Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi

Afyon Kocatepe University Journal of Science and Engineering

AKÜ FEMÜBİD **23** (2023) 031302 (638-647)

AKU J. Sci. Eng. **23** (2023) 031302 (638-647)

DOI: 10.35414/akufemubid.1182145

Araştırma Makalesi / Research Article

Genelleştirilmiş Kuaterniyonlar ve Matris Cebiri

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Geliş Tarihi: 29.09. 2022 Kabul Tarihi: 02.05.2023

Anahtar kelimeler
Genelleştirilmiş kuaterniyonlar; Hamilton operatörleri; Pauli matrisleri; Genelleştirilmiş üniter matrisler; Genelleştirilmiş ortogonal matrisler

Öz

Bu çalışmada, Hamilton operatörlerini kullanarak genelleştirilmiş kuaterniyon cebiri ile gerçel (kompleks) matris cebirleri arasındaki bağlantıyı kurduk. Genelleştirilmiş kuaterniyonların gerçel ve kompleks temeline karşılık gelen gerçel ve kompleks matrisler elde ettik. Ayrıca, gerçel ve kompleks matrislerin temel özelliklerini araştırdık. Genelleştirilmiş kuaterniyonlara karşılık gelen Pauli matrislerini elde ettik. Daha sonra, bu matrisler tarafından üretilen cebirin, genelleştirilmiş $E^3_{\alpha\beta}$ uzayı tarafından üretilen Clifford cebiri $Cl(E^3_{\alpha\beta})$ ile izomorf olduğunu gösterdik. Son olarak, genelleştirilmiş birim kuaterniyonlara karşılık gelen simplektik matrisler grubu, genelleştirilmiş birim matrisler grubu ve genelleştirilmiş ortogonal matrisler grubu arasındaki ilişkileri inceledik.

Generalized Quaternions and Matrix Algebra

Keywords
Generalized quaternions; Hamilton operators; Pauli matices; Generalized unitary matrices; Generalized orthogonal matrices

Abstract

In this paper, we established the connection between generalized quaternion algebra and real (complex) matrix algebras by using Hamilton operators. We obtained real and complex matrices corresponding to real and complex basis of the generalized quaternions. Also, we investigated the basis features of real and complex matrices. We get Pauli matrices corresponding to generalized quaternions. Then, we have shown that the algebra produced by these matrices is isomorphic to the Clifford algebra $Cl(E^3_{\alpha\beta})$ produced by generalized space $E^3_{\alpha\beta}$.Finally, we studied the relations among symplectic matrices group corresponding to generalized unit quaternions, generalized unitary matrices group, and generalized orthogonal matrices group.

1. Introduction and Preliminaries

Clifford algebra is a unital associative algebra generated by a vector space with a quadratic form. This algebra examines the properties of the vector spaces by using dot product, cross product, geometric product. Thus, this algebra is a useful and standard tool in dealing with geometric and physical problems. Clifford algebra can be generalized to the real numbers, complex numbers, quaternions, hyper-complex number systems. Clifford algebra's name comes from British mathematician W. K. Clifford (1876). The Clifford algebra generated by the vector space V with a scalar product on it is uniquely defined. In addition to the linear and associative properties of the scalar product, the following property must be provided for the x and y vectors in vector space V,

$$xy + yx = 2x \cdot y$$

where '$\cdot$' is a scalar product on V. The subset of the even degree elements of $Cl(V)$ defines the even subalgebra and denoted by $Cl^+(V)$. When $dimV = n$ then, $dimCl(V) = 2^n$ and $dimCl^+(V) = 2^{n-1}$ (Aragan *et al.* 1997, Catoni *et al.* 2005).

In this study, we defined Clifford algebra product on generalized space. Afterward, we have shown that even Clifford algebra of $E^3_{\alpha\beta}$ corresponds to generalized quaternion algebra. 3-dimensional non-degenerate vector space $E^3_{\alpha\beta}$ has a orthogonal basis $\{e_1, e_2, e_3\}$. The even Clifford algebra $Cl^+(E^3_{\alpha\beta}) = Cl_{p,q}; p + q = 3$ of the vector space has a orthogonal basis $\{1, e_1 = i, e_2 = j, e_1e_2 = k\}$, where $e_1^2 = -\alpha$, $e_2^2 = -\beta$ and $e_1e_2 = -e_2e_1$ (Ata and Savcı, 2021).

Generalized quaternions have emerged as a tool for studying quadratic forms. These quaternions are a natural generalization of Hamilton quaternions and split quaternions. Generalized quaternions can be either a Hamiltonian quaternion or a split quaternion, depending on the choice of numbers α and β. Therefore, it has a more general algebraic structure. For more detailed information on Hamiltonin and split quaternions, see (Alagöz 2012,Cockle 1849, Hamilton 1853, 1866, Özdemir and Ergin 2006, Sangwine and Le Bihan 2010). Ata and the Savcı have shown that unit generalized quaternions $H_{\alpha\beta}^3$ correspond to a rotation in generalized space $E_{\alpha\beta}^3$. They stated that this motion gives the rotational motion in 3-dimensional Euclidean space E^3 and in 3-dimensional Lorenz space E_1^3 for their special choices of $\alpha, \beta \in \mathbb{R}$ (Ata and Savcı, 2021). Ata and Yıldırım have obtained a different polar representation of generalized quaternions. And, they showed that a rotational motion in generalized space can be obtained as the product of two rotations in the same space by using this notation (Ata and Yıldırım, 2018). See detail (Ata *et al*. 2012, Jafari and Yaylı 2015, Lam 2005) for more information about generalized space and their algebraic properties.

