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The Macwilliams Identity for Lipschitz Weight Enumerators

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ABSTRACT

In this paper, the MacWilliams identity is stated for codes over quaternion integers with respect to the Lipschitz metric.

Keywords: *MacWilliams identity, Block codes, Weight enumerator, Lipschitz metric*

1. INTRODUCTION

The MacWilliams identity is one of the most important theorems in coding theory. It is well known that two of the most famous results in block code theory are the MacWilliams Identity Theorem and Equivalence Theorem [1,2]. Given the weight enumerator of an $[n,k,d]$ code, the MacWilliams identity allows one to obtain the weight enumerator of the dual $[n,n-k,d^{\perp}]$ code. The MacWilliams identity is very useful since weight distribution of high rate codes can be obtained from low rate codes. A well known version of the MacWilliams identity for codes with respect to the Hamming weight was presented in [3]. The more general version of this theorem are less often used in practical applications. The impact of this identity for practical as well as theoretical purposes is well known, see for instance [3].

Recently, the MacWilliams identity have been proven for different weights. For example, MacWilliams identity for m-spotty Lee weight enumerators was given in [4]. Complete weight enumerators and MacWilliams identities for linear codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ were obtained in [5].

In this study, we obtain MacWilliam identity for Lipschitz weight enumerators to obtain the MacWilliams identity for codes over integer quaternions (IQ) with respect to the Lipschitz distance.

Our approach is similar to the work of Huber in [9].

In what follows, we consider the following:

Definition 1. [12] Let $\mathbb{R}$ be the field of real numbers. The Hamilton Quaternion Algebra over $\mathbb{R}$ denoted by

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$H[\mathbb{R}]$ is the associative unital algebra given by the following representation:

$H[\mathbb{R}]$ is the free $\mathbb{R}$ module over the symbols $1, i, j, k,$ that is, $H[\mathbb{R}] = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$;

1 is the multiplicative identity;

$i^2 = j^2 = k^2 = -1$;

$ij = -ji = k, ki = -ik = j, jk = -kj = i$.

If $q = a_0 + a_1 i + a_2 j + a_3 k$ is a quaternion, its conjugate quaternion is $\overline{q} = a_0 - (a_1 i + a_2 j + a_3 k)$. The norm of q is $N(q) = q.\overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2$, which is multiplicative, that is, $N(q_1 q_2) = N(q_1)N(q_2)$. It should be noted that quaternions are not commutative. The ring of integer quaternions is $H[\mathbb{Z}] = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$. More information which is related with the arithmetic properties of $H[\mathbb{Z}]$ can be found in [10, pp. 57-71].

Definition 2. [11] Let $\pi \neq 0$ be an integer quaternion. Then, $q_1, q_2 \in H[\mathbb{Z}]$ are right congruent modulo π if there exist $\beta \in H[\mathbb{Z}]$ such that $q_1 - q_2 = \beta\pi$. It is denoted by

$$q_1 \equiv_r q_2 \ (mod\ \pi).$$

Theorem 1. [11] Let $\pi \in H[\mathbb{Z}]$. Then $H[\mathbb{Z}]_\pi$ has $N(\pi)^2$ elements.

Definition 3. [10] Let $\pi \neq 0$ be an integer quaternion. Given $\beta_1, \beta_2 \in H[\mathbb{Z}]_\pi$, the Lipschitz distance between β_1 and β_2 is computed as $|a_0| + |a_1| + |a_2| + |a_3|$ and denoted by $d_{Lip}(\beta_1, \beta_2)$,

where $\beta_1 - \beta_2 \equiv_r a_0 + a_1 i + a_2 j + a_3 k \ (mod\ \pi)$ with $|a_0| + |a_1| + |a_2| + |a_3|$ minimum.

The Lipschitz weight of $\beta_1 - \beta_2$ is defined as $w_{Lip}(\beta_1 - \beta_2) = d_{Lip}(\beta_1, \beta_2)$. More general information about Lipschiz distance, Lipschiz weight and Lipschitz integers can be found in [13-15].

