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On Stability Analysis of Riemann-Liouville Fractional Singular Systems with Delays

Erdal KORKMAZ [1]🖉, Meltem KAYA[1]
[1]Department of Mathematics, Faculty of Art And ScienceMus Alparslan University, Muş, Türkiye

🖂: e.korkmaz@alparslan.edu.tr [1]0000-0002-6647-9312 [2]0000-0003-1547-4658

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ABSTRACT

In this study, two lagged fractional order singular neutral differential equations are considered. Using the advantage of the association property of the Riemann -Liouville derivative, the derivative of the appropriate Lyapunov function is calculated. Then, with the help of LMI, sufficient conditions for asymptotic stability of zero solutions are obtained.

Keywords: Asymptotic stability, Fractional singular equations, Lyapunov method

Gecikmeli Riemann-Liouville Kesirli Singüler Sistemlerin Kararlılık Analizi

ÖZ

Bu çalışmada gecikmeli kesirli mertebeden singüler nötr iki diferansiyel denklem ele alınır. Riemann-Liouville türevin birleşme özelliğinin avanatajı kullanılarak uygun Lyapunov fonksiyonun türevi hesaplanır. Sonra LMI yardımıyla sıfır çözümlerin asimptotik kararlılığı için yeter şartlar elde edilir.

Anahtar Kelimeler: Asimptotik kararlılık, Kesirli singüler denklemler, Lyapunov metodu

INTRODUCTION

It is thought that the fractional derivative was first introduced in 1695 with the question asked by the Marquis de L'Hospital in the letter he sent to Gottfried Wilhelm Leibniz [1]. Books with a very high impact factor have been written on the fractional derivative and are the inspiration for the studies in the literature [1-4]. Especially in the last 20 years, studies on fractional derivatives continue to increase. A summary of some of these studies is given below.
In [5], Heymans and Podlubny show, through a series of examples in the field of viscoelasticity, that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives. In [6], Deng et al. consider the stability of a time-delayed n-dimensional fractional linear differential equation system by using the Laplace transform. In [7-9], the authors give sufficient conditions for the stability of certain fractional differential equation systems with the help of LMI. In [10,11] the authors investigate the stability of certain systems of fractional differential equations by using Lyapunov's second method. In [12], Aguila-Camacho et al. prove the

$$\frac{1}{2}{}_{t_0}^{C}D_t^q x^2(t) \le x(t){}_{t_0}^{C}D_t^q x(t), \quad \forall q \in (0,1)$$

inequality for the Caputo derivative. This inequality facilitates the application of Lyapunov's second method. In [13-27], the authors apply the Lyapunov method, which is generally an effective method, by considering the behavior of solutions of certain differential equations with or without fractional delay. Researchers can refer to references and their references for more information.

Preliminaries

In this section, definitions of the Riemann-Liouville fractional derivative and integral and some lemmas which will be used in the proof of the main results are given. The Riemann-Liouville fractional integral is defined as

$${}_{t_0}D_t^{-q}x(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x(s)\mathrm{d}s, \quad (q>0)$$

The Riemann-Liouville fractional derivative is defined as

$${}_{t_0}D_t^{q}x(t) = \frac{1}{\Gamma(n-q)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}\frac{x(s)}{(t-s)^{q+1-n}}\,\mathrm{d}s, \ (n-1 \le q < n).$$

Lemma 1 ([4]) If $p > q > 0$, then

$${}_{t_0}D_t^{q}\left({}_{t_0}D_t^{-p}x(t)\right) = {}_{t_0}D_t^{q-p}x(t) \tag{1}$$

holds for ‘’sufficiently good’’ functions $x(t)$. In particular; this relation holds if $x(t)$ is integrable.

Lemma 2 ([18]) Let $x(t) \in \boldsymbol{R}^{\boldsymbol{n}}$ be a vector of differentiable function. Then the following relationship holds for $\forall q \in (0,1)$

$$\frac{1}{2}{}_{t_0}D_t^{q}\left(x^T(t)Px(t)\right) \le x^T(t)P_{t_0}D_t^{q}x(t), \ \forall t \ge t_0 \tag{2}$$

where $P \in \boldsymbol{R}^{\boldsymbol{n}\times\boldsymbol{n}}$ is a constant, square, symmetric and positive semi definite matrix.

Lemma 3 ([3]) Let us first define $\Phi{:}\,\Phi(x_t) = x(t) - Cx(t-\tau)$. The operator Φ is said to be stable if the zero solution of the homogeneous difference equation $\Phi(x_t)=0,\ \ t\geq 0$ is uniformly asymptotically stable. Note that the operator Φ is stable if $\parallel C \parallel < 1$.

MAIN RESULTS

In this section, two different equation models of singular fractional order with delay arguments are discussed. The first of these equations;

$$E_{t_0}D_t^{\alpha}x(t)=Ax(t)+Bx\big(t-\tau_1(t)\big)+C_{t_0}D_t^{\alpha}x(t-\tau_2(t)) \quad (3)$$

where $0<\alpha<1$, $x(t)\in R^n$ is the state vector $E,A,B,C\in R^{n\times n}$ are constant matrices, for all $t>t_0$, $\tau_1(t),\tau_2(t)>0$ are time-varying delays.
The second equation is considered that

$$E_{t_0}D_t^{\alpha}x(t)=Ax(t)+B_1x\big(t-\tau_1(t)\big)+B_2x(t-\tau_2(t))+C_{t_0}D_t^{\alpha}x(t-\tau_3(t)) \quad (4)$$

where $0<\alpha<1$, $x(t)\in R^n$ is the state vector $E,A,B_1,B_2,C\in R^{n\times n}$ are constant matrices and $\tau_1(t),\tau_2(t),\tau_3(t)>0$ are time-varying delays for all $t>t_0$.

