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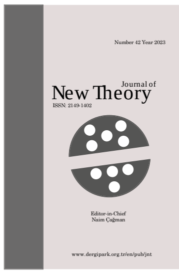

A Characterization of Semiprime Rings with Homoderivations

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Research Article

Abstract — This paper is focused on the commutativity of the laws of semiprime rings, which satisfy some algebraic identities involving homoderivations on ideals. It provides new and notable results that will interest researchers in this field, such as "$\mathfrak{R}$ contains a nonzero central ideal if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that $\delta(\mathfrak{I}) \subseteq Z$ where $\mathfrak{R}$ is a semiprime ring with center Z and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$". Moreover, the research also generalizes some results previously published in the literature, including derivation on prime rings using homoderivation on semiprime rings. It also demonstrates the necessity of hypotheses operationalized in theorems by an example. Finally, the paper discusses how the results herein can be further developed in future research.

Keywords *Semiprime rings, ideals, derivations, homoderivations*

Mathematics Subject Classification (2020) 11T99, 16W25

1. Introduction (Compulsory)

There is a growing literature on strong commutativity preserving (SCP) maps and derivations. Bell and Daif [1] first investigated the derivation of SCP maps on the ideal of a semiprime ring. Bresar [2] generalized this work to the Lie ideal of the ring. In [3], Ma and Xu handled this study for generalized derivations. Moreover, Koç and Gölbaşı [4] have been studied for the multiplicative generalized derivations by generalizing these conditions on the semiprime ring. In [5], Ali et al. showed that if $\mathfrak{R}$ is a semiprime ring and f is an endomorphism which is an SCP map on a nonzero ideal U of $\mathfrak{R}$, then f is commuting on U. Samman [6] proved that an epimorphism of a semiprime ring is strong commutativity preserving if and only if it is centralizing. Researchers have extensively studied derivations and SCP mappings in the context of operator algebras, prime rings, and semiprime rings. In [7], Melaibari et al. examined this condition for homoderivation. This paper investigated SCP maps for homoderivation in the ideal of a semiprime ring. In [8], Herstein showed that if $\mathfrak{R}$ is a prime ring of characteristics different from two and d is a nonzero derivation such that $d(\mathfrak{R}) \subseteq Z$, then $\mathfrak{R}$ must be commutative. This condition on the Lie ideal of the prime ring was discussed by Bergen et al. [9]. Gölbaşı and Koç [10] examined this condition for the (σ,τ)-Lie ideal of the prime ring. This condition is then examined for different subsets of the ring and different derivations. Ashraf et al. [11] proved that a prime ring $\mathfrak{R}$ must be commutative if $\mathfrak{R}$ satisfies the following condition: $f(x)f(y) = xy$ or $f(x)f(y) = yx$ where f is a generalized derivation of $\mathfrak{R}$, and $\mathfrak{I}$ is a nonzero two-sided ideal of $\mathfrak{R}$. In [12], the following conditions are examined by Alharfie and Muthana for homoderivation in the prime ring:

$$i.\, x\delta(y) \pm xy \in Z,\; ii.\, x\delta(y) \pm yx \in Z,\; iii.\, [\delta(x), y] \pm xy \in Z, \text{ and } iv.\, [\delta(x), y] \pm yx \in Z$$

This article aims to generalize the above conditions for homoderivation on an ideal semiprime ring.

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2. Preliminaries

Let $\mathfrak{R}$ be an associative ring with center Z. For any $x, y \in \mathfrak{R}$, the symbol $[x, y]$ stands for the commutator $xy - yx$, and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. Recall that a ring $\mathfrak{R}$ is a semiprime if $x\mathfrak{R}x = 0$ implies $x = 0$. An additive mapping $d: \mathfrak{R} \to \mathfrak{R}$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in \mathfrak{R}$. An additive mapping $\delta: \mathfrak{R} \to \mathfrak{R}$ is called a homoderivation if $\delta(xy) = \delta(x)\delta(y) + \delta(x)y + x\delta(y)$, for all $x, y \in \mathfrak{R}$ in [13]. For example, $\delta(x) = f(x) - x$, for all $x \in \mathfrak{R}$, where f is an endomorphism on $\mathfrak{R}$ if $\delta(x)\delta(y) = 0$, for all $x, y \in \mathfrak{R}$, then the homoderivation δ is a derivation. If $S \subseteq \mathfrak{R}$, then a mapping $f: \mathfrak{R} \to \mathfrak{R}$ preserves S if $f(S) \subseteq S$. A mapping $f: \mathfrak{R} \to \mathfrak{R}$ is zero-power valued on S if f preserves S and if, for each $x \in S$, there exists a positive integer $n(x) > 1$ such that $f^{n(x)} = 0$. Let S be a nonempty subset of $\mathfrak{R}$. A mapping F from $\mathfrak{R}$ to $\mathfrak{R}$ is called commutativity preserving on a subset S of $\mathfrak{R}$ if $[x, y] = 0$ implies $[F(x), F(y)] = 0$, for all $x, y \in S$. The mapping F is called an SCP on S if $[x, y] = [F(x), F(y)]$, for all $x, y \in S$.