Definition 4. A linear code C of length n over $H[\mathbb{Z}]_\pi$ is a submodule of $H[\mathbb{Z}]_\pi^n$.

2. THE MACWILLIAMS IDENTITY FOR CODES OVER INTEGER QUATERNIONS

Now we recall some notations and definitions on characters and weight enumerators needed in this paper.

Definition 5. [3] Let α be a primitive element of $\mathbb{F}_q$ where $q = p^m$ and p is a prime. An element β of $\mathbb{F}_q$ can be written uniquely in the form $\beta = \beta_0 + \beta_1\alpha + \beta_2\alpha^2 + \cdots + \beta_{m-1}\alpha^{m-1}$ where $\beta_i \in \mathbb{F}_p$ or equivalently as an $m-$tuple $\beta = (\beta, \beta_1, \cdots, \beta_{m-1}) \in \mathbb{F}_p^m$. Now for each fixed $\beta = (\beta_0, \beta_1, \cdots, \beta_{m-1})$, χ_β is a complex-valued mapping on $\mathbb{F}_q$ defined as

$$\chi_\beta(\gamma) = \xi^{\beta_0\gamma_0 + \beta_1\gamma_2 + \cdots + \beta_{m-1}\gamma_{m-1}}$$

all $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{m-1}) \in \mathbb{F}_q$. χ_β is called a character of $\mathbb{F}_q$.

Let γ be an element of the field $\mathbb{F}_{p^m}$. Using the primitive element γ, γ_1 can be represented as $\gamma = \sum_{t=0}^{m-1} g_t \gamma_1^t$ with g_t from $\mathbb{F}_p$. The character $\chi_i(\gamma)$ is defined using the primitive complex $p-th$ root ξ:

$$\chi_i(\gamma) = \xi^{g_i},$$

where $\xi = \exp\left(2\pi\sqrt{-1}/p\right), \pi = 3,14...$

The complete weight enumerator classifies the codewords of a linear code according to the number of times each field element ω_t appears in the codeword.

Definition 6. [3] The composition of $u = (u_0, \ u_1, \ \cdots \ , u_{n-1})$, denoted by $comp(u)$, is $s = (s_0, \ s_1, \ \cdots \ s_{q-1})$, where $s_t = s_t(u)$ is the number of components

u_t equals to ω_t. Thus it is obtained

$$\sum_{t=0}^{q-1} s_t(u) = n.$$

Let C be a linear $[n,k]$ code over $\mathbb{F}_q$. Then the complete weight enumerator of C is

$$W_C(z_0, z_1, \cdots, z_{q-1}) = \sum_{u \in C}\left(\prod_{t=0}^{q-1} z_t^{s_t(u)}\right),$$

where z_t are indeterminates and the sum extends over all compositions.

The MacWilliams theorem for complete weight enumerators ([3, pp.143-144, Thm 10]) then states:

Theorem 2. [3] The complete weight enumerator of the dual code $C^{\perp}$ can be obtained from the complete weight enumerator of the code C by replacing each z_t by

$$\sum_{s=0}^{q-1} \chi_1(\omega_t \omega_s) z_s$$

and dividing the result by the cardinality of C which is denoted by $|C|$.

The integer quaternion (IQ) weight enumerator of the dual code $C^{\perp}$ from the integer quaternion weight enumerator of the code C over $H[\mathbb{Z}]_\pi$ obtained as follows.