Theorem 4 The trivial solution of system (3) is asymptotically stable, if for all $t>t_0$, $\tau_i'(t)\leq d_i<1\ (i=1,2)$, $\tau_2(t)$ is a bounded function and there exist positive and symmetric definite matrices $P,Q,R_1,\ R_2$ such that the following LMI holds:

$$M=\begin{pmatrix}M_{11} & M_{12} & M_{13}\\ M_{12}^T & M_{22} & M_{23}\\ M_{13}^T & M_{23}^T & M_{33}\end{pmatrix}<0, \quad (5)$$

Where

$$M_{11}=PA+A^TP+Q+A^T(R_1+mR_2)A,$$
$$M_{12}=PB+A^T(R_1+mR_2)B,$$
$$M_{13}=PC+A^T(R_1+mR_2)C$$
$$M_{22}=B^T(R_1+mR_2)B-(1-d_1)Q,$$
$$M_{23}=B^T(R_1+mR_2)C,$$
$$M_{33}=C^T(R_1+mR_2)C-(1-d_2)R_1,$$

and m is a constant such that $|\tau_2(t)|\leq m$.

Proof. Let the Lyapunov-Krasovskii functional is defined by:

$$V(t)={}_{t_0}D_t^{\alpha-1}(x^T(t)P^TEx(t))+\int_{-\tau_2(t)}^{0}(E_{t_0}D_t^{\alpha}x(t+s))^TR_1(E_{t_0}D_t^{\alpha}x(t+s))ds+\int_{t-\tau_2(t)}^{t}\int_{\theta}^{t}(E_{t_0}D_s^{\alpha}x(s))^TR_2\left(E_{t_0}D_s^{\alpha}x(s)\right)dsd\theta+\int_{t-\tau_1(t)}^{t}x^T(s)Qx(s)ds. \quad (6)$$

With the help of Lemma 1, the derivative of $V(t)$ along the trajectories of (3) is obtained as follows:

$$\begin{aligned}\dot V(t)&={}_{t_0}D_t^{\alpha}\big(x^T(t)P^TEx(t)\big)+x^T(t)Qx(t)\\&-(1-\tau_1'(t))x^T(t-\tau_1(t))Qx(t-\tau_1(t))\\&+(E_{t_0}D_t^{\alpha}x(t))^TR_1(E_{t_0}D_t^{\alpha}x(t))\\&-(1-\tau_2'(t))(E_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TR_1(E_{t_0}D_t^{\alpha}x(t-\tau_2(t)))\\&+\tau_2(t)(E_{t_0}D_t^{\alpha}x(t))^TR_2(E_{t_0}D_t^{\alpha}x(t))\\&-(1-\tau_2'(t))\int_{t-\tau_2(t)}^{t}(E_{t_0}D_s^{\alpha}x(s))^TR_2(E_{t_0}D_s^{\alpha}x(s))ds\end{aligned}$$

Using Lemma 2, it is written as

$$\begin{aligned}\dot V(t)&\leq 2x^T(t)P^TE_{t_0}D_t^{\alpha}x(t)+x^T(t)Qx(t)\\&-(1-d_1)x^T(t-\tau_1(t))Qx(t-\tau_1(t))\\&+(E_{t_0}D_t^{\alpha}x(t))^TR_1(E_{t_0}D_t^{\alpha}x(t))\\&-(1-d_2)(E_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TR_1(E_{t_0}D_t^{\alpha}x(t-\tau_2(t)))\\&+m(E_{t_0}D_t^{\alpha}x(t))^TR_2\left(E_{t_0}D_t^{\alpha}x(t)\right).\end{aligned} \quad (7)$$

Note that

$$\begin{aligned}&2x^T(t)P^TE_{t_0}D_t^{\alpha}x(t)\\&=2x^T(t)P^T\big[Ax(t)+Bx(t-\tau_1(t))+C_{t_0}D_t^{\alpha}x(t-\tau_2(t))\big]\\&=x^T(t)(P^TA+A^TP)x(t)+2x^T(t)P^TBx(t-\tau_1(t))\\&+2x^T(t)P^TC_{t_0}D_t^{\alpha}x(t-\tau_2(t))\end{aligned} \quad (8)$$

and

$$\begin{aligned}&(E_{t_0}D_t^{\alpha}x(t))^TR_1(E_{t_0}D_t^{\alpha}x(t))+m(E_{t_0}D_t^{\alpha}x(t))^TR_2(E_{t_0}D_t^{\alpha}x(t))\\&=\big[Ax(t)+Bx\big(t-\tau_1(t)\big)+C_{t_0}D_t^{\alpha}x\big(t-\tau_2(t)\big)\big]^T(R_1+mR_2)\\&\quad\big[Ax(t)+Bx(t-\tau_1(t))+C_{t_0}D_t^{\alpha}x(t-\tau_2(t))\big]\\&=x^T(t)A^T(R_1+mR_2)Ax(t)+x^T(t)A^T(R_1+mR_2)Bx(t-\tau_1(t))\\&+x^T(t)A^T(R_1+mR_2)C_{t_0}D_t^{\alpha}x\big(t-\tau_2(t)\big)\\&+x^T(t-\tau_1(t))B^T(R_1+mR_2)Ax(t)\\&+x^T(t-\tau_1(t))B^T(R_1+mR_2)Bx(t-\tau_1(t))\\&+x^T(t-\tau_1(t))B^T(R_1+mR_2)C_{t_0}D_t^{\alpha}x(t-\tau_2(t))\\&+({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TC^T(R_1+mR_2)Ax(t)\\&+({}_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TC^T(R_1+mR_2)Bx(t-\tau_1(t))\\&+({}_{t_0}D_t^{\alpha}x\big(t-\tau_2(t)\big))^TC^T(R_1+mR_2)C_{t_0}D_t^{\alpha}x\big(t-\tau_2(t)\big).\end{aligned} \quad (9)$$

