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HYBRINOMIALS RELATED TO HYPER-FIBONACCI AND HYPER-LUCAS NUMBERS

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Abstract

Hybrid number system is a generalization of complex, hyperbolic and dual numbers. Hybrid numbers and hybrid polynomials have been the subject of much research in recent years. In this paper, hybrinomials related to hyper-Fibonacci and hyper-Lucas numbers are defined. Then some algebraic properties of newly defined hybrinomials are examined such as the recurrence relations and summation formulas. Also, the relation between hybrinomials related to hyper-Fibonacci and hyper-Lucas numbers is given. Additionally, hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers are defined by using the hybrinomials related to hyper-Fibonacci and hyper-Lucas numbers.

Keywords: Hyper-Fibonacci numbers, hyper-Lucas numbers, polynomials, hybrinomials

1. Introduction

The Fibonacci and Lucas number sequences are the most popular integer sequences which are the special cases of the Horadam number sequences. The Fibonacci and Lucas number sequences have a wide range of applications in mathematics [1-3, 7, 8, 12, 15, 25, 26] and they are defined by the recurrence relations, respectively ($n \geq 1$) [13]:

$$F_{n+1} = F_n + F_{n-1} \quad \text{with} \quad F_0 = 0, \quad F_1 = 1 \tag{1}$$

and

$$L_{n+1} = L_n + L_{n-1} \quad \text{with} \quad L_0 = 2, \quad L_1 = 1, \tag{2}$$

We encounter many generalizations of the Fibonacci and Lucas numbers in the literature [2, 3, 6, 8, 14, 18, 28]. Dil and Mező [6] introduced hyper-Fibonacci number $F_n^{(r)}$ and hyper-Lucas number $L_n^{(r)}$ as a generalization of the Fibonacci and Lucas numbers, by the formulas

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$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)} \quad \text{with} \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1 \tag{3}$$

and

$$L_n^{(r)} = \sum_{k=0}^{n} L_k^{(r-1)} \quad \text{with} \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \tag{4}$$

where r is a positive integer. The hyper-Fibonacci and hyper-Lucas numbers have the recurrence relations for $n \geq 1$ and $r \geq 1$ [6]:

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)} \tag{5}$$

and

$$L_n^{(r)} = L_{n-1}^{(r)} + L_n^{(r-1)}. \tag{6}$$

Horzum and Koçer [9] introduced the Horadam polynomials as a generalization of the Horadam numbers, and investigated some of their properties. Fibonacci polynomial $F_n(x)$ and Lucas polynomial $L_n(x)$ are defined by Catalan and Bicknell, respectively as

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad \text{with} \quad F_0(x) = 0, \quad F_1(x) = 1 \tag{7}$$

and

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \quad \text{with} \quad L_0(x) = 2, \quad L_1(x) = x, \tag{8}$$

where x is any variable quantity and $n \geq 2$. Hyper-Fibonacci polynomial $F_n^{(r)}(x)$ and hyper-Lucas polynomial $L_n^{(r)}(x)$ are defined by the formulas

$$F_n^{(r)}(x) = \sum_{s=0}^{n} F_s^{(r-1)}(x) \ \text{ with } \ F_n^{(0)}(x) = F_n(x),\ F_0^{(r)}(x) = 0,\ F_1^{(r)}(x) = 1 \tag{9}$$

and

$$L_n^{(r)}(x) = \sum_{s=0}^{n} L_s^{(r-1)}(x) \text{ with } \ L_n^{(0)}(x) = L_n(x),\ L_0^{(r)}(x) = 2,\ L_1^{(r)}(x) = x + 2r \tag{10}$$

for any variable quantity x and positive integer r [16]. For $r \geq 1$ and $n \geq 1$, the recurrence relations

$$F_n^{(r)}(x) = F_{n-1}^{(r)}(x) + F_n^{(r-1)}(x), \tag{11}$$

$$L_n^{(r)}(x) = L_{n-1}^{(r)}(x) + L_n^{(r-1)}(x) \tag{12}$$

are hold and also for $r \geq 1$ and $n \geq 2$, the relations

$$F_n^{(r)}(x) = xF_{n-1}^{(r)}(x) + F_{n-2}^{(r)}(x) + \binom{n+r-2}{r-1} \tag{13}$$