In this paper, real and complex matrix representations of a generalized quaternion are obtained with the help of Hamilton operators. The real and complex basis matrices corresponding to the real and complex basis vectors of the generalized quaternion algebra are obtained. Besides that the properties of these matrices are investigated. It is shown that second complex Hamilton matrices corresponding to a generalized unit quaternion are a generalized special unitary 2×2 matrix. These matrices have been obtained in exponential form using generalized Pauli matrices. It has been shown that the algebra produced by the generalized Pauli matrices is isomorphic to Cilfford algebra $Cl(E_{\alpha\beta}^3)$ and the Lie algebra of group $SU_{\alpha\beta}(2)$.

In this sections, we present some properties of the generalized quaternions.

Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ be in $\mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$, then the generalized metric tensor product is defined by $g(u, v) = \alpha u_1v_1 + \beta u_2v_2 + \alpha\beta u_3v_3$. It could be written matrix formas as;

$$g(u, v) = u^T \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} v = u^T G v.$$

Generalized metric tensor $g(u, v)$ has different names depending on the choice of numbers $\alpha, \beta \in \mathbb{R}$; $g(u, v)$ is called generalized inner product when $\alpha > 0, \beta > 0$,

$g(u, v)$ is called the generalized semi-Euclidean inner product when $\alpha > 0, \beta < 0$,

$g(u, v)$ is called Euclidean inner product when $\alpha = \beta = 1$,

$g(u, v)$ is called semi-Euclidean inner product when $\alpha = 1, \beta = -1$,

The vector space on $\mathbb{R}^3$ equipped with the generalized inner product, is called 3-dimensional generalized space denoted by $E_{\alpha\beta}^3$ or $E^3(\alpha, \beta)$.

If $\alpha = \beta = 1$, then $E^3(1,1) = E^3$ 3-dimensional Euclidean space If $\alpha = 1$ and $\beta = -1$, then $E^3(1, -1) = E_1^3$ 3-dimensional semi-Euclidean space (Minkowski space).

1.1 Generalized Quaternions

A generalized quaternion is defined as

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2 and a_3 are real numbers and i, j, k are quaternionic units which satisfy the equations

$$\begin{aligned} &i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta \\ &ij = k = -ji, \quad jk = \beta i = -kj \end{aligned}$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$. A generalized quaternion q is a sum of a scalar and a vector, where scalar part, $S_q = a_0$, and vector part $\mathbf{V}_q = a_1 i + a_2 j + a_3 k \in \mathbb{R}_{\alpha\beta}^3$. Therefore, $H_{\alpha\beta}$ forms 4-dimensional real space which contains the (real) axis $\mathbb{R}, H_{\alpha\beta} = \mathbb{R} \oplus E_{\alpha\beta}^3$.

Special cases:

1) If $\alpha = \beta = 1$, then $H_{\alpha\beta}$ is the algebra of real quaternions.

2) If $\alpha = 1, \beta = -1$, then $H_{\alpha\beta}$ is the algebra of split quaternions.

3) If $\alpha = 1, \beta = 0$, then $H_{\alpha\beta}$ is the algebra of semi quaternions.

4) If $\alpha=-1, \beta=0$, then $H_{\alpha\beta}$ is the algebra of split semi-quaternions.

5) If $\alpha=0, \beta=0$, then $H_{\alpha\beta}$ is the algebra of $1/4$ quaternions (Jafari 2012).

The addition rule for generalized quaternions, $H_{\alpha\beta}$, is:

$$p+q=(a_0+b_0)+(a_1+b_1)i+(a_2+b_2)j+(a_3+b_3)k$$

for $p=a_0+a_1i+a_2j+a_3k$ and $q=b_0+b_1i+b_2j+b_3k$.

This rule preserves the associativity and commutativity properties of addition, and provides a consistent behaviour for the subset of quaternions corresponding to real numbers, i.e

$$S_{p+q}=S_p+S_q=a_0+b_0.$$

The product of a scalar and a generalized quaternion is defined in a straight forward manner. If c is a scalar and $q \in H_{\alpha\beta}$,

$$cq=cS_q+c\mathbf{V}_q=(ca_0)1+(ca_1)i+(ca_2)j+(ca_3)k.$$

The multiplication rule for generalized quaternions is defined as

$$pq=S_pS_q-g(\mathbf{V}_p,\mathbf{V}_q)+S_p\mathbf{V}_q+S_q\mathbf{V}_q+\mathbf{V}_p\wedge\mathbf{V}_q$$

which could also be expressed as

$$pq=\begin{pmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Obviously, quaternion multiplication is associative and distributive with respect to addition and subtraction, but the commutative law does not hold in general (Grop *et al*. 2003).

$H_{\alpha\beta}$ with addition and multiplication has all the properties of a number field except commutativity of the multiplication. It is therefore called the skew field of generalized quaternions.

Some properties of generalized quaternions were given below;

1) The Hamilton conjugate of $q=a_0+a_1i+a_2j+a_3k=S_q+\mathbf{V}_q$ is

$$\overline{q}=a_0-(a_1i+a_2j+a_3k)=S_q-\mathbf{V}_q$$

2) The norm of q is defined as $N_q=|q\overline{q}|=|\overline{q}q|=\left|a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2\right|$. If $N_q=1$, then q is called unit generalized quaternion, for $N_q\neq 0$, $q_0=\frac{q}{N_q}$ is unit generalized quaternion.

If $N_q=\left|a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2\right|=1$, then q is called as unit generalized quaternion.

3) The inverse of generalized quaternion q is defined as $q^{-1}=\frac{\overline{q}}{N_q}, N_q\neq 0$.

If $I_q>0$, $I_q<0$, and $I_q=0$ for $<q,q>=I_q$, then q is called spacelike, timelike, and lightlike quaternion, respectively.

If generalized quaternion q is lightlike quaternion ($N_q=0$), then, q is not inversible.