By substituting the equations (8) and (9) in (7), it is obtained as

$$\begin{aligned}\dot V(t)&\leq x^T(t)(P^TA+A^TP)x(t)\\&+2x^T(t)P^TBx(t-\tau_1(t))\\&+2x^T(t)P^TC_{t_0}D_t^{\alpha}x\big(t-\tau_2(t)\big)+x^T(t)Qx(t)\\&-(1-d_1)x^T(t-\tau_1(t))Qx(t-\tau_1(t))\\&-(1-d_2)(E_{t_0}D_t^{\alpha}x(t-\tau_2(t)))^TR_1(E_{t_0}D_t^{\alpha}x(t-\tau_2(t)))\\&+x^T(t)A^T(R_1+mR_2)Ax(t)+x^T(t)A^T(R_1+mR_2)Bx(t-\tau_1(t))\end{aligned}$$

$+x^T(t)A^T(R_1+mR_2)C_{t_0}D_t^\alpha x(t-\tau_2(t))$
$+x^T(t-\tau_1(t))B^T(R_1+mR_2)Ax(t)$
$+x^T(t-\tau_1(t))B^T(R_1+mR_2)Bx(t-\tau_1(t))$
$+x^T(t-\tau_1(t))B^T(R_1+mR_2)C_{t_0}D_t^\alpha x(t-\tau_2(t))$
$+({}_{t_0}D_t^\alpha x(t-\tau_2(t)))^T C^T(R_1+mR_2)Ax(t)$
$+({}_{t_0}D_t^\alpha x(t-\tau_2(t)))^T C^T(R_1+mR_2)Bx(t-\tau_1(t))$
$+({}_{t_0}D_t^\alpha x(t-\tau_2(t)))^T C^T(R_1+mR_2)C_{t_0}D_t^\alpha x(t-\tau_2(t))$.

Thus it is written as

$$\dot V(t)\le \xi^T M\xi \qquad (10)$$

where

$$\xi=(x^T(t),x^T(t-\tau_1(t)),({}_{t_0}D_t^\alpha x(t-\tau_2(t)))^T)^T.$$

From (5) it is said that $\dot V(t)$ is negative definite, which means that the trival solution of system (3) is asymptotically stable.

Theorem 5 The trivial solution of system (3) is asymptotically stable, if for all $t>t_0$, $\tau_i'(t)\le d_i<1$ $(i=1,2)$, $\tau_2(t)$ bounded function and there exists positive and symmetric definite matrices P,Q_1,Q_2, R such that the following LMI satisfies:

$$N=\begin{pmatrix}N_{11} & N_{12} & N_{13}\\ N_{12}^T & N_{22} & N_{23}\\ N_{13}^T & N_{23}^T & N_{33}\end{pmatrix}<0, \qquad (11)$$

where

$$N_{11}=E^TPA+A^TPE+Q_1+Q_2+mA^TRA,$$
$$N_{12}=E^TPB+mA^TRB,$$
$$N_{13}=-A^TPC,$$
$$N_{22}=mB^TRB-(1-d_1)Q_1,$$
$$N_{23}=-B^TPC,$$
$$N_{33}=-(1-d_2)Q_2,$$

and m is a constant such that $|\tau_2(t)|\le m$.

Proof. Let the Lyapunov-Krasovskii functional is defined by:

$$V(t)={}_{t_0}D_t^{\alpha-1}((Ex(t)-Cx(t-\tau_2(t)))^TP^T(Ex(t)-Cx(t-\tau_2(t)))) + \int_{t-\tau_1(t)}^t x^T(s)Q_1x(s)ds+\int_{t-\tau_2(t)}^t x^T(s)Q_2x(s)ds + \int_{t-\tau_2(t)}^t\int_\theta^t ({}_{t_0}D_s^\alpha(Ex(s)-Cx(s-\tau_2(s))))^TR({}_{t_0}D_s^\alpha(Ex(s)-Cx(s-\tau_2(s))))dsd\theta. \qquad (12)$$

With the help of Lemma 1, the derivative of V (t) along the trajectories of (3) is obtained as follows:

$\dot V(t)={}_{t_0}D_t^\alpha((Ex(t)-Cx(t-\tau_2(t)))^TP(Ex(t)-Cx(t-\tau_2(t))))$
$+x^T(t)Q_1x(t)-(1-\tau_1'(t))x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))$
$+x^T(t)Q_2x(t)-(1-\tau_2'(t))x^T(t-\tau_2(t))Q_2x(t-\tau_2(t))$
$+\tau_2(t)({}_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t))))^TR({}_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t))))$
$-(1-\tau_2'(t))\int_{t-\tau_2(t)}^t({}_{t_0}D_s^\alpha(Ex(s)-Cx(s-\tau_2(s))))^TR({}_{t_0}D_s^\alpha(Ex(s)-Cx(s-\tau_2(s))))ds$