$$L_n^{(r)}(x) = xL_{n-1}^{(r)}(x) + L_{n-2}^{(r)}(x) - x\binom{n+r-2}{r-1} + 2\binom{n+r-1}{r-1} \tag{14}$$

are valid for the hyper-Fibonacci and hyper-Lucas polynomials [16]. Furthermore, if $r \geq 1$ and $n \geq 1$, then there are the summation formulas

$$\sum_{s=0}^{r} F_n^{(s)}(x) = F_{n+1}^{(r)}(x) + (1-x)F_n(x) - F_{n-1}(x) \tag{15}$$

and

$$\sum_{s=0}^{r} L_n^{(s)}(x) = L_{n+1}^{(r)}(x) + (1-x)L_n(x) - L_{n-1}(x) \tag{16}$$

and there are the relations between the hyper-Fibonacci and hyper-Lucas polynomials [16]:

$$xF_n^{(r)}(x) + L_n^{(r)}(x) = 2F_{n+1}^{(r)}(x) \tag{17}$$

and

$$xF_n^{(r)}(x) - L_n^{(r)}(x) = -2\left(F_{n-1}^{(r)}(x) + \binom{n+r-1}{r-1}\right). \tag{18}$$

Hybrid number set $\mathbb{K}$, which is a generalization of complex, hyperbolic and dual numbers, is introduced by Özdemir [19] as

$$\mathbb{K} = \{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = hi = \epsilon + i\}. \tag{19}$$

The hybrid number system has attracted the attention of many researchers, recently. So there are a lot of papers about the hybrid number system including the Horadam numbers and especially the special cases of Horadam numbers [4, 5, 10, 11, 17, 20-25, 27]. Let us give a brief overview of some of them: Szynal-Liana [21] defined the Horadam hybrid numbers and gave some of their properties such as the Binet formula, character and generating function. Fibonacci hybrid number FH_n and Lucas hybrid number LH_n are defined as

$$FH_n = F_n + iF_{n+1} + \epsilon F_{n+2} + hF_{n+3} \tag{20}$$

and

$$LH_n = L_n + iL_{n+1} + \epsilon L_{n+2} + hL_{n+3} \tag{21}$$

by using the Fibonacci and Lucas numbers [21]. Tan and Ait-Amrane [25] introduced the bi-periodic Horadam hybrid numbers as a generalization of the Horadam hybrid numbers. They investigated the generating function and the Binet formula for newly defined hybrid numbers and gave the relation between generalized bi-periodic Fibonacci hybrid numbers and generalized bi-periodic Lucas hybrid numbers. Kızılateş [11] defined the generalization of Fibonacci and Lucas hybrid numbers called q-Fibonacci and q-Lucas hybrid numbers, respectively. The author also gave some of their algebraic properties. The nth (p, q)-Fibonacci hybrid and (p, q)-Lucas hybrid numbers are defined by using (p, q)-Fibonacci and (p, q)-Lucas numbers, respectively, moreover the generating functions, Binet formulas, Catalan and Cassini identities are presented for these hybrid numbers [5]. Horadam hybrid polynomials are defined using the Horadam polynomials, by Kızılateş [10]. Fibonacci and Lucas hybrid polynomials, which are called Fibonacci and Lucas hybrinomials, are introduced by Szynal-Liana and Wloch [23] as follows:

$$FH_n(x) = F_n(x) + iF_{n+1}(x) + \epsilon F_{n+2}(x) + hF_{n+3}(x) \tag{22}$$

and

$$LH_n(x) = L_n(x) + iL_{n+1}(x) + \epsilon L_{n+2}(x) + hL_{n+3}(x), \tag{23}$$

where $F_n(x)$ and $L_n(x)$ are the ordinary Fibonacci and Lucas polynomials, respectively. The recurrence relation for the Fibonacci hybrinomials is $(n \geq 2)$

$$FH_n(x) = xFH_{n-1}(x) + FH_{n-2}(x), \tag{24}$$

with the initial conditions

$$FH_0(x) = i + \epsilon x + h(x^2 + 1), FH_1(x) = 1 + ix + \epsilon(x^2 + 1) + h(x^3 + 2x),$$

and the recurrence relation for the Lucas hybrinomials is $(n \geq 2)$

$$LH_n(x) = xLH_{n-1}(x) + LH_{n-2}(x) \tag{25}$$

with

$$LH_0(x) = 2 + ix + \epsilon(x^2 + 2) + h(x^3 + 3x)$$

and

$$LH_1(x) = x + i(x^2 + 2) + \epsilon(x^3 + 3x) + h(x^4 + 4x^2 + 2) \text{ [23].}$$

