \quad \text{and} \quad \begin{cases} k_1(s) = 0, \\ k_2(s) = 1, \\ k_3(s) = 0. \end{cases}$$

From the equation (2.3), we can write:

$$\eta_1 = \pm\sqrt{\frac{15}{34} + \frac{1}{2}\sin s - i\frac{4}{17}\cos s},$$
$$\eta_2 = \pm\sqrt{-\frac{15}{34} + \frac{1}{2}\sin s + i\frac{4}{17}\cos s}.$$

By using the equation (3.3), we get

$$\frac{d\eta}{ds} = \frac{i}{2}\widehat{\eta}.$$

One can easily check this result from the equation (3.17).

5. Conclusions

In this study, for the trajectories endowed with PAF, we investigated the spinor representations of PAF. Also, we found the spinor equations and spinor components of PAF vectors. Moreover, we examined this representation with respect to the geometric interpretations. In addition to these, we gave the relations between the spinor representations of PAF and Serret-Frenet frame. Afterwards, we provided an example in order to support given materials.

There is no doubt that this paper will lead to a new perspective and novelty for future studies regarding the spinors, which have a major place in mathematics and physics, and PAF, which is a new type interesting moving frame. Also some physical results and applications can be found by using the physical construction of spinors in the future studies.

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