Using Lemma 2 it is written as

$\dot V(t)\le 2(Ex(t)-Cx(t-\tau_2(t)))^TP_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t)))$
$+x^T(t)Q_1x(t)-(1-d_1)x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))$
$+x^T(t)Q_2x(t)-(1-d_2)x^T(t-\tau_2(t))Q_2x(t-\tau_2(t))$
$+m({}_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t))))^TR({}_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t))))$. (13)

Note that

$2(Ex(t)-Cx(t-\tau_2(t)))^TP_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t)))$
$=2(Ex(t)-Cx(t-\tau_2(t)))^TP(Ax(t)+Bx(t-\tau_1(t)))$
$=x^T(t)(E^TPA+A^TPE)x(t)-2x^T(t-\tau_2(t))C^TPAx(t)$
$+2x^T(t)E^TPBx(t-\tau_1(t))-2x^T(t-\tau_2(t))C^TPBx(t-\tau_1(t))$ (14)

and

$m({}_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t))))^TR({}_{t_0}D_t^\alpha(Ex(t)-Cx(t-\tau_2(t))))$
$=m[Ax(t)+Bx(t-\tau_1(t))]^TR[Ax(t)+Bx(t-\tau_1(t))]$
$=mx^T(t)A^TRAx(t)+mx^T(t)A^TRBx(t-\tau_1(t))+mx^T(t-\tau_1(t))B^TRAx(t)$
$+mx^T(t-\tau_1(t))B^TRBx(t-\tau_1(t))$ (15)

By substituting the equations (14) and (9) in (13), it is obtained as

$\dot V(t)\le x^T(t)(E^TPA+A^TPE)x(t)$
$-2x^T(t-\tau_2(t))C^TPAx(t)$
$+2x^T(t)E^TPBx(t-\tau_1(t))$
$-2x^T(t-\tau_2(t))C^TPBx(t-\tau_1(t))$
$+x^T(t)Q_1x(t)-(1-d_1)x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))$
$+x^T(t)Q_2x(t)-(1-d_2)x^T(t-\tau_2(t))Q_2x(t-\tau_2(t))$
$+mx^T(t)A^TRAx(t)+mx^T(t)A^TRBx(t-\tau_1(t))+mx^T(t-\tau_1(t))B^TRAx(t)$
$+mx^T(t-\tau_1(t))B^TRBx(t-\tau_1(t))$.

Thus it is written as

$$\dot V(t)\le\xi^TM\xi \qquad (16)$$

where

$$\xi = (x^T(t), x^T(t-\tau_1(t)), x^T(t-\tau_2(t)))^T.$$

From (11) it is said that $\dot{V}(t)$ is negative definite, which means that the trival solution of system (3) is asymptotically stable.

Theorem 6 The trivial solution of system (4) is asymptotically stable, if for all $t > t_0$, $\tau_i'(t) \le d_i < 1$ $(i = 1,2,3)$, $\tau_3(t)$ bounded function and there exists positive and symmetric definite matrices P, Q, R_1, R_2 such that the following LMI satisfies:

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12}^T & M_{22} & M_{23} & M_{24} \\ M_{13}^T & M_{23}^T & M_{33} & M_{34} \\ M_{14}^T & M_{24}^T & M_{34}^T & M_{44} \end{pmatrix} < 0, \qquad (17)$$

where

$$\begin{aligned} M_{11} &= PA + A^TP + 2Q + A^T(R_1 + mR_2)A, \\ M_{12} &= PB_1 + A^T(R_1 + mR_2)B_1, \\ M_{13} &= PB_2 + A^T(R_1 + mR_2)B_2, \\ M_{14} &= PC + A^T(R_1 + mR_2)C, \\ M_{22} &= B_1^T(R_1 + mR_2)B_1 - (1 - d_1)Q, \\ M_{23} &= B_1^T(R_1 + mR_2)B_2 \\ M_{24} &= B_1^T(R_1 + mR_2)C, \\ M_{33} &= B_2^T(R_1 + mR_2)B_2 - (1 - d_2)Q, \\ M_{34} &= B_2^T(R_1 + mR_2)C, \\ M_{44} &= C^T(R_1 + mR_2)C - (1 - d_3)R_1, \end{aligned}$$

and m is a constant such that $|\tau_3(t)| \le m$.

Proof. Let the Lyapunov-Krasovskii functional is defined by:

$$\begin{aligned} V(t) = {}_{t_0}D_t^{\alpha-1}\left(x^T(t)P^TEx(t)\right) + \\ \int_{t-\tau_1(t)}^{t} x^T(s)Qx(s)ds + \int_{t-\tau_2(t)}^{t} x^T(s)Qx(s)ds \\ + \int_{-\tau_3(t)}^{0} (E_{t_0}D_t^{\alpha}x(t+s))^TR_1(E_{t_0}D_t^{\alpha}x(t+s))ds \\ + \int_{t-\tau_3(t)}^{t}\int_{\theta}^{t} (E_{t_0}D_s^{\alpha}x(s))^TR_2\left(E_{t_0}D_s^{\alpha}x(s)\right)dsd\theta. \quad (18) \end{aligned}$$