Mersin and Bahşi [17] defined hyper-Fibonacci and hyper-Lucas hybrinomials as

$$HF_n^{(r)} = \sum_{s=0}^{n} HF_s^{(r-1)} \text{ with } HF_n^{(0)} = HF_n(x),\ HF_0^{(r)} = HF_0(x) \tag{26}$$

and

$$HL_n^{(r)} = \sum_{s=0}^{n} HL_s^{(r-1)} \text{ with } HL_n^{(0)} = HL_n(x),\ HL_0^{(r)} = HL_0(x), \tag{27}$$

where r is a positive integer, $HF_n(x)$ and $HL_n(x)$ are the ordinary Fibonacci and Lucas hybrinomials. The authors examined some of their properties such as the recurrence relations, summation formulas and generating functions. Also in [17] hyper-Fibonacci hybrid numbers and hyper-Lucas hybrid numbers are introduced by using the hyper-Fibonacci and hyper-Lucas hybrinomials for $x = 1$, respectively.

By the motivation of the above papers, this paper aims to define hybrinomials related to hyper-Fibonacci and hyper-Lucas numbers in a different way than in [17] and to examine some properties of these hybrinomials, such as the recurrence relations and summation formulas. Another aim is to introduce hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers by using newly defined hybrinomials.

2. Main results

Definition 2.1 Hybrinomials related to hyper-Fibonacci and hyper-Lucas numbers are defined as:

$$FH_n^{(r)}(x) = F_n^{(r)}(x) + iF_{n+1}^{(r)}(x) + \epsilon F_{n+2}^{(r)}(x) + hF_{n+3}^{(r)}(x) \tag{28}$$

and

$$LH_n^{(r)}(x) = L_n^{(r)}(x) + iL_{n+1}^{(r)}(x) + \epsilon L_{n+2}^{(r)}(x) + hL_{n+3}^{(r)}(x), \tag{29}$$

where $F_n^{(r)}(x)$ and $L_n^{(r)}(x)$ are the ordinary hyper-Fibonacci and hyper-Lucas polynomials, respectively.

The first few hybrinomials related to hyper-Fibonacci are

$$\begin{aligned}
FH_0^{(1)}(x) &= i + \epsilon(x+1) + h(x^2 + x + 2),\\
FH_1^{(1)}(x) &= 1 + i(x+1) + \epsilon(x^2 + x + 2) + h(x^3 + x^2 + 3x + 2),\\
FH_2^{(1)}(x) &= (x+1) + i(x^2 + x + 2) + \epsilon(x^3 + x^2 + 3x + 2) + h(x^4 + x^3 + 4x^2 + 3x + 3),
\end{aligned}$$

$$\begin{aligned}
FH_0^{(2)}(x) &= i + \epsilon(x+2) + h(x^2 + 2x + 4),\\
FH_1^{(2)}(x) &= 1 + i(x+2) + \epsilon(x^2 + 2x + 4) + h(x^3 + 2x^2 + 5x + 6),\\
FH_2^{(2)}(x) &= (x+2) + i(x^2 + 2x + 4) + \epsilon(x^3 + 2x^2 + 5x + 6) + h(x^4 + 2x^3 + 6x^2 + 8x + 9).
\end{aligned}$$

