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RESEARCH ARTICLE

THE NOWICKI CONJECTURE FOR TRACELESS GENERIC MATRIX ALGEBRAS

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ABSTRACT

A locally nilpotent linear derivation δ of the commutative polynomial algebra $K[X_d] = K[x_1, \dots, x_d]$ of rank d is called Weitzenböck. It is well known that the subalgebra $K[X_d]^\delta$ of $K[X_d]$ consisting of polynomials which are sent to zero by δ is finitely generated. Let the Weitzenböck derivation δ act on $K[X_d, Y_d]$ such that $\delta(y_i) = x_i$, $\delta(x_i) = 0$, $i = 1, \dots, d$. The explicit form of generators of the algebra $K[X_d, Y_d]^\delta$ was conjectured by Nowicki in 1994. In this study, we consider the Nowicki conjecture in the algebra W generated by two traceless generic matrices with entries from commutative associative unitary polynomial algebra with six variables, and obtain the free generators of the algebra W^δ of constants in this algebra.

Keywords: Algebra of constants, Generic matrix algebra, The Nowicki conjecture

1. INTRODUCTION

Let $K[X_d]$ be the free algebra of rank d in the variety of associative commutative unital algebras over a field K of characteristic zero, and let $GL_d(K)$ be the group of $d \times d$ invertible matrices so called the general linear group. The fourteenth of twenty three questions proposed by Hilbert [1] is the initiation of the classical invariant theory. The statement of the problem is that "Is the algebra $K[X_d]^H$ of invariants of any subgroup H of $GL_d(K)$ finitely generated?", which was negated by Nagata [2] in 1959. However, the algebra $K[X_d]^H$ is finitely generated for finite groups H via Noether [3].

Weitzenböck [4] utilized locally nilpotent derivations δ to approach the finite generation problem in 1932. Such derivations have been called the Weitzenböck derivations in the modern algebra papers, recently. He showed that the algebra $K[X_d]^\delta$ of constants is finitely generated, and using this algebraic technique, gave a partial affirmative answer to the fourteenth problem of Hilbert.

Let $K[X_d, Y_d] = K[x_1, \dots, x_d, y_1, \dots, y_d]$ be the polynomial algebra of rank $2d$, and let the Weitzenböck derivation δ ack on $K[X_d, Y_d]$ as $\delta(y_i) = x_i$, $\delta(x_i) = 0$, $i = 1, \dots, d$. In 1994, Nowicki [5] conjectured generators of the algebra $K[X_d, Y_d]^\delta$. The conjecture was proved by different mathematicians with distinct techniques [6,7,8,9] in 2008-2009. The Nowicki conjecture is that the algebra $K[X_d, Y_d]^\delta$ is generated by $x_1, \dots, x_n$ and the elements $x_i y_j - x_j y_j$, where $1 \leq i < j \leq d$.

In the period 2020-2022, many noncommutative nonassociative analogues of the Nowicki conjecture have been studied. One may count the Nowicki conjecture for the free metabelian Lie algebra F_{2d} of rank $2d$ [10], in which a finite generating set for the algebra $(F'_{2d})^\delta$ as a $K[X_d, Y_d]^\delta$-module was given. Additinally, the Nowicki conjecture was studied for the free metabelian associative algebra of rank $2d$ [11]. Also, generators were obtained for algebras of invariants in Grassmann algebras [11]. Finally, the free metabelian Possion algebra was considered in [12].

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Let X and Y be two traceless generic matrices with entries from polynomial algebra and W be the associative unital algebra generated by the set $\{X, Y\}$ over a field of characteristic zero. We consider the Nowicki conjecture for the algebra W and determine the generators of the algebra W^{δ} of constants of the Weitzenböck derivation $\delta: Y \to X \to 0$, in the present paper.