With the help of Lemma 1, the derivative of V (t) along the trajectories of (4) is obtained as follows:

$$\begin{aligned} \dot{V}(t) = {}_{t_0}D_t^{\alpha}\left(x^T(t)P^TEx(t)\right) + x^T(t)Qx(t) \\ -(1-\tau_1'(t))x^T(t-\tau_1(t))Qx(t-\tau_1(t)) \\ +x^T(t)Qx(t) - (1-\tau_2'(t))x^T(t-\tau_2(t))Qx(t \\ -\tau_2(t)) \\ +(E_{t_0}D_t^{\alpha}x(t))^TR_1(E_{t_0}D_t^{\alpha}x(t)) \\ -(1-\tau_3'(t))(E_{t_0}D_t^{\alpha}x(t-\tau_3(t)))^TR_1(E_{t_0}D_t^{\alpha}x(t \\ -\tau_3(t))) \\ +\tau_3(t)(E_{t_0}D_t^{\alpha}x(t))^TR_2(E_{t_0}D_t^{\alpha}x(t)) \\ -(1-\tau_3'(t))\int_{t-\tau_3(t)}^{t}(E_{t_0}D_s^{\alpha}x(s))^TR_2\left(E_{t_0}D_s^{\alpha}x(s)\right)ds. \end{aligned}$$

Using Lemma 2 it is written as

$$\begin{aligned} \dot{V}(t) \le 2x^T(t)P^TE_{t_0}D_t^{\alpha}x(t) + 2x^T(t)Qx(t) \\ -(1-d_1)x^T(t-\tau_1(t))Qx(t-\tau_1(t)) \\ -(1-d_2)x^T(t-\tau_2(t))Qx(t-\tau_2(t)) \\ +(E_{t_0}D_t^{\alpha}x(t))^TR_1(E_{t_0}D_t^{\alpha}x(t)) \\ -(1-d_3)(E_{t_0}D_t^{\alpha}x(t-\tau_3(t)))^TR_1(E_{t_0}D_t^{\alpha}x(t \\ -\tau_3(t))) \\ +m(E_{t_0}D_t^{\alpha}x(t))^TR_2(E_{t_0}D_t^{\alpha}x(t)) \quad (19) \end{aligned}$$

Note that

$$\begin{aligned} 2x^T(t)P^TE_{t_0}D_t^{\alpha}x(t) \\ = 2x^T(t)P^T\big[Ax(t) + B_1x(t-\tau_1(t)) + B_2x(t-\tau_2(t)) \\ + C_{t_0}D_t^{\alpha}x(t-\tau_3(t))\big] \\ = x^T(t)(P^TA + A^TP)x(t) + 2x^T(t)P^TB_1x(t-\tau_1(t)) \\ +2x^T(t)P^TB_2x(t-\tau_2(t)) + \\ 2x^T(t)P^TC_{t_0}D_t^{\alpha}x(t-\tau_3(t)) \quad (20) \end{aligned}$$

and

$$\begin{aligned} (E_{t_0}D_t^{\alpha}x(t))^TR_1(E_{t_0}D_t^{\alpha}x(t)) \\ + m(E_{t_0}D_t^{\alpha}x(t))^TR_2(E_{t_0}D_t^{\alpha}x(t)) \\ = \big[Ax(t) + B_1x(t-\tau_1(t)) + B_2x(t-\tau_2(t)) \\ + C_{t_0}D_t^{\alpha}x(t-\tau_3(t))\big]^T(R_1 + mR_2) \\ \big[Ax(t) + B_1x(t-\tau_1(t)) + B_2x(t-\tau_2(t)) \\ + C_{t_0}D_t^{\alpha}x(t-\tau_3(t))\big] \\ = x^T(t)A^T(R_1 + mR_2)Ax(t) + x^T(t)A^T(R_1 \\ + mR_2)B_1x(t-\tau_1(t)) \\ +x^T(t)A^T(R_1 + mR_2)B_2x(t-\tau_2(t)) + x^T(t)A^T(R_1 \\ + mR_2)C_{t_0}D_t^{\alpha}x(t-\tau_3(t)) \\ +x^T(t-\tau_1(t))B_1^T(R_1 + mR_2)Ax(t) + x^T(t \\ -\tau_1(t))B_1^T(R_1 + mR_2)B_1x(t \\ -\tau_1(t)) \\ +x^T(t-\tau_1(t))B_1^T(R_1 + mR_2)B_2x(t-\tau_2(t))) \\ +x^T(t-\tau_1(t))B_1^T(R_1 + mR_2)C_{t_0}D_t^{\alpha}x(t-\tau_3(t)) \\ +x^T(t-\tau_2(t))B_2^T(R_1 + mR_2)Ax(t) + x^T(t \\ -\tau_2(t))B_2^T(R_1 + mR_2)B_1x(t \\ -\tau_1(t)) \\ +x^T(t-\tau_2(t))B_2^T(R_1 + mR_2)B_2x(t-\tau_2(t)) \\ +x^T(t-\tau_2(t))B_2^T(R_1 + mR_2)C_{t_0}D_t^{\alpha}x(t-\tau_3(t)) \\ +({}_{t_0}D_t^{\alpha}x(t-\tau_3(t)))^TC^T(R_1 + mR_2)Ax(t) \\ +({}_{t_0}D_t^{\alpha}x(t-\tau_3(t)))^TC^T(R_1 + mR_2)B_1x(t-\tau_1(t)) \\ +({}_{t_0}D_t^{\alpha}x(t-\tau_3(t)))^TC^T(R_1 + mR_2)B_2x(t-\tau_2(t)) \\ +({}_{t_0}D_t^{\alpha}x(t-\tau_3(t)))^TC^T(R_1 + mR_2)C_{t_0}D_t^{\alpha}x(t- \\ \tau_3(t)). \quad (21) \end{aligned}$$