Proposition 2.1. Let $\mathfrak{R}$ be a semiprime ring. Then,

i. The center of $\mathfrak{R}$ contains no nonzero nilpotent elements.

ii. If P is a nonzero prime ideal of $\mathfrak{R}$ and $a, b \in \mathfrak{R}$ such that $a\,\mathfrak{R}\,b \subseteq P$, then either $a \in P$ or $b \in P$.

iii. The center of a nonzero one-sided ideal is in the center of $\mathfrak{R}$. In particular, any one-sided commutative ideal is included in the center of $\mathfrak{R}$.

Lemma 2.2. [14] If $\mathfrak{R}$ is a semiprime ring, then the center of a nonzero ideal of $\mathfrak{R}$ is contained in the center of $\mathfrak{R}$.

3. Main Results

This section investigates the aforesaid commutativity conditions for homoderivations in the semiprime ring.

Theorem 3.1. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that $\delta(\mathfrak{I}) \subseteq Z$.

PROOF.

By the hypothesis, we have

$$\delta(v_1) \in Z, \text{ for all } v_1 \in \mathfrak{I}$$

Commuting this term with $r \in \mathfrak{R}$, we obtain that

$$[\delta(v_1), r] = 0, \text{ for all } v_1 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing v_1 by $v_1 v_2$, $v_2 \in \mathfrak{I}$, in this equation and using the hypothesis, we get

$$\delta(v_1)[v_2, r] + [v_1, r]\delta(v_2) = 0$$

Taking r by v_1, we obtain that

$$\delta(v_1)[v_2, v_1] = 0 \tag{1}$$

Replacing v_2 by $v_2 r$, $r \in \mathfrak{R}$, in the last equation and using Equation 1, we have

$$\delta(v_1)v_2[r, v_1] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}, r \in \mathfrak{R} \tag{2}$$

Taking v_2 by $[r, v_1]t\delta(v_1)$, $t \in \mathfrak{R}$, we observe that

$$\delta(v_1)[r, v_1]t\delta(v_1)[r, v_1] = 0$$

and

$$\delta(v_1)[r, v_1]\mathfrak{R}\,\delta(v_1)[r, v_1] = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get

$$\delta(v_1)[r, v_1] = 0$$

That is,

$$\delta(v_1)\mathfrak{R}[\mathfrak{R}, v_1] = (0), \text{ for all } v_1 \in \mathfrak{I}$$

Since $\mathfrak{R}$ is a semiprime ring, it must contain a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$. If P is a typical member of $\wp$ and $v_1 \in \mathfrak{I}$, we have $[\mathfrak{R}, v_1] \subseteq P$ or $\delta(v_1) \subseteq P$ by Proposition 2.1 (*ii*). Define two additive subgroups $\mathcal{L} = \{v_1 \in \mathfrak{I} \mid [\mathfrak{R}, v_1] \subseteq P\}$ and $\mathcal{F} = \{v_1 \in \mathfrak{I} \mid \delta(v_1) \subseteq P\}$.

It is clear that $\mathfrak{I} = \mathcal{L} \cup \mathcal{F}$. Since a group cannot be a union of two of its subgroups, either $\mathcal{L} = \mathfrak{I}$ or $\mathcal{F} = \mathfrak{I}$. Therefore, we have

$$[\mathfrak{R}, \mathfrak{I}] \subseteq P \text{ or } \delta(\mathfrak{I}) \subseteq P$$

Thus, both cases together yield

$$[\mathfrak{R}, \mathfrak{I}]\delta(\mathfrak{I}) \subseteq P, \text{ for any } P \in \wp$$

Therefore,

$$[\mathfrak{R}, \mathfrak{I}]\delta(\mathfrak{I}) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$$

and

$$[\mathfrak{R}, \mathfrak{I}]\delta(\mathfrak{I}) = (0)$$

Hence,

$$[\mathfrak{R}, \mathfrak{R}\mathfrak{I}\delta(\mathfrak{I})\mathfrak{R}]\mathfrak{R}\mathfrak{R}\mathfrak{I}\delta(\mathfrak{I})\mathfrak{R} = (0)$$

This implies that $[\mathfrak{R}, \Pi]\mathfrak{R}\Pi = (0)$ where $\Pi = \mathfrak{R}\mathfrak{I}\delta(\mathfrak{I})\mathfrak{R}$ is a nonzero ideal of $\mathfrak{R}$ since $\delta(\mathfrak{I}) \neq (0)$. Then,

$$[\mathfrak{R}, \Pi]\mathfrak{R}[\mathfrak{R}, \Pi] = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get $[\mathfrak{R}, \Pi] = (0)$. Hence, $\Pi \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

The theorem below is proved for prime rings in Theorem 5 [15]. Here, it is generalized using semiprime rings.