4) The scalar product of two generalized quaternions, $p=S_p+\mathbf{V}_p$ and $q=S_q+\mathbf{V}_q$, is defined as;

$$\begin{aligned} <p,q> &= S_pS_q+g(\mathbf{V}_p,\mathbf{V}_q) \\ &= S_{p\overline{q}} \end{aligned}$$

The above expression defines a metric in $E^4_{\alpha\beta}$. From know on, for the calculation below, according to the selection of α and β, $H^4_{\alpha\beta}$ generalized quaternion space were taken identical to generalized Euclidean and semi-Euclidean space, where

$$E^4_{\alpha\beta}=\{q=(a_0,a_1,a_2,a_3)\in\mathbb{R}^4:<q,q> = a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2, \alpha,\beta \in\mathbb{R}\}$$

Generalized quaternions with scalar part zero are called pure quaternions and denoted by $H^{\circ}_{\alpha\beta}$. The set $H^{\circ}_{\alpha\beta}$ is a subalgebra of $H_{\alpha\beta}$ and it is identical to 3-dimensional generalized real linear Euclidean space $\mathbb{R}^3_{\alpha\beta}$.

5) Polar form: Let $\alpha,\beta>0$, then every generalized quaternion $q=a_0+a_1i+a_2j+a_3k$ can be written in the form

$$q=r(\cos\theta+\overleftarrow{u}\sin\theta), \qquad 0\leq\theta\leq 2\pi$$

with

$$r=\sqrt{N_q}=\sqrt{a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2},$$

and

$$\cos\theta=\frac{a_0}{r}, \qquad \sin\theta=\frac{\sqrt{\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2}}{r}.$$

The unit vector $\overleftarrow{u}$ is given by

$$\mathrm{u}=\frac{a_1i+a_2j+a_3k}{\sqrt{\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2}}$$

where $\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2\neq 0$.

Let $\alpha>0, \beta<0$, then

$q = a_0 + a_1 i + a_2 j + a_3 k$ is generalized quaternion, in polar coordinates, q is given by

$q = r(\cosh\varphi + \mathrm{u}\sinh\varphi)$, (the generalized quaternion is called timelike generalized quaternion with spacelike vectorial part), where $< q, q >< 0$, $g(\mathbf{V}_p, \mathbf{V}_q) > 0$, $\cosh\varphi = \frac{a_0}{r}$ and

$$\sinh\varphi = \frac{\sqrt{\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2}}{r}, \varphi \in \mathbb{R}$$

(Ata and Yıldırım 2018).

1.2 Hamilton Operators in Generalized Quaternion

We will study, the real and complex Hamilton operators and their properties, corresponding to generalized quaternions, via matrices.

Let H^+ and H^- be linear transformations,

$$H^+: \begin{array}{l} H_{\alpha\beta} \to H_{\alpha\beta} \\ q \to H^+(q) \end{array} \quad \text{and} \quad H^-: \begin{array}{l} H_{\alpha\beta} \to H_{\alpha\beta} \\ q \to H^-(q) \end{array}$$

are endomorphisms.

Left and right notations of $H_{\alpha\beta}$ algebra can be given equation (1) and equation (2), respectively;

$$H^+(q): \begin{array}{l} H_{\alpha\beta} \to H_{\alpha\beta} \\ x \to H^+(q)(x) = qx \end{array} \tag{1}$$

and

$$H^-(q): \begin{array}{l} H_{\alpha\beta} \to H_{\alpha\beta} \\ x \to H^-(q)(x) = xq \end{array} \tag{2}$$

Those maps are called as Hamilton operators

As known, each finite dimensional associative A algebra over the K field, isomorphic to subalgebra of $M_n(K)$ algebra. Thus, one can find accurate representation of the A algebra of the $M_n(K)$ algebra.

For the $H_{\alpha\beta}$ generalized quaternion algebra $H^+(q)$ and $H^-(q)$ transformations are isomorphisms,

$$H^+: H_{\alpha\beta} \to M_4(\mathbb{R}),$$

$$H^+(q) = \begin{pmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}$$

and

$$H^-: H_{\alpha\beta} \to M_4(\mathbb{R}),$$

$$H^-(q) = \begin{pmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}$$

where $q = a_0 + a_1 i + a_2 j + a_3 k \in H_{\alpha\beta}$. $H^+(q)$ and $H^-(q)$ matrices are called real Hamiltonian matrices of generalized quaternions. The product of the generalized quaternions (p and q) can be shown as matrices product;

$$qp = H^+(q)p \text{ and } qp = H^-(p)q$$

Matrices derived by Hamilton matrices $H^+(q)$ and $H^-(q)$ are generalized pseudo-orthogonal matrices which satisfy the following properties,

i) $(H^+(q))^T \varepsilon (H^+(q)) = N(q)\varepsilon$,

ii) $(H^-(q))^T \varepsilon (H^-(q)) = N(q)\varepsilon$,

iii) $H^+(q)$ and $H^-(q)$ are generalized orthonormal matrices if and only if q is a generalized unit quaternions,

$$qp = H^+(q)p \text{ and } pq = H^-(p)q$$

(Jafari and Yaylı 2015).

1.3 Fundamental Real Matrices of the Generalized Quaternions

The scope of this section is to find out how many real fundamental matrices that generalized quaternions have .

Let I_4 be a 4×4 identity matrix and H_1, J_1, K_1 be 4×4 real matrices. Hence, the first fundamental matrix of q can be given as

$$H^+(q) = a_0 I_4 + a_1 H_1 + a_2 J_1 + a_3 K_1 \tag{3}$$

Where $H_1 = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

$$J_1 = \begin{pmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The second fundamental matrix;

$$H^-(q) = a_0 I_4 + a_1 H_2 + a_2 J_2 + a_3 K_2 \tag{4}$$

Where $H_2 = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -1 & 0 \end{pmatrix}$,

$$J_2 = \begin{pmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & \beta & 0 \\ 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

A matrix $S = \begin{pmatrix} 0 & \alpha s_1 & \beta s_2 & \alpha\beta s_3 \\ -s_1 & 0 & -\beta s_4 & \beta s_5 \\ -s_2 & \alpha s_4 & 0 & -\alpha s_6 \\ -s_3 & -s_5 & s_6 & 0 \end{pmatrix}$ called a generalized skew-symmetric matrix if $S^T\varepsilon = -\varepsilon S$, where

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{pmatrix}, \alpha, \beta \in \mathbb{R}$$

(Jafari and Yaylı 2015).