By substituting the equations (20) and (21) in (19), it is obtained as

$$\begin{aligned} \dot{V}(t) \le x^T(t)(P^TA + A^TP)x(t) + 2x^T(t)P^TB_1x(t \\ -\tau_1(t)) + 2x^T(t)P^TB_2x(t-\tau_2(t)) \end{aligned}$$

$+2x^T(t)P^T C_{t_0}D_t^\alpha x(t-\tau_3(t)) + 2x^T(t)Qx(t) - (1 - d_1)x^T(t-\tau_1(t))Qx(t-\tau_1(t))$
$-(1-d_2)x^T(t-\tau_2(t))Qx(t-\tau_2(t))$
$-(1-d_3)(E_{t_0}D_t^\alpha x(t-\tau_3(t)))^T R_1(E_{t_0}D_t^\alpha x(t - \tau_3(t)))$
$+x^T(t)A^T(R_1+mR_2)Ax(t) + x^T(t)A^T(R_1 + mR_2)B_1x(t-\tau_1(t))$
$+x^T(t)A^T(R_1+mR_2)B_2x(t-\tau_2(t)) + x^T(t)A^T(R_1 + mR_2)C_{t_0}D_t^\alpha x(t-\tau_3(t))$
$+x^T(t-\tau_1(t))B_1^T(R_1+mR_2)Ax(t) + x^T(t - \tau_1(t))B_1^T(R_1+mR_2)B_1x(t - \tau_1(t))$
$+x^T(t-\tau_1(t))B_1^T(R_1+mR_2)B_2x(t-\tau_2(t)))$
$+x^T(t-\tau_1(t))B_1^T(R_1+mR_2)C_{t_0}D_t^\alpha x(t-\tau_3(t))$
$+x^T(t-\tau_2(t))B_2^T(R_1+mR_2)Ax(t) + x^T(t - \tau_2(t))B_2^T(R_1+mR_2)B_1x(t - \tau_1(t))$
$+x^T(t-\tau_2(t))B_2^T(R_1+mR_2)B_2x(t-\tau_2(t))$
$+x^T(t-\tau_2(t))B_2^T(R_1+mR_2)C_{t_0}D_t^\alpha x(t-\tau_3(t))$
$+({}_{t_0}D_t^\alpha x(t-\tau_3(t)))^T C^T(R_1+mR_2)Ax(t)$
$+({}_{t_0}D_t^\alpha x(t-\tau_3(t)))^T C^T(R_1+mR_2)B_1x(t-\tau_1(t))$
$+({}_{t_0}D_t^\alpha x(t-\tau_3(t)))^T C^T(R_1+mR_2)B_2x(t-\tau_2(t))$
$+({}_{t_0}D_t^\alpha x(t-\tau_3(t)))^T C^T(R_1+mR_2)C_{t_0}D_t^\alpha x(t - \tau_3(t)).$

Thus it is written as

$$\dot{V}(t) \le \xi^T M\xi \tag{22}$$

where

$$\xi = (x^T(t), x^T(t-\tau_1(t)), x^T(t-\tau_2(t)), ({}_{t_0}D_t^\alpha x(t - \tau_3(t)))^T)^T.$$

From (17) it is said that $\dot{V}(t)$ is negative definite, which means that the trival solution of system (4) is asymptotically stable.

Theorem 7 The trivial solution of system (23) is asymptotically stable, if for all $t > t_0$, $\tau_i'(t) \le d_i < 1$ $(i = 1,2)$, $\tau_3(t)$ bounded function and there exists positive and symmetric definite matrices P, Q_1, Q_2, R such that the following LMI satisfies:

$$N = \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{12}^T & N_{22} & N_{23} \\ N_{13}^T & N_{23}^T & N_{33} \end{pmatrix} < 0 \tag{23}$$

where

$$N_{11} = E^TPA + A^TPE + Q_1 + Q_2 + mA^TRA,$$
$$N_{12} = E^TPB + mA^TRB,$$
$$N_{13} = -A^TPC,$$
$$N_{22} = mB^TRB - (1-d_1)Q_1,$$
$$N_{23} = -B^TPC,$$
$$N_{33} = -(1-d_2)Q_2,$$

and m is a constant such that $|\tau_3(t)| \le m$.

Proof. Let the Lyapunov-Krasovskii functional is defined by:

$$V(t) = {}_{t_0}D_t^{\alpha-1}((Ex(t) - Cx(t-\tau_3(t)))^T P^T(Ex(t) - Cx(t-\tau_3(t))))$$
$$+ \int_{t-\tau_1(t)}^t x^T(s)Q_1x(s)ds + \int_{t-\tau_2(t)}^t x^T(s)Q_2x(s)ds + \int_{t-\tau_3(t)}^t x^T(s)Q_3x(s)ds$$
$$+ \int_{t-\tau_3(t)}^t \int_\theta^t ({}_{t_0}D_s^\alpha(Ex(s) - Cx(s - \tau_3(s))))^T R({}_{t_0}D_s^\alpha(Ex(s) - Cx(s-\tau_3(s))))dsd\theta.$$