Theorem 3.2. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that

i. $\delta([\mathfrak{I}, \mathfrak{I}]) = (0)$ or

ii. $[\delta(\mathfrak{I}), \mathfrak{I}] \subseteq Z$ or

iii. $[\delta(\mathfrak{I}), \delta(\mathfrak{I})] = (0)$, $\mathfrak{I}\delta^2(\mathfrak{I}) \neq (0)$, and $\delta(\mathfrak{I}) \subseteq \mathfrak{I}$

Proof.

i. By the hypothesis, we have

$$\delta([v_1, v_2]) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_1$ in this equation, we get

$$\delta([v_1, v_2]v_1) = 0$$

Since δ is a homoderivation, we have

$$\delta([v_1, v_2])\delta(v_1) + \delta([v_1, v_2])v_1 + [v_1, v_2]\delta(v_1) = 0$$

By the hypothesis, we get

$$[v_1, v_2]\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Using the same arguments in the proof of Theorem 3.1, we find that $\mathfrak{R}$ contains a nonzero central ideal.

ii. By the hypothesis, we get

$$[\delta(v_1), v_2] \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is,

$$\big[[\delta(v_1), v_2], r\big] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing v_2 by $\delta(v_1)v_2$ in the last equation and using this equation, we have

$$0 = [\delta(v_1)[\delta(v_1), v_2], r] = [\delta(v_1), r][\delta(v_1), v_2]$$

Taking r by $v_2 r$ in this equation, we see that

$$[\delta(v_1), v_2]r[\delta(v_1), v_2] = 0$$

By the semiprimeness of $\mathfrak{R}$, we get

$$[\delta(v_1), v_2] = 0$$

That is, $\delta(v_1) \in Z(\mathfrak{I})$, for all $v_1 \in \mathfrak{I}$. By Lemma 2.2, we get $\delta(v_1) \in Z$, for all $v_1 \in \mathfrak{I}$. By Theorem 3.1, we conclude that $\mathfrak{R}$ contains a nonzero central ideal.

iii. By the hypothesis, we get

$$[\delta(v_1), \delta(v_2)] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $v_2\delta(v_1)$ in this equation and using this equation, we have

$$[\delta(v_1), v_2]\delta^2(v_1) = 0$$

Replacing v_2 by $rv_2, r \in \mathfrak{R}$ in the last equation and using this equation, we have

$$[\delta(v_1), r]v_2\delta^2(v_1) = 0$$

That is,

$$[\delta(v_1), \mathfrak{R}]\mathfrak{R}\mathfrak{I}\delta^2(v_1) = (0), \text{ for all } v_1 \in \mathfrak{I}$$

Since $\mathfrak{R}$ is a semiprime ring, we must contain a family $\wp = \{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$. If P is a typical member of $\wp$ and $v_1 \in \mathfrak{I}$, by Proposition 2.1 (*ii*), we have

$$[\delta(v_1), \mathfrak{R}] \subseteq P \text{ or } \mathfrak{I}\delta^2(v_1) \subseteq P$$

Define two additive subgroups

$$\mathcal{L} = \{v_1 \in \mathfrak{I} \mid [\delta(v_1), \mathfrak{R}] \subseteq P\} \text{ and } \mathcal{F} = \{v_1 \in \mathfrak{I} \mid \mathfrak{I}\delta^2(v_1) \subseteq P\}$$

It is clear that $\mathfrak{I} = \mathcal{L} \cup \mathcal{F}$. Since a group cannot be a union of its two subgroups, either $\mathcal{L} = \mathfrak{I}$ or $\mathcal{F} = \mathfrak{I}$. Then,

$$[\delta(\mathfrak{I}), \mathfrak{R}] \subseteq P \text{ or } \mathfrak{I}\delta^2(\mathfrak{I}) \subseteq P$$

Thus, both cases together yield

$$[\delta(\mathfrak{I}), \mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) \subseteq P, \text{ for any } P \in \wp$$

Therefore,

$$[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha = \{0\}$$

and

$$[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

That is,

$$\begin{aligned}(0) &= [\delta(\mathfrak{I}\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &= [\delta(\mathfrak{I})\delta(\mathfrak{I})+\delta(\mathfrak{I})\mathfrak{I}+\mathfrak{I}\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &= [\delta(\mathfrak{I})\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+[\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+[\mathfrak{I}\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &= [\delta(\mathfrak{I}),\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I})+\delta(\mathfrak{I})[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+[\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+\mathfrak{I}[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})\\ &\quad +[\mathfrak{I},\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I})\end{aligned}$$

Using $\delta(\mathfrak{I}) \subseteq \mathfrak{I}$, we have

$$[\delta(\mathfrak{I}),\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I})+\delta(\mathfrak{I})[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+[\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+\mathfrak{I}[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I})+[\delta(\mathfrak{I}),\mathfrak{R}]\delta(\mathfrak{I})\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

Since $[\delta(\mathfrak{I}),\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$, we get

$$[\delta(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

This implies that

$$\left[\delta\big(\mathfrak{I}\delta(\mathfrak{I})\big)\mathfrak{I},\mathfrak{R}\right]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

and

$$[\mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{I}\delta^2(\mathfrak{I}) = (0)$$

Hence,

$$[\mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I},\mathfrak{R}]\mathfrak{R}\mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I} = (0)$$

This implies that $[\Pi,\mathfrak{R}]\mathfrak{R}\Pi = (0)$ where $\Pi = \mathfrak{I}\delta^2(\mathfrak{I})\mathfrak{I}$ is a nonzero ideal of $\mathfrak{R}$ since $\mathfrak{I}\delta^2(\mathfrak{I}) \neq (0)$. Then,

$$[\Pi,\mathfrak{R}]\mathfrak{R}[\Pi,\mathfrak{R}] = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get $[\Pi,\mathfrak{R}] = (0)$. Hence, $\Pi \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

The proof of the following result differs from that of Theorem 2.2 in [7].