The first few hybrinomials related to hyper-Lucas numbers are

$$\begin{aligned}
LH_0^{(1)}(x) &= 2 + i(x+2) + \epsilon(x^2 + x + 4) + h(x^3 + x^2 + 4x + 4),\\
LH_1^{(1)}(x) &= (x+2) + i(x^2 + x + 4) + \epsilon(x^3 + x^2 + 4x + 4) + h(x^4 + x^3 + 5x^2 + 4x + 6),\\
LH_2^{(1)}(x) &= (x^2 + x + 4) + i(x^3 + x^2 + 4x + 4) + \epsilon(x^4 + x^3 + 5x^2 + 4x + 6)\\
&\qquad + h(x^5 + x^4 + 6x^3 + 5x^2 + 9x + 6),
\end{aligned}$$

$$\begin{aligned}
LH_0^{(2)}(x) &= 2 + i(x+4) + \epsilon(x^2 + 2x + 8) + h(x^3 + 2x^2 + 6x + 12),\\
LH_1^{(2)}(x) &= (x+4) + i(x^2 + 2x + 8) + \epsilon(x^3 + 2x^2 + 6x + 12)\\
&\qquad + h(x^4 + 2x^3 + 7x^2 + 10x + 18),\\
LH_2^{(2)}(x) &= (x^2 + 2x + 8) + i(x^3 + 2x^2 + 6x + 12) + \epsilon(x^4 + 2x^3 + 7x^2 + 10x + 18)\\
&\qquad + h(x^5 + 2x^4 + 8x^3 + 12x^2 + 19x + 24).
\end{aligned}$$

The hybrinomials defined in Definition 2.1 gives the following definition of hybrid numbers for x=1:

Definition 2.2 Hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers are defined as

$$FH_n^{(r)} = F_n^{(r)} + iF_{n+1}^{(r)} + \epsilon F_{n+2}^{(r)} + hF_{n+3}^{(r)} \tag{30}$$

and

$$LH_n^{(r)} = L_n^{(r)} + iL_{n+1}^{(r)} + \epsilon L_{n+2}^{(r)} + hL_{n+3}^{(r)}, \tag{31}$$

where $F_n^{(r)}$ and $L_n^{(r)}$ are the ordinary hyper-Fibonacci and hyper-Lucas numbers, respectively.

Next two tables contain some values of the hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers, respectively.

Table 1: The first few values of the hybrid hyper-Fibonacci numbers $FH_n^{(r)}$.

	$r=0$	$r=1$	$r=2$	$r=3$
$n=0$	$i+\epsilon+2h$	$i+2\epsilon+4h$	$i+3\epsilon+7h$	$i+4\epsilon+11h$
$n=1$	$1+i+2\epsilon+3h$	$1+2i+4\epsilon+7h$	$1+3i+7\epsilon+14h$	$1+4i+11\epsilon+25h$
$n=2$	$1+2i+3\epsilon+5h$	$2+4i+7\epsilon+12h$	$3+7i+14\epsilon+26h$	$4+11i+25\epsilon+51h$
$n=3$	$2+3i+5\epsilon+8h$	$4+7i+12\epsilon+20h$	$7+14i+26\epsilon+46h$	$11+25i+51\epsilon+97h$
$n=4$	$3+5i+8\epsilon+13h$	$7+12i+20\epsilon+33h$	$14+26i+46\epsilon+79h$	$25+51i+97\epsilon+176h$

Table 2: The first few values of the hybrid hyper-Lucas numbers $LH_n^{(r)}$.

	$r=0$	$r=1$	$r=2$	$r=3$
$n=0$	$2+i+3\epsilon+4h$	$2+3i+6\epsilon+10h$	$2+5i+11\epsilon+21h$	$2+7i+18\epsilon+39h$
$n=1$	$1+3i+4\epsilon+7h$	$3+6i+10\epsilon+17h$	$5+11i+21\epsilon+38h$	$7+18i+39\epsilon+77h$
$n=2$	$3+4i+7\epsilon+11h$	$6+10i+17\epsilon+28h$	$11+21i+38\epsilon+66h$	$18+39i+77\epsilon+143h$
$n=3$	$4+7i+11\epsilon+18h$	$10+17i+28\epsilon+46h$	$21+38i+66\epsilon+112h$	$39+77i+143\epsilon+255h$
$n=4$	$7+11i+18\epsilon+29h$	$17+28i+46\epsilon+75h$	$38+66i+112\epsilon+187h$	$77+143i+255\epsilon+442h$

Next, we give our results.