With the help of Lemma 1, the derivative of V (t) along the trajectories of (4) is obtained as follows:

$$\dot{V}(t) = {}_{t_0}D_t^\alpha((Ex(t) - Cx(t-\tau_3(t)))^T P(Ex(t) - Cx(t-\tau_3(t))))$$
$$+x^T(t)Q_1x(t) - (1-\tau_1'(t))x^T(t-\tau_1(t))Q_1x(t - \tau_1(t))$$
$$+x^T(t)Q_2x(t) - (1-\tau_2'(t))x^T(t-\tau_2(t))Q_2x(t - \tau_2(t))$$
$$+x^T(t)Q_3x(t) - (1-\tau_3'(t))x^T(t-\tau_3(t))Q_2x(t - \tau_3(t))$$
$$+\tau_3(t)({}_{t_0}D_t^\alpha(Ex(t) - Cx(t-\tau_3(t))))^T R({}_{t_0}D_t^\alpha(Ex(t) - Cx(t-\tau_3(t))))$$
$$-(1-\tau_3'(t))\int_{t-\tau_3(t)}^t ({}_{t_0}D_s^\alpha(Ex(s) - Cx(s-\tau_3(s))))^T R({}_{t_0}D_s^\alpha(Ex(s) - Cx(s-\tau_3(s))))ds$$

Using Lemma 2 it is written as

$$\dot{V}(t) \le 2(Ex(t) - Cx(t-\tau_3(t)))^T P_{t_0}D_t^\alpha(Ex(t) - Cx(t-\tau_3(t)))$$
$$+x^T(t)Q_1x(t) - (1-d_1)x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))$$
$$+x^T(t)Q_2x(t) - (1-d_2)x^T(t-\tau_2(t))Q_2x(t - \tau_2(t))$$
$$+x^T(t)Q_3x(t) - (1-d_3)x^T(t-\tau_3(t))Q_3x(t - \tau_3(t))$$
$$+m({}_{t_0}D_t^\alpha(Ex(t) - Cx(t-\tau_3(t))))^T R({}_{t_0}D_t^\alpha(Ex(t) - Cx(t-\tau_3(t)))). \tag{24}$$

Note that

$$2(Ex(t) - Cx(t-\tau_3(t)))^T P_{t_0}D_t^\alpha(Ex(t) - Cx(t - \tau_3(t)))$$
$$= 2(Ex(t) - Cx(t-\tau_3(t)))^T P(Ax(t) + B_1x(t - \tau_1(t)) + B_2x(t-\tau_2(t)))$$
$$= x^T(t)(E^TPA + A^TPE)x(t) - 2x^T(t - \tau_3(t))C^TPAx(t)$$
$$+2x^T(t)E^TPB_1x(t-\tau_1(t)) + 2x^T(t)E^TPB_2x(t - \tau_2(t))$$
$$-2x^T(t-\tau_3(t))C^TPB_1x(t-\tau_1(t)) - 2x^T(t - \tau_3(t))C^TPB_2x(t-\tau_2(t)) \tag{25}$$

and

$$m({}_{t_0}D_t^{\alpha}(Ex(t) - Cx(t-\tau_3(t))))^T R({}_{t_0}D_t^{\alpha}(Ex(t) - Cx(t-\tau_3(t))))$$
$$= m[Ax(t) + B_1x(t-\tau_1(t)) + B_2x(t-\tau_2(t))]^T R[Ax(t) + B_1x(t-\tau_1(t)) + B_2x(t-\tau_2(t))]$$
$$= mx^T(t)A^TRAx(t) + mx^T(t)A^TRB_1x(t-\tau_1(t)) + mx^T(t)A^TRB_2x(t-\tau_2(t))$$
$$+mx^T(t-\tau_1(t))B_1^TRAx(t) + mx^T(t-\tau_1(t))B_1^TRB_1x(t-\tau_1(t))$$
$$+mx^T(t-\tau_1(t))B_1^TRB_2x(t-\tau_2(t)) + mx^T(t-\tau_2(t))B_2^TRAx(t)$$
$$+mx^T(t-\tau_2(t))B_2^TRB_1x(t-\tau_1(t)) + mx^T(t-\tau_2(t))B_2^TRB_2x(t-\tau_2(t)). \quad (26)$$

By substituting the equations (25) and (26) in (24), it is obtained as

$$\dot{V}(t) \le x^T(t)(E^TPA + A^TPE)x(t)$$
$$-2x^T(t-\tau_3(t))C^TPAx(t)$$
$$+2x^T(t)E^TPB_1x\big(t-\tau_1(t)\big)$$
$$+2x^T(t)E^TPB_2x(t-\tau_2(t))$$
$$-2x^T(t-\tau_3(t))C^TPB_1x(t-\tau_1(t)) - 2x^T(t-\tau_3(t))C^TPB_2x(t-\tau_2(t))$$
$$+x^T(t)Q_1x(t) - (1-d_1)x^T(t-\tau_1(t))Q_1x(t-\tau_1(t))$$
$$+x^T(t)Q_2x(t) - (1-d_2)x^T(t-\tau_2(t))Q_2x(t-\tau_2(t))$$
$$+x^T(t)Q_3x(t) - (1-d_3)x^T(t-\tau_3(t))Q_3x(t-\tau_3(t))$$
$$+mx^T(t)A^TRAx(t) + mx^T(t)A^TRB_1x(t-\tau_1(t)) + mx^T(t)A^TRB_2x(t-\tau_2(t))$$
$$+mx^T(t-\tau_1(t))B_1^TRAx(t) + mx^T(t-\tau_1(t))B_1^TRB_1x(t-\tau_1(t))$$
$$+mx^T(t-\tau_1(t))B_1^TRB_2x(t-\tau_2(t)) + mx^T(t-\tau_2(t))B_2^TRAx(t)$$
$$+mx^T(t-\tau_2(t))B_2^TRB_1x(t-\tau_1(t)) + mx^T(t-\tau_2(t))B_2^TRB_2x(t-\tau_2(t))$$

Thus, it is written as

$$\dot{V}(t) \le \xi^T M \xi \quad (27)$$

Where

$$\xi = (x^T(t), x^T(t-\tau_1(t)), x^T(t-\tau_2(t)), x^T(t-\tau_3(t)))^T.$$

From (23) it is said that $\dot{V}(t)$ is negative definite, which means that the trival solution of system (4) is asymptotically stable.