Corollary 3.3. Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$, and δ a nonzero and zero-power valued homoderivation on $\mathfrak{I}$. If δ is an SCP on $\mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal.

Proof.

By the hypothesis, we get

$$[\delta(v_1),\delta(v_2)] = [v_1,v_2], \text{ for all } v_1,v_2 \in \mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathfrak{I}$, in the last equation, we have

$$[\delta(v_1),\delta(v_2)]\delta(v_3)+\delta(v_2)[\delta(v_1),\delta(v_3)]+[\delta(v_1),\delta(v_2)]v_3+\delta(v_2)[\delta(v_1),v_3]+[\delta(v_1),v_2]\delta(v_3)+v_2[\delta(v_1),\delta(v_3)] = [v_1,v_2]v_3+v_2[v_1,v_3]$$

Using the hypothesis, we obtain that

$$[v_1,v_2]\delta(v_3)+\delta(v_2)[v_1,v_3]+\delta(v_2)[\delta(v_1),v_3]+[\delta(v_1),v_2]\delta(v_3) = 0$$

That is,

$$[v_1+\delta(v_1),v_2]\delta(v_3)+\delta(v_2)[\delta(v_1)+v_1,v_3]=0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n>1$ such that $\delta^n(v_1)=0$, for all $v_1\in\mathfrak{I}$. Replacing v_1 by $v_1-\delta(v_1)+\delta^2(v_1)+\cdots+(-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we get

$$[v_1,v_2]\delta(v_3)+\delta(v_2)[v_1,v_3]=0$$

Replacing v_3 by v_1 in the last equation, we get

$$[v_1,v_2]\delta(v_1)=0, \text{ for all } v_1,v_2\in\mathfrak{I}$$

The rest of the proof is the same as Equation 1. This completes the proof.

Theorem 3.4. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero homoderivation δ on $\mathfrak{I}$ such that

i. $\delta(\mathfrak{I}\circ\mathfrak{I})=(0)$ or

ii. $\delta(\mathfrak{I})\circ\mathfrak{I}\subseteq Z$

PROOF.

i. We have

$$\delta(v_1\circ v_2)=0, \text{ for all } v_1,v_2\in\mathfrak{I}$$

Relacing v_2 by v_2v_1 in the above equation, we get

$$\delta(v_1\circ v_2)\delta(v_1)+\delta(v_1\circ v_2)v_1+(v_1\circ v_2)\delta(v_1)=0$$

Using the hypothesis, we get

$$(v_1\circ v_2)\delta(v_1)=0, \text{ for all } v_1,v_2\in\mathfrak{I}$$

Taking v_2 by rv_2, $r\in\mathfrak{R}$, in the last equation, we get

$$[v_1,r]v_2\delta(v_1)=0, \text{ for all } v_1,v_2\in\mathfrak{I}, r\in\mathfrak{R}$$

Using the same arguments in the proof of Theorem 3.1, we find that $\mathfrak{R}$ contains a nonzero central ideal.

ii. We get

$$\delta(v_1)\circ v_2\in Z, \text{ for all } v_1,v_2\in\mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3\in\mathfrak{I}$, in this equation, we have

$$(\delta(v_1)\circ v_2)v_3+v_2[v_3,\delta(v_1)]\in Z$$

That is,

$$[(\delta(v_1)\circ v_2)v_3+v_2[v_3,\delta(v_1)],r]=0, \text{ for all } v_1,v_2,v_3\in\mathfrak{I}, r\in\mathfrak{R}$$

and

$$[(\delta(v_1)\circ v_2),r]v_3+(\delta(v_1)\circ v_2)[v_3,r]+v_2\big[[v_3,\delta(v_1)],r\big]+[v_2,r][v_3,\delta(v_1)]=0$$

Using the hypothesis, we observe that

$$(\delta(v_1)\circ v_2)[v_3,r]+v_2\big[[v_3,\delta(v_1)],r\big]+[v_2,r][v_3,\delta(v_1)]=0$$

Taking r by v_3 in the above equation, we have

$$v_2\big[[v_3,\delta(v_1)],v_3\big]+[v_2,v_3][v_3,\delta(v_1)]=0$$

Replacing v_2 by $\delta(v_1)[\delta(v_1),v_3]$ in the last equation, we get

$$\delta(v_1)[\delta(v_1),v_3]\big[[v_3,\delta(v_1)],v_3\big]+[\delta(v_1)[\delta(v_1),v_3],v_3][v_3,\delta(v_1)]=0$$