2. Some Properties of Real Hamilton Matrices in Generalized Quaternions

The set of all the $H_{\alpha\beta}$ generalized quaternions is a form of 4-dimensional real vector space which contains the real axis $\mathbb{R}$ and 3-dimensional real linear space $E^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = \mathbb{R} \oplus E^3_{\alpha\beta}$. So, $q \in \mathbb{R}E^4_{\alpha\beta}$ (where $E^4_{\alpha\beta}$ is generalized 4-dimensional real vector space) vector and q generalized quaternion can be match, which is denoted as "$\cong$". Here

$$q = a_0 + a_1 i + a_2 j + a_3 k \cong q = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

2.1 Generator Fundamental Real Matrices of Generalized Quaternion

In this part, we obtained different fundamental matrices using the triplet (H_1, J_1, K_1) with their negations. Firstly, we generated ordered triples which satisfy condition 3. We found that there were totally two possible choices for the first element of the triplet. Then, deleting the chosen matrix and its negation, we had four choices for the second element of the triplet, leaving the third element to be determined by the product of the first and second elements. Thus, we got a totally 8 choices for the first system. By the same way, we got a totally 8 choices for the second system $((H_2, J_2, K_2)$. Then, we had 16 ordered triples for each system. Hence, we obtain 16 different fundamental matrices using the ordered triples which satisfy condition 3 or 4.

For the convenience in working, we will express fundamental matrices $H^{+i}(q)$ and $H^{-i}(q)$, $0 \leq i \leq 7$ obtained from ordered triples (H_s, J_s, K_s), $(H_s, -J_s, -K_s)$, $(H_s, -J_s, K_s)$, $(H_s, J_s, -K_s)$, $(-H_s, J_s, -K_s)$, $(-H_s, -J_s, K_s)$, $(-H_s, J_s, K_s)$, $(-H_s, -J_s, -K_s)$, respectively, for s=1,2.

2.2 Basic Properties of The Fundamental Real Matrices

For a given generalized quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$, we can write the conjugate of q as below,

$$\overline{q} \cong \overline{\mathbf{q}} = \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} = Cq, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

and

$$S_q \cong a_0 e_1, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{V}_q \cong q_* = \begin{pmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Let q and p be the generalized quaternions and $a, b \in \mathbb{R}$, then the following identities hold: $H^{\pm i}$ represents H^{+i} and H^{-i} together.

1. $q = p \Leftrightarrow H^{\pm i}(q) = H^{\pm i}(p),\ 0 \leq i \leq 7,$

2. $H^{\pm i}(p + q) = H^{\pm i}(p) + H^{\pm i}(q)$, $H^{\pm i}(pq) = H^{\pm i}(p) H^{\pm i}(q)$,

3. $H^{+i}(q) H^{-i}(p) = H^{-i}(p) H^{+i}(q)$,

4. $H^{+i}(p) H^{+i}(q) = H^{+i}((H^{+i}(q))(p))$, $H^{\pm i}(ap + bq) = aH^{\pm i}(p) + bH^{\pm i}(q)$,

5. $(H^{+i}(\overline{q}))^T = C(H^{-i}(q))^T C$,

6. $tr(H^{\pm i}(q)) = 4S_q$, $det(H^{\pm i}(q)) = \|q\|^2$.

The 3th identity can be proved by simple matrix computation. Naturally, these identities are closely related to basic properties of generalized quaternion algebra. For example, the identity

$$H^{\pm i}(ap + bq) = aH^{\pm i}(p) + bH^{\pm i}(q)$$

is connected to the linearity of left (or right) multiplication. Similarly, the identity

$$H^{+i}(p)H^{+i}(q) = H^{+i}((H^{+i}(q))(p))$$

is related to the associative law for generalized quaternion multiplication. Furthermore, applying the matrix approach leads to a convenient and concise way of writing proofs. We can illustrate this point by considering three well-known identities $|q\overline{q}| = \|q\|^2$, $\|qp\|^2 = \|q\|^2\|p\|^2$ and $\overline{qp} = \overline{p}\overline{q}$.

$$\begin{aligned} |q\overline{q}| &\cong |H^{+0}(q)(\overline{q})| = |H^{-0}(\overline{q})(p)| = |q|^2 e_1, \\ &\cong \|q\|^2 = \left\|a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2\right\|^2 \\ \|qp\|^2 &\cong \left|\left|H^{+0}(q)(p)\right|\right|^2 = \|q\|^2\|p\|^2 \end{aligned}$$

and

$$\begin{aligned} \overline{qp} &\cong C(H^{+0}(q)(p)) = (CH^{+0}(q))(p) \\ &= H^{-0}(\overline{q})(\overline{p}) = \overline{p}\overline{q}. \end{aligned}$$

Now, we define the linear transformation representing multiplication of fundamental matrices in $H_{\alpha\beta}$. Let q be a generalized quaternion, then $\gamma^{+i}(q): H_{\alpha\beta} \to H_{\alpha\beta}$ and $\gamma^{-i}(q): H_{\alpha\beta} \to H_{\alpha\beta}$ are defined as follows:

$$\begin{aligned} \gamma^{+i}(q)(x) &= H^{+i}(q)(x), \\ \gamma^{-i}(q)(x) &= H^{-i}(q)(x), \\ x &\in H_{\alpha\beta}. \end{aligned}$$