Theorem 3. Let C be a linear code of length n over $H[\mathbb{Z}]_\pi$. Then, the relation between the IQ weight enumerator of C and its dual is given by

$$W_{C^{\perp}}\left(z_0, z_1, \ldots, z_{\left(N(\pi)^2-1\right)/8}\right) = \frac{1}{|C|} W_C\left(z_0, z_1, \ldots, \tilde{z}_{\left(N(\pi)^2-1\right)/8}\right),$$

where

$$z_0 = z_0 + 8\sum_{s=1}^{\left(N(\pi)^2-1\right)/8} z_s,$$

$$\tilde{z}_1 = z_0 + \sum_{s=1}^{\left(N(\pi)^2-1\right)/8}\left(\sum_{r \in \{\mp 1, \mp i, \mp j, \mp k\}} \chi_1(r\omega_s)\right) z_s,$$

and

$$\tilde{z}_t = z_0 + \sum_{s=\left(N(\pi)^2-1\right)/8-(t-2)}^{\left(N(\pi)^2-1\right)/8}\left(\sum_{r \in \{\mp 1, \mp i, \mp j, \mp k\}} \chi_1(r\omega_s)\right) z_{s+t-1-\left(N(\pi)^2-1\right)/8} \quad (t \geq 2)$$

$$+ \sum_{s=t}^{\left(N(\pi)^2-1\right)/8}\left(\sum_{r \in \{\mp 1, \mp i, \mp j, \mp k\}} \chi_1(r\omega_{s-t+1})\right) z_s,$$

Proof: Before giving the proof of Theorem 3, we give a partition of $H[\mathbb{Z}]_\pi$.

Let π be a prime integer quaternion. Then, we have partition of $H[\mathbb{Z}]_\pi$ as follows:

$$H[\mathbb{Z}]_\pi = \{0\} \cup G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5 \cup G_6 \cup G_7 \cup G_8.$$

We set $\omega_0 = 0$ and $\omega_1 = 1$. G_1 contains $\left(N(\pi)^2 - 1\right)/8$ elements $\omega_t, t = 1, 2, \ldots, \left(N(\pi)^2 - 1\right)/8$ in a fixed way such that for $t = 1, 2, \ldots, \left(N(\pi)^2 - 1\right)/8$ we have

$$\begin{aligned} G_2 &= -G_1, \\ G_3 &= iG_1, \\ G_4 &= -iG_1, \\ G_5 &= jG_1, \\ G_6 &= -jG_1, \\ G_7 &= kG_1, \\ G_8 &= -kG_1, \end{aligned}$$

where $lx_1 \neq x_2$ for all $x_1, x_2 \in G_1$ and all $l \in \{\pm 1, \pm i, \pm j, \pm k\}$. The Lipschitz weight of a vector u over $H[\mathbb{Z}]_\pi$ is defined as $quat(u) = \left(g_0, \quad g_1, \quad \cdots \quad, g_{\left(N(\alpha)^2-1\right)/8} \right)$. This means that the integer quaternion enumerator does not distinguish between the eight elements $\mp\omega, \mp i\omega, \mp j\omega, \mp k\omega.$

Using Theorem 2 and above partition of $H[\mathbb{Z}]_\pi$, we have

$$\begin{aligned} \tilde{z}_0 &= \sum_{s=0}^{q-1} \chi_1\left(\omega_0 \omega_s\right) z_s \\ &= z_0 + \sum_{s=1}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r \in \{\pm 1, \pm i, \pm j, \pm k\}} \chi_1\left(\omega_0 \omega_s\right) z_s \right) \\ &= z_0 + \sum_{s=1}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r \in \{\pm 1, \pm i, \pm j, \pm k\}} \chi_1(0) z_s \right) \\ &= z_0 + 8 \sum_{s=1}^{\left(N(\pi)^2-1\right)/8} z_s, \end{aligned}$$

$$\begin{aligned} \tilde{z}_1 &= \sum_{s=0}^{q-1} \chi_1\left(\omega_1 \omega_s\right) z_s \\ &= z_0 + \sum_{s=1}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r \in \{\pm 1, \pm i, \pm j, \pm k\}} \chi_1\left(r \omega_1 \omega_s\right) \right) z_s \\ &= z_0 + \sum_{s=1}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r \in \{\pm 1, \pm i, \pm j, \pm k\}} \chi_1\left(r \omega_s\right) \right) z_s \end{aligned}$$

and

$$\tilde{z}_t = z_0 + \sum_{s=\left(N(\pi)^2-1\right)/8-(t-2)}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r\in\{\mp1,\mp i,\mp j,\mp k\}} \chi_1(r\omega_s) \right) z_{s+t-1-\left(N(\pi)^2-1\right)/8}$$

$$+ \sum_{s=t}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r\in\{\mp1,\mp i,\mp j,\mp k\}} \chi_1(r\omega_{s-t+1}) \right) z_s. \quad (t \geq 2)$$