That is,

$$\delta(v_1)[\delta(v_1),v_3]\big[[v_3,\delta(v_1)],v_3\big]+\delta(v_1)\big[[\delta(v_1),v_3],v_3\big][v_3,\delta(v_1)]+[\delta(v_1),v_3][\delta(v_1),v_3][v_3,\delta(v_1)]=0$$

Using the above equation, we get

$$[\delta(v_1),v_3][\delta(v_1),v_3][v_3,\delta(v_1)]=0$$

and

$$([\delta(v_1),v_3])^3=0$$

The semiprime ring contains no nonzero nilpotent elements. Thus, $[\delta(v_1),v_3]=0$, for all $v_1,v_3\in\mathfrak{I}$. By Theorem 3.2 (*ii*), we get $\mathfrak{R}$ contains a nonzero central ideal.

Theorem 3.5. Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero and zero-power valued homoderivation δ on $\mathfrak{I}$ such that, for all $v_1,v_2\in\mathfrak{I}$,

i. $\delta(v_1)\delta(v_2)=v_1v_2$ or

ii. $\delta(v_1)\delta(v_2)=v_2v_1$ or

iii. $\delta(v_1)\delta(v_2)=[v_1,v_2]$ or

iv. $\delta(v_1)\delta(v_2)=v_1\circ v_2$ or

v. $\delta([v_1,v_2])=[\delta(v_1),v_2]$ or

vi. $\delta(v_1\circ v_2)=\delta(v_1)\circ v_2$

PROOF.

i. By the hypothesis, we get

$$\delta(v_1)\delta(v_2)=v_1v_2,\ \text{for all } v_1,v_2\in\mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3\in\mathfrak{I}$, in this equation, we have

$$\delta(v_1)\delta(v_2)v_3+\delta(v_1)\delta(v_2)\delta(v_3)+\delta(v_1)v_2\delta(v_3)=v_1v_2v_3$$

Using the hypothesis, we see that

$$v_1v_2\delta(v_3)+\delta(v_1)v_2\delta(v_3)=0 \tag{3}$$

That is,

$$\big(v_1+\delta(v_1)\big)v_2\delta(v_3)=0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n(x)>1$ such that $\big(\delta^{n(x)}\big)(x)=0$, for all $x\in\mathfrak{I}$. Replacing v_1 by $v_1-\delta(v_1)+\delta^2(v_1)+\cdots+(-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we get

$$v_1v_2\delta(v_3)=0$$

Replacing v_1 by $v_1v_2\delta(v_3)rv_1$ in this equation, we get

$$v_1v_2\delta(v_3)rv_1v_2\delta(v_3)=0$$

That is,

$$v_1v_2\delta(v_3)\mathfrak{R}v_1v_2\delta(v_3)=(0)$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$v_1 v_2 \delta(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_1 by $[r, v_3]$ in the last equation, we have

$$[r, v_3]v_2\delta(v_3) = 0, \text{ for all } v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

and

$$[\mathfrak{R}, v_3]\mathfrak{I}\delta(v_3) = 0, \text{ for all } v_3 \in \mathfrak{I}$$

The rest of the proof is the same as Equation 2. This completes the proof.

ii. By the hypothesis, we have

$$\delta(v_1)\delta(v_2) = v_2 v_1, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in this equation, we get

$$\delta(v_1)\delta(v_2)\delta(v_3) + \delta(v_1)\delta(v_2)v_3 + \delta(v_1)v_2\delta(v_3) = v_2 v_3 v_1$$

and

$$\delta(v_1)\delta(v_2)\delta(v_3) + \delta(v_1)\delta(v_2)v_3 + \delta(v_1)v_2\delta(v_3) = v_2 v_1 v_3 - v_2 v_1 v_3 + v_2 v_3 v_1$$

Using the hypothesis, we obtain that

$$\delta(v_1)v_2\delta(v_3) = -v_2 v_1 v_3 + v_2 v_3 v_1 - v_2 v_1 \delta(v_3) \tag{4}$$

Replacing v_2 by rv_2, $r \in \mathfrak{R}$, in this equation, we obtain that

$$\delta(v_1)rv_2\delta(v_3) = -rv_2 v_1 v_3 + rv_2 v_3 v_1 - rv_2 v_1 \delta(v_3)$$

Using Equation 4 in this equation, we get

$$\delta(v_1)rv_2\delta(v_3) = r\delta(v_1)v_2\delta(v_3), \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing r by $\delta(v_3)$ in the above equation, we find that

$$\delta(v_1)\delta(v_3)v_2\delta(v_3) = \delta(v_3)\delta(v_1)v_2\delta(v_3)$$

Using the hypothesis, we observe that

$$v_3 v_1 v_2 \delta(v_3) = v_1 v_3 v_2 \delta(v_3), \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