Let q be a unit generalized quaternion;

$$\left|\gamma^{+i}(q)(x)\right| = \left|H^{+i}(q)(x)\right| = \left|H^{+i}(q)\right||x| = |x|,$$

and

$$\left|\gamma^{-i}(q)(x)\right| = \left|H^{+i}(q)(x)\right| = \left|H^{+i}(q)\right||x| = |x|,$$

where $\gamma^{+i}(q)$ and $\gamma^{-i}(q)$ are generalized orthogonal transformations of $H_{\alpha\beta}$. Thus, for unit generalized quaternions q and p, the mapping $C_{p,q}: H_{\alpha\beta} \to H_{\alpha\beta}$ is defined by

$$C_{p,q} = \gamma^{+i}(q) \circ \gamma^{-i}(p) = \gamma^{-i}(p) \circ \gamma^{+i}(q)$$

Similarly, the complex Hamilton operators corresponding to generalized quaternions are calculated as follows

$$\begin{aligned} q &= a_0 + a_1 i + a_2 j + a_3 k \\ &= (a_0 + a_1 i) + (a_2 + a_3 i)j \end{aligned}$$

$q = z + wj$, where $z = a_0 + a_1 i$, $w = a_2 j + a_3 k$, and $z, w \in \mathbb{C}$. Hence, the generalized quaternion q is conceivable pair of complex number, i.e. $H_{\alpha\beta} \cong \mathbb{C}^2$. The complex standard base of the generalized quaternion algebra is $\{1, j\}$,

$$\begin{aligned} H_{\mathbb{C}}^{+}(q)(1) &= q \cdot 1 &&= z + wj, \\ H_{\mathbb{C}}^{+}(q)(j) &= q \cdot j &&= (z + wj) \cdot j = zj + wj^2 \\ & &&= -\beta w + zj \end{aligned}$$

from the above two equations;

$$H_{\mathbb{C}}^{+}(q) = \begin{pmatrix} z & w \\ -\beta w & z \end{pmatrix}, \alpha = 1, \beta \in \mathbb{R}$$

Fundamental complex matrices of generalized quaternions;

$$H_{\mathbb{C}}^{+}(q) = zI_2 + wH_1;$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H_1 = \begin{pmatrix} 0 & -\beta \\ 1 & 0 \end{pmatrix}$ and

$$H_1^2 = -\beta I_2.$$

Similarly, second fundamental complex matrices;

$$\begin{aligned} H_{\mathbb{C}}^{-}(q)(1) &= 1 \cdot q &&= z + wj, \\ H_{\mathbb{C}}^{-}(q)(j) &= j \cdot q &&= j \cdot (z + wj) = jz + jwj \\ & &&= \overline{z}j + \overline{w}j^2 \\ & &&= \overline{z}j - \beta\overline{w} \\ & &&= -\beta\overline{w} + \overline{z}j \end{aligned}$$

from the above equations;

$$H_{\mathbb{C}}^{-}(q) = \begin{pmatrix} z & w \\ -\beta\overline{w} & \overline{z} \end{pmatrix}$$

Using the $H_{\alpha\beta} \cong \mathbb{C}^2$ relation, mentioned above, p and q quaternion product of generalized quaternions p, q (right product of p with q) according to matrix product $H_{\mathbb{C}}^{-}(q) \cdot p$, such as, for $q = z_1 + w_1 j$ and $p = z_2 + w_2 j$,

$$\begin{aligned} qp &= (z_1 + w_1 j)(z_2 + w_2 j) \\ &= (z_1 z_2 - \beta\overline{w}_1 w_2) + (w_1 z_2 + \overline{z}_1 w_2)j \\ &= (z_1 z_2 - \beta w_1 \overline{w}_2, z_1 w_2 + w_1 \overline{z}_2) \end{aligned}$$

and, for $H_{\mathbb{C}}^{-}(q) = \begin{bmatrix} z_1 & w_1 \\ -\beta\overline{w}_1 & \overline{z}_1 \end{bmatrix}$, $p = z_2 + w_2 j = \begin{bmatrix} z_2 \\ w_2 \end{bmatrix}$, then

$$\begin{aligned} qp = H_{\mathbb{C}}^{-}(p) \cdot q &= \begin{bmatrix} z_2 & w_2 \\ -\beta\overline{w}_2 & \overline{z}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ w_1 \end{bmatrix} \\ &= \begin{bmatrix} z_1 z_2 + \ w_1 w_2 \\ -\beta\overline{w}_2 z_1 + w_1 \overline{z}_2 \end{bmatrix} \end{aligned}$$

In this respect, quaternion multiplication can be presented with second Hamilton complex matrices.

If one take $q = z + wj$, $z, w \in \mathbb{C}$ as a generalized unit quaternion, then second fundamental Hamiltonian complex matrix $H_{\mathbb{C}}^{-}(q) = \begin{bmatrix} z & w \\ -\beta\overline{w} & \overline{z} \end{bmatrix}$ is equivalent $\left(N_q\right)^2 = |z|^2 + \beta|w|^2 = 1$, where $\beta \neq 0$. In this case, matrix $H_{\mathbb{C}}^{-}(q)$ is a generalized special unitary matrix, i.e.

$$\left(\overline{H_{\mathbb{C}}^{-}(q)}\right)^{T} \epsilon H_{\mathbb{C}}^{-}(q) = \epsilon \text{ and } det\left(H_{\mathbb{C}}^{-}(q)\right) = 1,$$

$$\epsilon = \begin{bmatrix} \beta & 0 \\ 0 & 1 \end{bmatrix}$$

where ϵ is 2×2 diagonal matrix, obtained using generalized inner product and base $\{1, j\}$. Those matrices generate $SU_{\alpha\beta}(2)$ which is a group of generalized special unitary 2×2 matrices.