We now define the character function χ_1 . Let π be a prime integer quaternion and $p(x)\in\mathbb{Z}_{N(\pi)}[x]$ be a monic irreducible polynomial of degree 2. Then, $H[\mathbb{Z}]_\pi$ becomes isomorphic to $\mathbb{Z}_{N(\pi)}[x]/\langle p(x)\rangle$ since $H[\mathbb{Z}]_\pi$ has the basis $\{e_1,e_2\}\subset\{1,i,j,k\}$ and $\mathbb{Z}_{N(\pi)}[x]/\langle p(x)\rangle$ has the basis $\{1,\beta\}$, where β is a root of the polynomial $p(x)$. We set a bijective function f between $H[\mathbb{Z}]_\pi$ and $\mathbb{Z}_{N(\pi)}[x]/\langle p(x)\rangle$ such that

$$f(b) = a_1 + a_2\beta,$$

where $b\in H[\mathbb{Z}]_\pi$ and $a_1,a_2\in\mathbb{Z}_{N(\pi)}$. Hence, we define the character function χ_1 as

$$\chi_1(b) = \xi^{a_1}.$$

Hence, the proof is completed.

Example 1 Let $p=3, \pi=1+i+j$. Then

$$\begin{aligned} H[\mathbb{Z}]_\pi &= G_0\cup G_1\cup G_2\cup G_3\cup G_4\cup G_5\cup G_6\cup G_7\cup G_8 \\ &= \{0\}\cup\{1\}\cup\{-1\}\cup\{i\}\cup\{-i\}\cup\{j\}\cup\{-j\}\cup\{k\}\cup\{-k\}. \end{aligned}$$

We set $\omega_0=0,\ \omega_1=1$. Take the monic irreducible polynomial $p(x)=x^2+x+2\in\mathbb{Z}_3[x]$. Let β be the root of x^2+x+2. $\{1,i\}$ is a basis of $H[\mathbb{Z}]_{1+i+j}$ and $\{1,\beta\}$ is a basis of $\mathbb{F}_9\cong\mathbb{Z}_3[x]/\langle x^2+x+2\rangle$. Thus, we can define the function f from $H[\mathbb{Z}]_{1+i+j}$ to $\mathbb{F}_9\cong\mathbb{Z}_3[x]/\langle x^2+x+2\rangle$ by $f(1)=1, f(i)=\beta$. Now let us consider $[2,1,2]-$ code C over $H[\mathbb{Z}]_\pi$. Assume that the code C which is a left ideal of $H[\mathbb{Z}]_\pi\times H[\mathbb{Z}]_\pi$ is generated by the matrix $(1,1)$. Then, the IQ weight enumerator of C is $W_C=z_0^2+8z_1^2$. According to Theorem 3, we have

$$\tilde{z}_0 = z_0 + 8z_1$$

and

$$\begin{aligned} \tilde{z}_1 &= z_0 + \sum_{s=1}^{\left(N(\pi)^2-1\right)/8} \left(\sum_{r\in\{\pm1,\pm i,\pm j,\pm k\}} \chi_1(rw_s) \right) z_s \\ &= z_0 + \left(\sum_{r\in\{\pm1,\pm i,\pm j,\pm k\}} \chi_1(r) \right) z_1 \\ &= z_0 + \left(\chi_1(1)+\chi_1(-1)+\chi_1(i)+\chi_1(-i)+\chi_1(j)+\chi_1(-j)+\chi_1(k)+\chi_1(-k)\right) z_1 \\ &= z_0 + \left(\xi+\xi^2+\xi^0+\xi^0+\xi^2+\xi+\xi+\xi^2\right) z_1 \\ &= z_0 - z_1. \end{aligned}$$