Theorem 2.1 $FH_n^{(r)}(x)$ and $LH_n^{(r)}(x)$ have the recurrence relations for $r \geq 1$ and $n \geq 1$:

i. $FH_n^{(r)}(x) = FH_{n-1}^{(r)}(x) + FH_n^{(r-1)}(x)$,

ii. $LH_n^{(r)}(x) = LH_{n-1}^{(r)}(x) + LH_n^{(r-1)}(x)$.

Proof.

i. The proof is similar to the proof of (ii).

ii. Considering Definition 2.1 and equation (12), we have

$$\begin{aligned} LH_{n-1}^{(r)}(x) + LH_n^{(r-1)}(x) &= L_{n-1}^{(r)}(x) + iL_n^{(r)}(x) + \epsilon L_{n+1}^{(r)}(x) + hL_{n+2}^{(r)}(x) \\ &\quad + L_n^{(r-1)}(x) + iL_{n+1}^{(r-1)}(x) + \epsilon L_{n+2}^{(r-1)}(x) + hL_{n+3}^{(r-1)}(x) \\ &= \left(L_{n-1}^{(r)}(x) + L_n^{(r-1)}(x)\right) + i\left(L_n^{(r)}(x) + L_{n+1}^{(r-1)}(x)\right) \\ &\quad + \epsilon\left(L_{n+1}^{(r)}(x) + L_{n+2}^{(r-1)}(x)\right) + h\left(L_{n+2}^{(r)}(x) + L_{n+3}^{(r-1)}(x)\right) \\ &= L_n^{(r)}(x) + iL_{n+1}^{(r)}(x) + \epsilon L_{n+2}^{(r)}(x) + hL_{n+3}^{(r)}(x) \\ &= LH_n^{(r)}(x). \end{aligned}$$

Corollary 2.1 The hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers have the recurrence relations for $r \geq 1$ and $n \geq 1$:

i. $FH_n^{(r)} = FH_{n-1}^{(r)} + FH_n^{(r-1)}$,

ii. $LH_n^{(r)} = LH_{n-1}^{(r)} + LH_n^{(r-1)}$.

Theorem 2.2 If $r \geq 1$ and $n \geq 2$, then there are the recurrence relations for $FH_n^{(r)}(x)$ and $LH_n^{(r)}(x)$, respectively:

i. $$FH_n^{(r)}(x) = xFH_{n-1}^{(r)}(x) + FH_{n-2}^{(r)}(x) + \binom{n+r-2}{r-1} + i\binom{n+r-1}{r-1} + \epsilon\binom{n+r}{r-1} + h\binom{n+r+1}{r-1}$$

ii. $$LH_n^{(r)}(x) = xLH_{n-1}^{(r)}(x) + LH_{n-2}^{(r)}(x) - \binom{n+r-2}{r-1}(x-2h) - \binom{n+r-1}{r-1}(xi-2) - \binom{n+r}{r-1}(x\epsilon - 2i) - \binom{n+r+1}{r-1}(xh-2\epsilon).$$

Proof.

i. By using Definition 2.1 and equation (13), we have

$$xFH_{n-1}^{(r)}(x) + FH_{n-2}^{(r)}(x) + \binom{n+r-2}{r-1} + i\binom{n+r-1}{r-1} + \epsilon\binom{n+r}{r-1} + h\binom{n+r+1}{r-1}$$

$$= x\left(F_{n-1}^{(r)}(x) + iF_n^{(r)}(x) + \epsilon F_{n+1}^{(r)}(x) + hF_{n+2}^{(r)}(x)\right)$$

$$+\left(F_{n-2}^{(r)}(x) + iF_{n-1}^{(r)}(x) + \epsilon F_n^{(r)}(x) + hF_{n+1}^{(r)}(x)\right)$$

$$+\binom{n+r-2}{r-1} + i\binom{n+r-1}{r-1} + \epsilon\binom{n+r}{r-1} + h\binom{n+r+1}{r-1}$$

$$= xF_{n-1}^{(r)}(x) + F_{n-2}^{(r)}(x) + \binom{n+r-2}{r-1} + i\left(xF_{n}^{(r)}(x) + F_{n-1}^{(r)}(x) + \binom{n+r-1}{r-1}\right)$$

$$+\epsilon\left(xF_{n+1}^{(r)}(x) + F_{n}^{(r)}(x) + \binom{n+r}{r-1}\right) + h\left(xF_{n+2}^{(r)}(x) + F_{n+1}^{(r)}(x) + \binom{n+r+1}{r-1}\right)$$

$$= F_{n}^{(r)}(x) + iF_{n+1}^{(r)}(x) + \epsilon F_{n+2}^{(r)}(x) + hF_{n+3}^{(r)}(x).$$

ii. The proof is similar to the proof of (i).