If we take $\beta < 0$, then $SU_{\alpha\beta}(2)$ is a group of generalized special unitary hyperbolic matrices. Consequently, it is an isomorphism of the group $S_{\alpha\beta}^3$ onto the group $SU_{\alpha\beta}(2)$. Using the matrix $H_{\mathbb{C}}^{-}(q)$, we can obtain matrices corresponding to unit quaternions $\{1, i, j, k\}$ as follows:

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$j = \begin{bmatrix} 0 & 1 \\ -\beta i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -\dfrac{i}{\alpha} \\ \dfrac{\beta}{\alpha} i & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & 1 \\ -\beta i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ \beta & 0 \end{bmatrix}.$$

Let

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ \beta & 0 \end{bmatrix},$$

$$\sigma_2 = \begin{bmatrix} 0 & -\dfrac{i}{\alpha} \\ \dfrac{\beta}{\alpha} i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then

$$\begin{aligned} H_{\mathbb{C}}^{-}(q) &= \begin{bmatrix} z & w \\ -\beta\overline{w} & \overline{z} \end{bmatrix} = \begin{bmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -\beta(a_2 - a_3 i) & a_0 - a_1 i \end{bmatrix} \\ &= a_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ -\beta & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 1 \\ \beta i & 0 \end{bmatrix} \\ &= a_0\sigma_0 + a_1 i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_2 i \begin{bmatrix} 0 & -\dfrac{i}{\alpha} \\ \dfrac{\beta}{\alpha} i & 0 \end{bmatrix} + a_3 i \begin{bmatrix} 0 & 1 \\ \beta & 0 \end{bmatrix} \\ &= a_0\sigma_0 + i(a_3\sigma_1 + a_2\sigma_2 + a_1\sigma_3), \end{aligned}$$

where the matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ are generalized Paula matrices. These matrices are generalized Hamilton matrices and have zero traces. if we chose $a_0 = e_0$, $a_3 = e_1$, $a_2 = e_2$ and $a_1 = e_3$ then we can write $H_{\mathbb{C}}^{-}(q) = a_0 e_0 + i(e_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$ where e_0, e_1, e_2 and e_3 are called Euler parameters corresponding to the rotation specified by $H_{\mathbb{C}}^{-}(q)$. Since det($H_{\mathbb{C}}^{-}(q)$)= 1 so we can

$$H_{\mathbb{C}}^{-}(q) = \cos\theta\sigma_0 + i\sin\theta(\beta\sigma_3 + \gamma\sigma_2 + \delta\sigma_1),$$

where $(\beta, \gamma, \delta) = \frac{1}{\sin\theta}(a_3, a_2, a_1)$. If we use the known exponential function for matrices $A = \beta\sigma_3 + \gamma\sigma_2 + \delta\sigma_1$ we get

$$H_{\mathbb{C}}^{-}(q) = \cos\theta\sigma_0 + i\sin\theta A = e^{i\theta A}.$$

Since the matrix A is a generalized Hamilton matrices and has zero traces, matrix iA is a generalized skew symmetric Hamilton matrix and has zero traces.

The group $SU_{\alpha\beta}(2)$ is the Lie group of generalized unitary matrices which determinant is 1. The Lie algebra of $SU_{\alpha\beta}(2)$ is a $3-$dimesional generalized real algebra spanned by the set $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ and denoted by

$su_{\alpha\beta}(2) = span\{i\sigma_1, i\sigma_2, i\sigma_3\}$. As a result, the above exponential formula is obtained from the exponential transformation defined from Lie group $SU_{\alpha\beta}(2)$ to Lie algebra $su_{\alpha\beta}(2)$. Generalized vector space $E_{\alpha\beta}^3$ is a Lie algebra with the cross product defined on itself. Thus,

$$\exp = \quad E_{\alpha\beta}^3 \to SU_{\alpha\beta}(2)$$

$$\theta\vec{v} \to \exp(\theta\vec{v}) = (\cos\theta, \sin\theta\vec{v})$$

the transformation is isomorphism where $\vec{v}$ is a unit vector in space $E_{\alpha\beta}^3$ and $\theta \in \mathbb{R}$. The algebra produced by generalized Paulo matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ is isomorphic to the $Cl(E_{\alpha\beta}^3)$ Clifford algebra of $E_{\alpha\beta}^3$ and the algebra produced by matrices $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is isomorphic to the algebra of generalized quaternions $H_{\alpha\beta}$

In the following section, we showed matrix representation of the generalized 3-sphere $S_{\alpha\beta}^3$. Also, its relations with other matrix groups are given.

3. Matrices Representation of The Generalized Unit 3-Sphere $S_{\alpha\beta}^3$

Let $S_{\alpha\beta}^3$ denotes the unit generalized quaternions set in $H_{\alpha\beta}$, generalized quaternions space. Hence,

$$S_{\alpha\beta}^3 = \left\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \middle| x_0^2 + \alpha x_1^2 + \beta x_2^2 + \alpha\beta x_3^2 \right| = 1, \alpha, \beta \in \mathbb{R}\right\} \subset E_{\alpha\beta}^4.$$

The set has a group structure with generalized quaternions product.

Symplectic group over the generalized quaternion matrices is defined with the following set;

$$Sp_{\alpha\beta}(n) = \left\{ \begin{matrix} A \in M_n(H_{\alpha\beta}) | < Ax, Ay >= \\ < x, y >, \forall x, y \in H_{\alpha\beta}^n \end{matrix} \right\},$$

if we choose $n = 1$, then;

$$Sp_{\alpha\beta}(1) = \{q \in H_{\alpha\beta} | N(q) = 1\},$$

namely, the set becomes group of all unit length generalized quaternions. Therefore, if $Sp_{\alpha\beta}(1) = S_{\alpha\beta}^3$ then $Sp_{\alpha\beta}(1)$ is unit sphere in $E_{\alpha\beta}^4$ generalized space.