Note that $1+\xi+\xi^2=0$. Hence, we obtain the IQ weight enumerator $W_{C^\perp}$ of the dual code $C^\perp$ from the IQ weight enumerator W_C of the code C as follows:

$$W_{C^\perp}(z_0,z_1)=\frac{1}{|C|}W_C\left(\tilde{z}_0,\tilde{z}_1\right)=\frac{1}{9}\left[\left(z_0+8z_1\right)^2+8\left(z_0-z_1\right)^2\right]=z_0^2+8z_1^2.$$

Example 2 Let $p=5,\pi=2+i$. Then

$$\begin{aligned}H[\mathbb{Z}]_\pi &= G_0\cup G_1\cup G_2\cup G_3\cup G_4\cup G_5\cup G_6\cup G_7\cup G_8\\ &=\{0\}\cup\{1,1+j,1+k\}\cup\{-1,-1-j,-1-k\}\cup\{i,i+k,i-j\}\cup\{-i,-i-k,-i+j\}\\ &\cup\{j,-1+j,i+j\}\cup\{-j,1-j,-i-j\}\cup\{k,-i+k,-1+k\}\cup\{-k,i-k,1-k\}.\end{aligned}$$

We set $\omega_0=0,\omega_1=1,\omega_2=1+j,\omega_3=1+k$. Take the monic polynomial $p(x)=x^2+2x+3$. Let β be the root of $p(x)$. $\{1,j\}$ is a basis of $H[\mathbb{Z}]_{2+i}$ and $\{1,\beta\}$ is a basis of $\mathbb{F}_{25}\cong\mathbb{Z}_5[x]/\langle x^2+2x+3\rangle$. Thus, we can define the function $f:H[\mathbb{Z}]_{2+i}\to\mathbb{F}_{25}$ by $f(1)=1,\ f(j)=\beta$. Let us consider $[3,1,3]-$ code C over $H[\mathbb{Z}]_{2+i}$. Assume that the code C which is a left ideal of $H[\mathbb{Z}]_\pi^3$ is generated by the matrix G

$$G=\begin{pmatrix}1, & 1, & 1\end{pmatrix}.$$

The generator matrix H of the dual code $C^\perp$ can taken as

$$H=\begin{pmatrix}1, & -1, & 0\\ 0 & 1, & -1\end{pmatrix}.$$

The integer quaternion (IQ) weight enumerator of C is

$$W_C=z_0^3+8z_1^3+8z_2^3+8z_3^3.$$

According to Theorem 3, we get

$$\tilde{z}_0=z_0+8z_1+8z_2+8z_3,$$

$$\begin{aligned}\tilde{z}_1&=z_0+\sum_{s=1}^{3}\left(\sum_{r\in\{\pm1,\pm i,\pm j,\pm k\}}\chi_1(rw_s)\right)z_s\\
&=z_0+\sum_{s=1}^{3}\begin{pmatrix}\chi_1(w_s)+\chi_1(-w_s)+\chi_1(iw_s)+\chi_1(-iw_s)\\ +\chi_1(jw_s)+\chi_1(-jw_s)+\chi_1(kw_s)+\chi_1(-kw_s)\end{pmatrix}z_s\\
&=z_0+\left(\chi_1(1)+\chi_1(-1)+\chi_1(i)+\chi_1(-i)+\chi_1(j)+\chi_1(-j)+\chi_1(k)+\chi_1(-k)\right)z_1\\
&+\begin{pmatrix}\chi_1(1+j)+\chi_1(-1-j)+\chi_1(i(1+j))+\chi_1(-i(1+j))\\ +\chi_1(j(1+j))+\chi_1(-j(1+j))+\chi_1(k(1+j))+\chi_1(-k(1+j))\end{pmatrix}z_2\\
&+\begin{pmatrix}\chi_1(1+k)+\chi_1(-1-k)+\chi_1(i(1+k))+\chi_1(-i(1+k))\\ +\chi_1(j(1+k))+\chi_1(-j(1+k))+\chi_1(k(1+k))+\chi_1(-k(1+k))\end{pmatrix}z_3\\
&=z_0+\left(\xi+\xi^4+\xi^3+\xi^2+\xi^0+\xi^0+\xi^0+\xi^0\right)z_1\\
&+\left(\xi+\xi^4+\xi^3+\xi^2+\xi^4+\xi+\xi^2+\xi^3\right)z_2\\
&+\left(\xi+\xi^4+\xi^3+\xi^2+\xi^3+\xi^2+\xi^4+\xi\right)z_3\\
&=z_0+3z_1-2z_2-2z_3,\end{aligned}$$