Corollary 2.2 If $r \geq 1$ and $n \geq 2$, then there are the recurrence relations for the hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers, respectively:

i. $$FH_{n}^{(r)} = FH_{n-1}^{(r)} + FH_{n-2}^{(r)} + \binom{n+r-2}{r-1} + i\binom{n+r-1}{r-1} + \epsilon\binom{n+r}{r-1} + h\binom{n+r+1}{r-1},$$

ii. $$LH_{n}^{(r)} = LH_{n-1}^{(r)} + LH_{n-2}^{(r)} - \binom{n+r-2}{r-1}(1-2h) - \binom{n+r-1}{r-1}(i-2) - \binom{n+r}{r-1}(\epsilon-2i) - \binom{n+r+1}{r-1}(h-2\epsilon).$$

Theorem 2.3 The summation formulas

i. $$\sum_{s=0}^{n} FH_{s}^{(r)}(x) = FH_{n}^{(r+1)}(x) - \left(iF_{0}^{(r+1)}(x) + \epsilon F_{1}^{(r+1)}(x) + hF_{2}^{(r+1)}(x)\right),$$

ii. $$\sum_{s=0}^{n} LH_{s}^{(r)}(x) = LH_{n}^{(r+1)}(x) - \left(iL_{0}^{(r+1)}(x) + \epsilon L_{1}^{(r+1)}(x) + hL_{2}^{(r+1)}(x)\right)$$

are hold for $FH_{n}^{(r)}(x)$ and $LH_{n}^{(r)}(x)$, respectively.

Proof.

i. The proof is similar to the proof of (ii).

ii. We use the mathematical induction method on n. By using Definition 2.1 and equation (12), we have

$$
\begin{aligned}
&LH_0^{(r+1)}(x)-\left(iL_0^{(r+1)}(x)+\epsilon L_1^{(r+1)}(x)+hL_2^{(r+1)}(x)\right)\\
&=L_0^{(r+1)}(x)+iL_1^{(r+1)}(x)+\epsilon L_2^{(r+1)}(x)+hL_3^{(r+1)}(x)\\
&\qquad\qquad\qquad\qquad-\left(iL_0^{(r+1)}(x)+\epsilon L_1^{(r+1)}(x)+hL_2^{(r+1)}(x)\right)\\
&=L_0^{(r+1)}(x)+i\left(L_1^{(r+1)}(x)-L_0^{(r+1)}(x)\right)+\epsilon\left(L_2^{(r+1)}(x)-L_1^{(r+1)}(x)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+h\left(L_3^{(r+1)}(x)-L_2^{(r+1)}(x)\right)\\
&=L_0^{(r)}(x)+iL_1^{(r)}(x)+\epsilon L_2^{(r)}(x)+hL_3^{(r)}(x)\\
&=LH_0^{(r)}(x).
\end{aligned}
$$

Thus, the result is true for $n=0$. Assume that the result is true for $n=k$. Then, we have

$$\sum_{s=0}^{k} LH_s^{(r)}(x)=LH_k^{(r+1)}(x)-\left(iL_0^{(r+1)}(x)+\epsilon L_1^{(r+1)}(x)+hL_2^{(r+1)}(x)\right).$$

We must show that the result is true for $n=k+1$.

$$
\begin{aligned}
\sum_{s=0}^{k+1} LH_s^{(r)}(x)\quad &=\sum_{s=0}^{k} LH_s^{(r)}(x)+LH_{k+1}^{(r)}(x)\\
&=LH_k^{(r+1)}(x)-\left(iL_0^{(r+1)}(x)+\epsilon L_1^{(r+1)}(x)+hL_2^{(r+1)}(x)\right)+LH_{k+1}^{(r)}(x)\\
&=LH_{k+1}^{(r+1)}(x)-\left(iL_0^{(r+1)}(x)+\epsilon L_1^{(r+1)}(x)+hL_2^{(r+1)}(x)\right).
\end{aligned}
$$

This completes the proof.