That is,

$$[v_1, v_3]v_2\delta(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_1 by rv_1 in the last equation and using this equation, we get

$$[r, v_3]v_1 v_2 \delta(v_3) = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Taking v_1 by $v_2\delta(v_3)t[r, v_3]$, $t \in \mathfrak{R}$, in the last equation, we observe that

$$[r, v_3]v_2\delta(v_3)t[r, v_3]v_2\delta(v_3) = 0$$

and

$$[r, v_3]v_2\delta(v_3)\mathfrak{R}[r, v_3]v_2\delta(v_3) = (0)$$

By the semiprimeness of $\mathfrak{R}$, we get

$$[r, v_3]v_2\delta(v_3) = 0, \text{ for all } v_2, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

The rest of the proof is the same as Equation 2. This completes the proof.

iii. By the hypothesis, we get

$$\delta(v_1)\delta(v_2) = [v_1, v_2], \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_1 by $v_1 v_3$ in this equation, we have

$$\delta(v_1)\delta(v_3)\delta(v_2) + \delta(v_1)v_3\delta(v_2) + v_1\delta(v_3)\delta(v_2) = [v_1, v_2]v_3 + v_1[v_3, v_2]$$

Using the hypothesis, we see that

$$[v_1, v_3]\delta(v_2) + \delta(v_1)v_3\delta(v_2) = [v_1, v_2]v_3$$

Taking v_2 by v_1 in the above equation, we find that

$$[v_1, v_3]\delta(v_1) + \delta(v_1)v_3\delta(v_1) = 0$$

By the hypothesis, we have

$$\delta(v_1)\delta(v_3)\delta(v_1) + \delta(v_1)v_3\delta(v_1) = 0 \tag{5}$$

That is,

$$\delta(v_1)(\delta(v_3) + v_3)\delta(v_1) = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_3 by $v_3 - \delta(v_3) + \delta^2(v_3) + \cdots + (-1)^{n-1}\delta^{n-1}(v_3)$ in this equation, we get

$$\delta(v_1)v_3\delta(v_1) = 0$$

That is,

$$\delta(v_1)v_3\mathfrak{R}\delta(v_1)v_3 = (0), \text{ for all } v_1, v_3 \in \mathfrak{I}$$

By the semiprimeness of $\mathfrak{R}$, we have

$$\delta(v_1)v_3 = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}$$

Taking v_3 by $r[\delta(v_1), v_3]$, $r \in \mathfrak{R}$, in this equation, we obtain that

$$\delta(v_1)r[\delta(v_1), v_3] = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R} \tag{6}$$

Replacing r by $v_3 r$ in Equation 6, we find that

$$\delta(v_1)v_3 r[\delta(v_1), v_3] = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R} \tag{7}$$

Multiplying Equation 6 on the left by v_3, we have

$$v_3\delta(v_1)r[\delta(v_1), v_3] = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R} \tag{8}$$

Subtracting Equation 7 from Equation 8, we arrive at

$$[\delta(v_1), v_3]r[\delta(v_1), v_3] = 0$$

Since $\mathfrak{R}$ is a semiprime ring, we get $[\delta(v_1), v_3] = 0$, for all $v_1, v_3 \in \mathfrak{I}$. By Theorem 3.2 (*ii*), we have $\mathfrak{R}$ contains a nonzero central ideal.

iv. By the hypothesis, we have

$$\delta(v_1)\delta(v_2) = v_1 \circ v_2, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_1 by $v_1 v_3$, $v_3 \in \mathfrak{I}$, in this equation, we get

$$\delta(v_1)\delta(v_3)\delta(v_2) + \delta(v_1)v_3\delta(v_2) + v_1\delta(v_3)\delta(v_2) = v_1(v_3 \circ v_2) - [v_1, v_2]v_3$$

Using the hypothesis, we obtain that

$$\delta(v_1)\delta(v_3)\delta(v_2) + \delta(v_1)v_3\delta(v_2) = -[v_1, v_2]v_3$$

Replacing v_2 by v_1 in this equation, we obtain that

$$\delta(v_1)\delta(v_3)\delta(v_1) + \delta(v_1)v_3\delta(v_1) = 0$$

The rest of the proof is the same as Equation 5. This completes the proof.

v. We obtain that

$$\delta([v_1, v_2]) = [\delta(v_1), v_2], \text{ for all } v_1, v_2 \in \mathfrak{I}$$

This implies that

$$[\delta(v_1), \delta(v_2)] + [\delta(v_1), v_2] + [v_1, \delta(v_2)] = [\delta(v_1), v_2], \text{ for all } v_1, v_2 \in \mathfrak{I}$$

and

$$[\delta(v_1), \delta(v_2)] + [v_1, \delta(v_2)] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is,

$$[\delta(v_1) + v_1, \delta(v_2)] = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_1 by $v_1 - \delta(v_1) + \delta^2(v_1) + \cdots + (-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we get

$$[v_1, \delta(v_2)] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Theorem 3.2 (*ii*) concludes that $\mathfrak{R}$ contains a nonzero central ideal.