$\tilde{z}_2 = z_0 - 2z_1 + 3z_2 - 2z_3,$

and

$\tilde{z}_3 = z_0 - 2z_1 - 2z_2 + 3z_3.$

Here, $1+\xi+\xi^2+\xi^3+\xi^4=0$. Hence, we obtain the IQ weight enumerator $W_{C^\perp}$ of the dual code $C^\perp$ from the IQ weight enumerator W_C of the code C as

$$W_{C^\perp}(z_0,z_1,z_2,z_3) = \frac{1}{|C|} W_C(\tilde{z}_0,\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)$$

$$= \frac{1}{25}\begin{bmatrix} \left(z_0+8z_1+8z_2+8z_3\right)^3 + 8\left(z_0+3z_1-2z_2-2z_3\right)^3 \\ +8\left(z_0-2z_1+3z_2-2z_3\right)^3 + 8\left(z_0-2z_1-2z_2+3z_3\right)^3 \end{bmatrix}.$$

$$= z_0^3 + 24z_1^3 + 24z_2^3 + 24z_3^3 + 24z_0z_1^2 + 24z_0z_2^2 + 24z_0z_3^2 + 48z_1z_2^2 \\ +48z_1z_3^2 + 48z_2z_3^2 + 48z_1^2z_2 + 48z_1^2z_3 + 48z_2^2z_3 + 192z_1z_2z_3$$

Replacing $z_3 \rightarrow z_2$, since $w_{Lip}(\omega_2) = w_{Lip}(\omega_3) = 2$, we get the quaternion spectrum (Q-S) of $C^\perp$ as $W_{C^\perp}(Q-S) = z_0^3 + 24z_0z_1^2 + 48z_0z_2^2 + 24z_1^3 + 144z_2^3 + 96z_1^2z_2 + 288z_1z_2^2$.

In Table I, the 25 codewords are given in the first column. The complete weight enumerator of C is given in the column $C-S$ of Table I. The IQ weight enumerator of C is contained in the column IQ-S and finally the quaternion enumerator can be found in the column Q-S. Using our technique, one can directly obtain the IQ weight enumerator of a code C. Therefore, the IQ weight is not complete weight enumerator.

3. CONCLUSION

In this paper, we proved the MacWilliams identity for the Lipschitz metric. In fact, the Lipschitz metric can be seen as a four-dimensional generalization of the Lee metric.

Table 1: [3,1,3]-code over $H[\mathbb{Z}]_{2+i}$.

Codewords	C-S	IQ-S	Q-S
$(0,0,0)$	z_0^3	z_0^3	z_0^3
$(1,1,1)$	z_1^3	z_1^3	z_1^3
$(-1,-1,-1)$	z_2^3	z_1^3	z_1^3
(i,i,i)	z_3^3	z_1^3	z_1^3
$(-i,-i,-i)$	z_4^3	z_1^3	z_1^3
(j,j,j)	z_5^3	z_1^3	z_1^3
$(-j,-j,-j)$	z_6^3	z_1^3	z_1^3
(k,k,k)	z_7^3	z_1^3	z_1^3