Corollary 2.3 There are the summation formulas for the hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers, respectively:

$$\textbf{i.}\sum_{s=0}^{n} FH_s^{(r)}=FH_n^{(r+1)}-\left(iF_0^{(r+1)}+\epsilon F_1^{(r+1)}+hF_2^{(r+1)}\right),$$

$$\textbf{ii.}\sum_{s=0}^{n} LH_s^{(r)}=LH_n^{(r+1)}-\left(iL_0^{(r+1)}+\epsilon L_1^{(r+1)}+hL_2^{(r+1)}\right).$$

Theorem 2.4 For $r\geq 1$ and $n\geq 1$, the summation formulas

$$\textbf{i.}\ \sum_{s=0}^{r} FH_n^{(s)}(x)=FH_{n+1}^{(r)}(x)+(1-x)FH_n(x)-FH_{n-1}(x),$$

$$\textbf{ii.}\ \sum_{s=0}^{r} LH_n^{(s)}(x)=LH_{n+1}^{(r)}(x)+(1-x)LH_n(x)-LH_{n-1}(x)$$

are hold.

Proof.

i. By using Definition 2.1 and equation (15), we have

$$\sum_{s=0}^{r} FH_n^{(s)}(x) = \sum_{s=0}^{r} \left(F_n^{(s)}(x) + iF_{n+1}^{(s)}(x) + \epsilon F_{n+2}^{(s)}(x) + hF_{n+3}^{(s)}(x)\right)$$

$$= \sum_{s=0}^{r} F_n^{(s)}(x) + i\sum_{s=0}^{r} F_{n+1}^{(s)}(x) + \epsilon \sum_{s=0}^{r} F_{n+2}^{(s)}(x) + h\sum_{s=0}^{r} F_{n+3}^{(s)}(x)$$

$$= \left[F_{n+1}^{(r)}(x) + (1-x)F_n(x) - F_{n-1}(x)\right] + i\left[F_{n+2}^{(r)}(x) + (1-x)F_{n+1}(x) - F_n(x)\right]$$

$$+\epsilon\left[F_{n+3}^{(r)}(x) + (1-x)F_{n+2}(x) - F_{n+1}(x)\right]$$

$$+h\left[F_{n+4}^{(r)}(x) + (1-x)F_{n+3}(x) - F_{n+2}(x)\right].$$

Thus,

$$\sum_{s=0}^{r} FH_n^{(s)}(x) = \left(F_{n+1}^{(r)}(x) + iF_{n+2}^{(r)}(x) + \epsilon F_{n+3}^{(r)}(x) + hF_{n+4}^{(r)}(x)\right)$$
$$+(1-x)(F_n(x) + iF_{n+1}(x)+\epsilon F_{n+2}(x) + hF_{n+3}(x))$$
$$-\left(F_{n-1}(x) + iF_n(x) + \epsilon F_{n+1}(x) + hF_{n+2}(x)\right)$$

$$= FH_{n+1}^{(r)}(x) + (1-x)FH_n(x) - FH_{n-1}(x).$$

ii. The proof is similar to the proof of (i).

Corollary 2.4 If $r \geq 1$ and $n \geq 1$, then

i. $$\sum_{s=0}^{r} FH_n^{(s)} = FH_{n+1}^{(r)} - FH_{n-1},$$

ii. $$\sum_{s=0}^{r} LH_n^{(s)} = FL_{n+1}^{(r)} - LH_{n-1}$$

are valid.

Theorem 2.5 If $r \geq 1$ and $n \geq 1$, then there are the relations between $FH_n^{(r)}(x)$ and $LH_n^{(r)}(x)$*:*

i. $$xFH_n^{(r)}(x) + LH_n^{(r)}(x) = 2FH_{n+1}^{(r)}(x),$$

ii. $$xFH_n^{(r)}(x) - LH_n^{(r)}(x) = -2\left(FH_{n-1}^{(r)}(x) + \binom{n+r-1}{r-1} + i\binom{n+r}{r-1} + \epsilon\binom{n+r+1}{r-1} + h\binom{n+r+2}{r-1}\right).$$