REFERENCES

[1] Podlubny I. Fractional Differential Equations, Academic Press, New York, 1999.

[2] Hale J.K. Ordinary Differantial Equations, Wiley Interscience, New York, 1969.

[3] Hale J.K. Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[4] Kilbas A., Srivastava H., Trujillo J. Theory and Application of Fractional Differential Equations, Elsevier, New York, 2006.

[5] Heymans N., Podlubny I. Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheologica Acta, 45 765–771, 2006.

[6] Deng W.H., Li C.P., Lü J.H. Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dynamics, 48 409–416, 2007.

[7] Lu J.G., Chen G.R. Robust stability and stabilization of fractional-order interval systems: An LMI approach, IEEE Transactions on Automatic Control, 54 1294–1299, 2009.

[8] Qian D., Li, C., Agarwal R.P., Wong P.J.Y. Stability analysis of fractional differential system with Riemann-Liouville derivative, Mathematical and Computer Modelling, 52 862–874, 2010.

[9] Qian W., Li T., Cong S., Fei S.M. Improved stability analysis on delayed neural networks with linear fractional uncertainties, Applied Mathematics and Computation, 217 3596–3606, 2010 .

[10] Li Y., Chen Y.Q., Podlubny I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability, Computers & Mathematics with Applications, 59 1810–1821, 2010.

[11] Liu S., Li X., Jiang W., Zhou X.F. Mittag-Leffler stability of nonlinear fractional neutral singular systems, Communications in Nonlinear Science and Numerical Simulation, 17 3961–3966, 2012.

[12] Aguila-Camacho N., Duarte-Mermoud M., Gallegos J. Lyapunov functions for fractional order systems, Communications in Nonlinear Science and Numerical Simulation, 19 2951–2957, 2014.

[13] Li H., Zhong S., Li H. Asymptotic stability analysis of fractional-order neutral systems with time delay, Advances in Difference Equations, 2015 325–335, 2015.

[14] Chen L. P., He Y.G., Chai Y., Wu R.C. New results on stability and stabilization of a class of nonlinear fractional-order systems, Nonlinear Dynamics, 75 633–641, 2014.

[15] Liu S., Li X., Zhou X.F., Jiang W. Synchronization analysis of singular dynamical networks with unbounded time-delays, Advances in Difference Equations, 193 1–9, 2015.

[16] Duarte-Mermoud M., Aguila-Camacho N., Gallegos J., Castro-Linares R. Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems, Communications in Nonlinear Science and Numerical Simulation, 22 650–659, 2015.

[17] Liu S., Jiang W., Li X., Zhou X.F. Lyapunov stability analysis of fractional nonlinear systems, Applied Mathematics Letters, 51 13–19, 2016.

[18] Liu S., Wu X., Zhou X.F., Jiang W. Asymptotical stability of Riemann-Liouville fractional nonlinear systems, Nonlinear Dynamics, 86 65–71, 2016.

[19] Liu S., Zhou X.F., Li X., Jiang W. Stability of fractional nonlinear singular systems and its applications in synchronization of complex dynamical networks, Nonlinear Dynamics, 84 2377–2385, 2016.

[20] Chen L.P., Liu C., Wu R.C., He Y.G., Chai Y. Finite-time stability criteria for a class of fractional-order neural networks with delay, Neural Computing and Applications, 27 549–556, 2016.

[21] Liu S., Wu X., Zhang Y.J., Yang R. Asymptotical stability of Riemann–Liouville fractional neutral systems, Applied Mathematics Letters, 69 168–173, 2017.

[22] Liu S., Zhou X.F., Li X., Jiang W. Asymptotical stability of Riemann-Liouville fractional singular systems with multiple time-varying delays, Applied Mathematics Letters, 65 32–39, 2017.

[23] Priyadharsini S., Govindaraj V. Asymptotic stability of caputo fractional singular differential systems with multiple delays. Discontinuity, Nonlinearity, and complexity, 7 243-251, 2018.

[24] Korkmaz E., Özdemir A. On Stability of Fractional Differential Equations with Lyapunov Functions, Muş Alparslan Üniversitesi Fen Bilimleri Dergisi, 7, 635–638, 2019.

[25] Korkmaz E., Ozdemir A., Yildirim K. Asymptotical Stability of Riemann-Liouville Nonlinear Fractional Neutral Neural Networks with Time-Varying Delays, Journal of Mathematics, 2022 2022.

[26] Aydı, A., Korkmaz E. Introduce Gâteaux and Frêchet Derivatives in Riesz Spaces, Applications and Applied Mathematics: An International Journal (AAM), 15 16 2020.

[27] Altun Y., Tunç C. On the asymptotic stability of a nonlinear fractional-order system with multiple variable delays, Applications and Applied Mathematics, 15 458–468, 2020.