In Section 3, we defined an isomorphism between a group of generalized unit quaternions $S_{\alpha\beta}^3$ and a group of generalized special unitary matrices $SU_{\alpha\beta}(2)$. Now using the following proposition, we will establish a correspondence between a group of generalized unit quaternions $S_{\alpha\beta}^3$ and a group of generalized special orthogonal matrices $SO_{\alpha\beta}(3)$. A group of generalized orthogonal matrices $O_{\alpha\beta}(3)$ can be given as follow;

$$O_{\alpha\beta}(3) = \left\{ \begin{matrix} A \in M_3(\mathbb{R}) | A^T \varepsilon A = |A| \varepsilon, \\ \varepsilon = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix} \\ \alpha, \beta \in \mathbb{R}, \alpha \neq 0 \text{ or } \beta \neq 0 \end{matrix} \right\}.$$

If we take $\alpha > 0$ and $\beta < 0$, then the group is to be generalized semi-orthogonal group. $SO_{\alpha\beta}(3)$ is a group of generalized special orthogonal matrices, which is a subgroup of generalized orthogonal matrices group. $SO_{\alpha\beta}(3)$ can be given as follows;

$$SO_{\alpha\beta}(3) = \{A \in O_{\alpha\beta}(3) | \det A = |A| = 1\}.$$

Teorem 4.1: ρ: $S_{\alpha\beta}^3 \to SO_{\alpha\beta}(3)$, $\rho(q) = rqr^{-1}, r \in H_{\alpha\beta}$ is a surjective homomorphism and $Ker\rho = \{-1,1\}$.

Proof. The set of $H_{\alpha\beta}$ the generalized quaternions composes a group under the quaternion multiplication. $S_{\alpha\beta}^3$ and $SO_{\alpha\beta}(3)$ sets are subgroups of $H_{\alpha\beta}$. For any $p, q \in H_{\alpha\beta}$,

$$\begin{aligned} \rho(pq) = rpqr^{-1} &= rp(r^{-1}r)qr^{-1} \\ &= (rpr^{-1})(rqr^{-1}) = \rho(p)\rho(q) \end{aligned}$$

so, ρ defines a group homomorphism.

If $r \neq 0$ then $|\rho(q)| = |rqr^{-1}| = |q|$, then ρ mapping is a linear isometry,

if $r = S_r$ (i.e. $r \in \mathbb{R}$) then $\rho(q) = rqr^{-1} = q$,

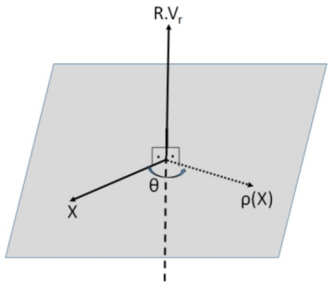
if $r = V_r$ (i.e. $r \in span\{i, j, k\}$ =Im $H_{\alpha\beta}$) then $r = S_r + V_r$ and $V_r = r - S_r$ thus,

$$\begin{aligned} \rho(V_r) &= rV_r r^{-1} = r(r - S_r)r^{-1} \\ &= rrr^{-1} - rS_r r^{-1} \\ &= r - S_r \\ &= V_r \end{aligned}$$

From the above equations, 3-dimensional space spanned by $\{i, j, k\}$ remains invariant under ρ transformation. Also, ρ is an isometries, thus the plane, which is perpendicular to V_r, remains invariant by ρ. Hence the restriction of ρ to space which is $span\{i, j, k\}$, determines rotation with angle θ around the axis $\mathbb{R}V_r$. This statement is illustrated in Figure a.

Thus, spatial rotations can be obtained from generalized quaternionic multiplication restricted to $E_{\alpha\beta}^3 = ImH_{\alpha\beta}$. These consist of the group of all linear isometries of $E_{\alpha\beta}^3$ (leaving the origin fixed), that is, they make up to the generalized orthogonal group $O_{\alpha\beta}(3)$. For simplicity, we restrict ourselves to direct linear isometries that constitute the generalized special orthogonal group $SO_{\alpha\beta}(3)$, a subgroup of $O_{\alpha\beta}(3)$. Thus, if $\pm 1 \neq r \in S_{\alpha\beta}^3$, then r defines a rotation, an element of $SO_{\alpha\beta}(3)$. On the other hand, $r = \pm 1$ defines the identity elements in $SO_{\alpha\beta}(3)$ so that ρ maps into $S_{\alpha\beta}^3$. It is clear that ρ is a homomorphism of group, and by what we just said, $\pm$ are in the kernel ρ. We know that ρ is into since all elements in $S_{\alpha\beta}(3)$ are rotations. It remains to show that the kernel of ρ is exactly $\{\pm 1\}$. Let $r \in$Kerρ, that is, $rqr^{-1} = q$ for all $q \in ImH_{\alpha\beta}$. Equivalently, r commits with all vectorial part of generalized quaternions. Writing this condition out in terms of i, j and k, we obtain that r must be real. Since it is in $S_{\alpha\beta}^3$, it must be one of ± 1.

Theorem 4.1 implies that the group $S_{\alpha\beta}^3$ of generalized unit quaternions module the normal subgroup $\{\pm 1\}$ isomorphic with the group $SO_{\alpha\beta}(3)$ of direct spatial linear isometries. The quaternions group $S_{\alpha\beta}^3/\{\pm 1\}$ is, by definition, the group of right (or left) cosets of $\{\pm 1\}$. A right coset containing $q \in S_{\alpha\beta}^3$, thus has the form $\{\pm 1\}q = \{\pm q\}$. Thus, topologically, $S_{\alpha\beta}^3/\{\pm 1\}$ can be consider as a model for the generalized projective space $R_{\alpha\beta}P^3$ as in (Ata and Yaylı 2009). By Theorem 4.1, $R_{\alpha\beta}P^3$ can be identified by the group of direct spatial isometries $SO_{\alpha\beta}(3)$. Thus, we obtain $S_{\alpha\beta}^3/\{\pm 1\} \cong R_{\alpha\beta}P^3$. These relationships can be illustrated in Fig.b.