Proof. From Definition 2.1, equations (17) and (18), we have the proofs as follows:

i.

$$\begin{aligned}
&FH_n^{(r)}(x) + LH_n^{(r)}(x) \\
&= x\left(F_n^{(r)}(x) + iF_{n+1}^{(r)}(x) + \epsilon F_{n+2}^{(r)}(x) + hF_{n+3}^{(r)}(x)\right) \\
&\qquad + \left(L_n^{(r)}(x) + iL_{n+1}^{(r)}(x) + \epsilon L_{n+2}^{(r)}(x) + hL_{n+3}^{(r)}(x)\right) \\
&= \left(xF_n^{(r)}(x) + L_n^{(r)}(x)\right) + i\left(xF_{n+1}^{(r)}(x) + L_{n+1}^{(r)}(x)\right) \\
&\qquad + \epsilon\left(xF_{n+2}^{(r)}(x) + L_{n+2}^{(r)}(x)\right) + h\left(xF_{n+3}^{(r)}(x) + L_{n+3}^{(r)}(x)\right) \\
&= 2F_{n+1}^{(r)}(x) + 2iF_{n+2}^{(r)}(x) + 2\epsilon F_{n+3}^{(r)}(x) + 2hF_{n+4}^{(r)}(x) \\
&= 2FH_{n+1}^{(r)}(x),
\end{aligned}$$

ii.

$$\begin{aligned}
&xFH_n^{(r)} - LH_n^{(r)} \\
&= x\left(F_n^{(r)}(x) + iF_{n+1}^{(r)}(x) + \epsilon F_{n+2}^{(r)}(x) + hF_{n+3}^{(r)}(x)\right) \\
&\qquad - \left(L_n^{(r)}(x) + iL_{n+1}^{(r)}(x) + \epsilon L_{n+2}^{(r)}(x) + hL_{n+3}^{(r)}(x)\right) \\
&= \left(xF_n^{(r)}(x) - L_n^{(r)}(x)\right) + i\left(xF_{n+1}^{(r)}(x) - L_{n+1}^{(r)}(x)\right) \\
&\qquad + \epsilon\left(xF_{n+2}^{(r)}(x) - L_{n+2}^{(r)}(x)\right) + h\left(xF_{n+3}^{(r)}(x) - L_{n+3}^{(r)}(x)\right) \\
&= -2\left(F_{n-1}^{(r)}(x) + \binom{n+r-1}{r-1}\right) - 2i\left(F_n^{(r)}(x) + \binom{n+r}{r-1}\right) \\
&\qquad - 2\epsilon\left(F_{n+1}^{(r)}(x) + \binom{n+r+1}{r-1}\right) - 2h\left(F_{n+2}^{(r)}(x) + \binom{n+r+2}{r-1}\right) \\
&= -2\left(FH_{n-1}^{(r)}(x) + \binom{n+r-1}{r-1} + i\binom{n+r}{r-1} + \epsilon\binom{n+r+1}{r-1}\right. \\
&\qquad \left. + h\binom{n+r+2}{r-1}\right).
\end{aligned}$$

Corollary 2.5 If $r \geq 1$ and $n \geq 1$, then there are the relations between the hybrid hyper-Fibonacci and hybrid hyper-Lucas numbers:

i. $FH_n^{(r)} + LH_n^{(r)} = 2FH_{n+1}^{(r)}$,

ii. $FH_n^{(r)} - LH_n^{(r)} = -2\left(FH_{n-1}^{(r)} + \binom{n+r-1}{r-1} + i\binom{n+r}{r-1} + \epsilon\binom{n+r+1}{r-1} + h\binom{n+r+2}{r-1}\right)$.

3. Conclusion

Hyper-Fibonacci and hyper-Lucas numbers are the generalizations of the well-known Fibonacci and Lucas number sequences which have huge number applications in many branches of science. The hybrid number system is also the generalization of complex, hyperbolic, and dual numbers which are attracted attention, recently. In the present paper, we defined the new hybrinomials and hybrid numbers related to hyper-Fibonacci and hyper-Lucas numbers. We obtained the recurrence relations and summation formulas for newly defined hybrinomials and hybrid numbers. We also gave the relationship between these hybrinomials and the relationship between these hybrid numbers.

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