vi. We get

$$\delta(v_1 \circ v_2) = \delta(v_1) \circ v_2, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

If this expression is edited, we have

$$\delta(v_1) \circ \delta(v_2) + \delta(v_1) \circ v_2 + v_1 \circ \delta(v_2) = \delta(v_1) \circ v_2$$

and

$$\delta(v_1) \circ \delta(v_2) + v_1 \circ \delta(v_2) = 0$$

That is,

$$(\delta(v_1) + v_1) \circ \delta(v_2) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Since δ is zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_1 by $v_1 - \delta(v_1) + \delta^2(v_1) + \cdots + (-1)^{n-1}\delta^{n-1}(v_1)$ in this equation, we obtain that

$$v_1 \circ \delta(v_2) = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

By Theorem 3.4 (*ii*), we get $\mathfrak{R}$ contains a nonzero central ideal.

Theorem 3.6. Let $\mathfrak{R}$ be a 2-torsion free semiprime ring and $\mathfrak{I}$ a nonzero ideal of $\mathfrak{R}$. Then, $\mathfrak{R}$ contains a nonzero central ideal, if $\mathfrak{R}$ admits a nonzero and zero-power valued homoderivation δ on $\mathfrak{I}$ such that, for all $v_1, v_2 \in \mathfrak{I}$,

i. $v_1\delta(v_2) + v_1v_2 \in Z$ or

ii. $v_1\delta(v_2) + v_2v_1 = 0$ or

iii. $v_1\delta(v_2) \pm v_1 \circ v_1 = 0$ or

iv. $[\delta(v_1), v_2] \pm v_1v_2 = 0$ or *v.* $[\delta(v_1), v_2] \pm v_2 v_1 = 0$

PROOF.

i. By the hypothesis, we get

$$v_1\delta(v_2) + v_1 v_2 \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is,

$$v_1(\delta(v_2) + v_2) \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_2 by $v_2 - \delta(v_2) + \delta^2(v_2) + \cdots + (-1)^{n-1}\delta^{n-1}(v_2)$ in this equation, we obtain that

$$v_1 v_2 \in Z, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Commuting this term with $r \in \mathfrak{R}$, we obtain that

$$0 = [v_1 v_2, r] = [v_1, r]v_2 + v_1[v_2, r]$$

Replacing v_1 by $v_3 v_1$, $v_3 \in \mathfrak{I}$, in this equation and using this equation, we get

$$[v_3, r]v_1 v_2 = 0$$

Taking v_2 by $t[v_3, r]v_1$ in this equation, we have

$$[v_3, r]v_1 t[v_3, r]v_1 = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r, t \in \mathfrak{R}$$

That is,

$$[v_3, r]v_1\mathfrak{R}[v_3, r]v_1 = (0), \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Since $\mathfrak{R}$ is a semiprime, we have

$$[v_3, r]v_1 = 0, \text{ for all } v_1, v_3 \in \mathfrak{I}, r \in \mathfrak{R}$$

Replacing v_1 by $t[v_3, r]$ in the last equation, we get

$$[v_3, r]t[v_3, r] = 0, \text{ for all } v_3 \in \mathfrak{I}, r, t \in \mathfrak{R}$$

By the semiprimeness of $\mathfrak{R}$, we have $[v_3, r] = 0$, for all $v_3 \in \mathfrak{I}, r \in \mathfrak{R}$. Thus, $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

ii. We get

$$v_1\delta(v_2) + v_2 v_1 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_1 v_2$, $v_2 \in \mathfrak{I}$, in this equation and using this equation, we have

$$v_1\delta(v_1)(\delta(v_2) + v_2) = 0$$

Since δ is a zero-power valued on $\mathfrak{I}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{I}$. Replacing v_2 by $v_2 - \delta(v_2) + \delta^2(v_2) + \cdots + (-1)^{n-1}\delta^{n-1}(v_2)$ in this equation, we obtain that

$$v_1\delta(v_1)v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $rv_1\delta(v_1)$, $r \in \mathfrak{R}$, in the last equation, we have

$$v_1\delta(v_1)rv_1\delta(v_1) = 0, \text{ for all } v_1 \in \mathfrak{I}, r \in \mathfrak{R}$$

By the semiprimeness of $\mathfrak{R}$, we get

$$v_1\delta(v_1) = 0, \text{ for all } v_1 \in \mathfrak{I} \tag{9}$$

By the hypothesis, we get

$$v_1\delta(v_1) + v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{J}$$

Using Equation 9, we obtain that

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{J} \tag{10}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we observe that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{J}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathfrak{J}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{J}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{J}$$

That is, $[\mathfrak{J}, \mathfrak{J}] = (0)$. By Lemma 2.2, we get $\mathfrak{J} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal. This completes the proof.

iii. By the hypothesis, we get

$$v_1\delta(v_2) \pm v_1 \circ v_2 = 0$$

Replacing v_2 by v_2v_1 in this equation, we get

$$v_1\delta(v_2)\delta(v_1) + v_1\delta(v_2)v_1 + v_1v_2\delta(v_1) \pm (v_1 \circ v_2)v_1 = 0$$