Rotation in generalized quaternions

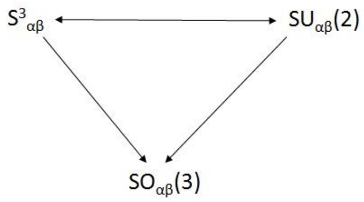


Diagram of the relations among the groups

Example 1: Let $q_1 = \frac{1}{4} + \frac{\sqrt{7}}{4}i + \frac{1}{4}j + \frac{\sqrt{3}}{4}k$ be a unit generalized quaternion and $\alpha = 1, \beta = 2$. The rotation matrix is

$$R_1 = \begin{bmatrix} 0 & \dfrac{\sqrt{7}-\sqrt{3}}{4} & \dfrac{\sqrt{21}+1}{4} \\ \dfrac{\sqrt{7}+\sqrt{3}}{8} & -\dfrac{5}{8} & \dfrac{\sqrt{7}+2\sqrt{3}}{8} \\ \dfrac{\sqrt{21}-1}{8} & \dfrac{\sqrt{7}+2\sqrt{3}}{8} & -\dfrac{1}{8} \end{bmatrix}$$

R_1 is a generalized orthogonal matrix, i.e $R_1^T \varepsilon_1 R_1 = \varepsilon_1$ and $det(R_1) = 1$ where $\varepsilon_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, q_1 unit generalized quaternion corresponds to the complex matrix

$$q_1 = \frac{1}{4} + \frac{\sqrt{7}}{4}i + \frac{1}{4}j + \frac{\sqrt{3}}{4}k$$
$$q_1 = (\frac{1}{4} + \frac{\sqrt{7}}{4}i) + (\frac{1}{4} + \frac{\sqrt{3}}{4}i)j$$
$$q_1 = z_1 + w_1 j \text{where} z_1 = \frac{1}{4} + \frac{\sqrt{7}}{4}i \text{and} w_1 = \frac{1}{4} + \frac{\sqrt{3}}{4}i$$

substituting z_1, w_1 in the matrix $H_{\mathbb{C}}^{-}(q) = \begin{pmatrix} z_1 & w_1 \\ -\beta\overline{w}_1 & \overline{z}_1 \end{pmatrix}$ the complex matrix is represented as:

$$H^{-}(q_1) = \begin{bmatrix} \dfrac{1}{4} + \dfrac{\sqrt{7}}{4}i & \dfrac{1}{4} + \dfrac{\sqrt{3}}{4}i \\ -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i & \dfrac{1}{4} - \dfrac{\sqrt{7}}{4}i \end{bmatrix}$$

$H_{\mathbb{C}}^{-}(q_1)$ is a generalized special unitary matrix, i.e. $\left(\overline{H_{\mathbb{C}}^{-}(q_1)}\right)^T \epsilon_1 H_{\mathbb{C}}^{-}(q_1) = \epsilon_1$ and $det(H_{\mathbb{C}}^{-}(q_1)) = 1$, where $\epsilon_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 2: Let $q_2 = \sqrt{2} + 0i + \frac{1}{2}j + \frac{1}{2}k$ be a unit generalized quaternion and $\alpha = 1$, $\beta = -2$. The rotation matrix is

$$R_2 = \begin{bmatrix} 3 & 2\sqrt{2} & -2\sqrt{2} \\ \sqrt{2} & 2 & -1 \\ -\sqrt{2} & -1 & 2 \end{bmatrix}$$

R_2 is a generalized orthogonal matrix, i.e $R_2^T \varepsilon_2 R_2 = \varepsilon_2$ and $det(R_2) = 1$ where $\varepsilon_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, q_2 unit generalized quaternion corresponds to the complex matrix

$$q_2 = \sqrt{2} + 0i + \frac{1}{2}j + \frac{1}{2}k$$
$$q_2 = \sqrt{2} + (\frac{1}{2} + \frac{1}{2}i)j$$
$$q_2 = z_2 + w_2 j \text{where} z_2 = \sqrt{2} \text{and} w_2 = \frac{1}{2} + \frac{1}{2}i$$

substituting z_2, w_2 in the matrix $H_{\mathbb{C}}^{-}(q) = \begin{pmatrix} z_2 & w_2 \\ -\beta\overline{w}_2 & \overline{z}_2 \end{pmatrix}$ the complex matrix is represented as:

$$H^{-}(q_2) = \begin{bmatrix} \sqrt{2} & \dfrac{1}{2} + \dfrac{1}{2}i \\ 1 - i & \sqrt{2} \end{bmatrix}$$

$H_{\mathbb{C}}^{-}(q_2)$ is a generalized special unitary matrix, i.e. $\left(\overline{H_{\mathbb{C}}^{-}(q_2)}\right)^T \epsilon_2 H_{\mathbb{C}}^{-}(q_2) = \epsilon_2$ and $det(H_{\mathbb{C}}^{-}(q_2)) = 1$, where

$$\epsilon_2 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

4 Conclusion

Generalized quaternions, which are a natural expansion of quaternions and split quaternions, have attracted the interest of researchers in recent years. Many authors considered generalized quaternions from different aspects.

In this study, different matrix representations of generalized quaternions were given and the relations between them were investigated. These relations between generalized quaternions and matrices mean that all known concepts and formulas of matrix algebra can be transferred to generalized quaternion algebra. This will provide a great convenience for scientists who will work on generalized quaternion algebra.

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