$(-k,-k,-k)$	z_8^3	z_1^3	z_1^3
$(1+j,1+j,1+j)$	z_9^3	z_2^3	z_2^3
$(-1-j,-1-j,-1-j)$	z_{10}^3	z_2^3	z_2^3
$(1-j,1-j,1-j)$	z_{11}^3	z_2^3	z_2^3
Codewords	C-S	IQ-S	Q-S
$(-1+j,-1+j,-1+j)$	z_{12}^3	z_2^3	z_2^3
$(1+k,1+k,1+k)$	z_{13}^3	z_3^3	z_2^3
$(-1-k,-1-k,-1-k)$	z_{14}^3	z_3^3	z_2^3
$(1-k,1-k,1-k)$	z_{15}^3	z_3^3	z_2^3
$(-1+k,-1+k,-1+k)$	z_{16}^3	z_3^3	z_2^3
$(i+k,i+k,i+k)$	z_{17}^3	z_2^3	z_2^3
$(-i-k,-i-k,-i-k)$	z_{18}^3	z_2^3	z_2^3
$(i-k,i-k,i-k)$	z_{19}^3	z_2^3	z_2^3
$(-i+k,-i+k,-i+k)$	z_{20}^3	z_2^3	z_2^3
$(i+j,i+j,i+j)$	z_{21}^3	z_3^3	z_2^3
$(-i-j,-i-j,-i-j)$	z_{22}^3	z_3^3	z_2^3
$(i-j,i-j,i-j)$	z_{23}^3	z_3^3	z_2^3
$(-i+j,-i+j,-i+j)$	z_{24}^3	z_3^3	z_2^3

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] F. J. MacWilliams, "Combinatorial Problems of Elementary Abelian Groups," Ph.D. dissertation, Harvard Univ., Cambridge, MA, 1962.

[2] F. J. MacWilliams, "A theorem on the distribution of weights in a systematic code," Bell Syst. Tech. J., vol. 42, pp. 79-94, 1963.

[3] F. J. Macwilliams and N. J. Sloane, "The Theory of Error Correcting Codes", North Holland Pub. Co., 1977.

[4] Ş. İrfan, "MacWilliams identity for $m-$spotty Lee weight enumerators," Appl. Math. Lett., 23 (2010) 13-16.

[5] B. Yıldız and S. Karadeniz., "Linear codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$," Des. Codes Cryptogr., (2010) 54:61–81.

[6] J. A. Wood, Duality for modules over finite rings and applications to coding theory, Amer. J. Math. 121 (1999), 555–575.

[7] T. Honold and I. Landjev, MacWilliams Identities for codes over frobenius ring, Finite Fields and application Springer pp. 276-292, 2000.

[8] V. Zinoviev, T. Ericson, On Fourier-Invariant Partitions of Finite Abelian Groups and the MacWilliams Identity for Group Codes, Problems of Information Transmission, 32, No. 1, 1996.

[9] K. Huber, "The MacWilliams Theorem for Two-Dimensional Modulo Metrics," AAECC, 41-48, 1997.

[10] C. Martinez, E. Stafford, R. Beivide and E. Gabidulin. "Perfect Codes over Lipschitz Integers". IEEE Int. Symposium on Information Theory, ISIT'07.

[11] C. Martinez, R. Beivide, and E. M. Gabidulin, "Perfect Codes from Cayley Graphs over Lipschitz Integers," IEEE Trans. Inform.Theory, vol. 55, pp. 3552-3562, August, 2009.

[12] G. Davidoff, P. Sarnak, A. Valette, "Elementary Number Theory, Group Theory, and Ramanujan Graphs", Cambridge University Pres, 2003.

[13] M. Güzeltepe, "Codes over Hurwitz integers", Disc. Math. Vol. 313/5, pp. 704-714, 2013.

[14] M. Güzeltepe, O. Heden, "Perfect Mannheim, Lipschitz and Hurwitz weight codes", Math. Communications, Vol. 19/2 pp. 253-276, 2014.

[15] O. Heden, M. Güzeltepe, "On perfect 1-$\mathcal{E}$ error-correcting codes", Math. Communications, Vol. 20/1 pp. 23-35, 2015.