Using the hypothesis, we get

$$v_1\delta(v_2)\delta(v_1) + v_1v_2\delta(v_1) = 0$$

That is,

$$v_1(\delta(v_2) + v_2)\delta(v_1) = 0$$

Since δ is zero-power valued on $\mathfrak{J}$, there exists an integer $n > 1$ such that $\delta^n(x) = 0$, for all $x \in \mathfrak{J}$. Replacing v_2 by $v_2 - \delta(v_2) + \delta^2(v_2) + \cdots + (-1)^{n-1}\delta^{n-1}(v_2)$ in this equation, we obtain that

$$v_1v_2\delta(v_1) = 0, \text{ for all } v_1, v_2 \in \mathfrak{J}$$

Taking v_2 by $\delta(v_1)rv_1$, $r \in \mathfrak{R}$, in the above equation, we see that

$$v_1\delta(v_1)rv_1\delta(v_1) = 0$$

By the semiprimeness of $\mathfrak{R}$, we have

$$v_1\delta(v_1) = 0, \text{ for all } v_1 \in \mathfrak{J}$$

By the hypothesis and using this equation, we have

$$v_1 \circ v_1 = 0, \text{ for all } v_1 \in \mathfrak{J} \tag{11}$$

and

$$2v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{J}$$

Since $\mathfrak{R}$ is a 2-torsion free, we have

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{J}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we see that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

iv. We get

$$[\delta(v_1), v_2] \pm v_1 v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Taking v_2 by $v_2 v_1$ in this equation, we get

$$[\delta(v_1), v_2]v_1 + v_2[\delta(v_1), v_1] \pm v_1 v_2 v_1 = 0$$

By the hypothesis, we get

$$v_2[\delta(v_1), v_1] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I} \tag{12}$$

Replacing v_2 by $[\delta(v_1), v_1]r$, $r \in \mathfrak{R}$, in this equation, we get

$$[\delta(v_1), v_1]r[\delta(v_1), v_1] = 0$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$[\delta(v_1), v_1] = 0, \text{ for all } v_1 \in \mathfrak{I}$$

By the hypothesis, we get

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we see that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_2 v_3$, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal.

v. By the hypothesis, we get

$$[\delta(v_1), v_2] \pm v_2 v_1 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $v_1 v_2$ in this equation, we have

$$v_1[\delta(v_1), v_2] + [\delta(v_1), v_1]v_2 \pm v_1 v_2 v_1 = 0$$

That is,

$$[\delta(v_1), v_1]v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by $r[\delta(v_1), v_1]$, $r \in \mathfrak{R}$, in this equation, we get

$$[\delta(v_1), v_1]r[\delta(v_1), v_1] = 0$$

Since $\mathfrak{R}$ is a semiprime ring, we have

$$[\delta(v_1), v_1] = 0, \text{ for all } v_1 \in \mathfrak{I}$$

By the hypothesis, we get

$$v_1^2 = 0, \text{ for all } v_1 \in \mathfrak{I}$$

Replacing v_1 by $v_1 + v_2$ in this equation, we see that

$$v_1 \circ v_2 = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

Replacing v_2 by v_2v_3, $v_3 \in \mathfrak{I}$, in the above expression and using this, we get

$$[v_1, v_2]v_3 = 0, \text{ for all } v_1, v_2, v_3 \in \mathfrak{I}$$

Replacing v_3 by $r[v_1, v_2]$ in this equation, we have

$$[v_1, v_2]\mathfrak{R}[v_1, v_2] = (0)$$

Since $\mathfrak{R}$ is a semiprime ring, we get

$$[v_1, v_2] = 0, \text{ for all } v_1, v_2 \in \mathfrak{I}$$

That is, $[\mathfrak{I}, \mathfrak{I}] = (0)$. By Lemma 2.2, we get $\mathfrak{I} \subseteq Z$. We conclude that $\mathfrak{R}$ contains a nonzero central ideal. We complete the proof.

Example 3.7. Let $\mathfrak{R} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$, $\mathfrak{I} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix} | a, b \in \mathbb{R} \right\}$, and $\delta: \mathfrak{R} \to \mathfrak{R}$ be a map defined by

$$\delta \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & b \\ c & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

Then, it is easy to verify that δ is an homoderivation of $\mathfrak{R}$, $\mathfrak{I}$ is an ideal of $\mathfrak{R}$, and $\mathfrak{R}$ is not a semiprime ring. The commutativity conditions given in Theorem 3.5 are satisfied. However, we have $\mathfrak{I} \nsubseteq Z$. We conclude that $\mathfrak{R}$ does not contain a nonzero central ideal.

4. Conclusion

The present study has shown some essential properties of a nonzero ideal of a semiprime ring with homoderivation. Moreover, it has exemplified the necessity of the hypotheses used in theorems. In future research, some well-known results in derivation can be applied to homoderivation and generalized homoderivation. Besides, the findings herein could help to uncover properties of homoderivations in Lie ideals or square-closed Lie ideals.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

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