2. PRELIMINARIES

Let $\boldsymbol{K}$ be a field of characteristic zero, $\boldsymbol{K}[\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3}]$ be the polynomial algebra generated by six algebraically independent commuting variables. We fix the traceless generic matrices

$$\boldsymbol{X} = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, \ \boldsymbol{Y} = \begin{pmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{pmatrix},$$

over the field $\boldsymbol{K}$. Let $\boldsymbol{W}$ be the free associative algebra generated by $\boldsymbol{X}$ and $\boldsymbol{Y}$. It is well known by [13] that the center of $\boldsymbol{W}$ is a free $\boldsymbol{K}[\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}]$-module generated by $\boldsymbol{I}, \boldsymbol{X}, \boldsymbol{Y}, [\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{XY} - \boldsymbol{YX}$, where $\boldsymbol{I}$ is the $\boldsymbol{2} \times \boldsymbol{2}$ identity matrix, and

$$\begin{aligned} \boldsymbol{t} &= \text{trace}(\boldsymbol{X}^2) = 2(x_1^2 + x_2 x_3)\boldsymbol{I} = 2\boldsymbol{X}^2, \\ \boldsymbol{u} &= \text{trace}(\boldsymbol{Y}^2) = 2(y_1^2 + y_2 y_3)\boldsymbol{I} = 2\boldsymbol{Y}^2, \\ \boldsymbol{v} &= \text{trace}(\boldsymbol{XY}) = (2x_1 y_1 + x_2 y_3 + x_3 y_2)\boldsymbol{I} = \boldsymbol{XY} + \boldsymbol{YX}, \end{aligned}$$

that are algebraically independent variables.

Let $\boldsymbol{\delta}$ be the locally nilpotent linear derivation of $\boldsymbol{W}$ sending $\boldsymbol{Y}$ to $\boldsymbol{X}$, and $\boldsymbol{X}$ to the zero matrix. As an analogue of the Nowicki conjecture in $\boldsymbol{W}$, we give a free generating set for the algebra

$$\boldsymbol{W}^{\boldsymbol{\delta}} = \{\boldsymbol{p} \in \boldsymbol{W} : \boldsymbol{\delta}(\boldsymbol{p}) = \boldsymbol{0}\},$$

which is a $\boldsymbol{K}[\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}]^{\boldsymbol{\delta}}$-module. An easy observation gives that

$$\begin{aligned} \boldsymbol{\delta}(\boldsymbol{t}) &= \boldsymbol{\delta}(2\boldsymbol{X}^2) = \boldsymbol{0}, \\ \boldsymbol{\delta}(\boldsymbol{u}) &= \boldsymbol{\delta}(2\boldsymbol{Y}^2) = 2\boldsymbol{XY} + 2\boldsymbol{YX} = 2\boldsymbol{v}, \\ \boldsymbol{\delta}(\boldsymbol{v}) &= \boldsymbol{\delta}(\boldsymbol{XY} + \boldsymbol{YX}) = \boldsymbol{XX} + \boldsymbol{XX} = \boldsymbol{t}. \end{aligned}$$

It is known, see e.g. [5], that $\boldsymbol{K}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]^{\boldsymbol{\delta}: z \to y \to x \to 0} = \boldsymbol{K}[\boldsymbol{x}, \boldsymbol{y}^2 - 2\boldsymbol{xz}]$ for algebraically independent variables $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. Therefore,

$$\boldsymbol{K}[\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}]^{\boldsymbol{\delta}: Y \to X \to 0} = \boldsymbol{K}[\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}]^{\boldsymbol{\delta}: u \to 2v \to 2t \to 0} = \boldsymbol{K}[\boldsymbol{t}, \boldsymbol{v}^2 - \boldsymbol{tu}].$$

In the next section, we provide free generators for the $\boldsymbol{K}[\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{v}]^{\boldsymbol{\delta}}$-module $\boldsymbol{W}^{\boldsymbol{\delta}}$.

3. MAIN RESULTS

We start by the constants in the submodule $K[t, u, v]X \oplus K[t, u, v]Y$ of W.

Lemma 1. $(K[t, u, v]X \oplus K[t, u, v]Y)^{\delta} \subset W^{\delta}$ is generated by X and $vX - tY$ as a $K[t, u, v]^{\delta}$-module.

Proof. Let $p(X, Y) = p_1(t, u, v)X + p_2(t, u, v)Y \in K[t, u, v]X \oplus K[t, u, v]Y$ such that $\delta\big(p(X, Y)\big) = 0$. Then we get that

$$0 = \delta(p_1(t,u,v))X + \delta(p_2(t,u,v))Y + p_2(t,u,v)X$$

or

$$\delta(p_1(t,u,v)) + p_2(t,u,v) = 0\,, \qquad \delta(p_2(t,u,v)) = 0,$$

in the free $K[t,u,v]$-submodule generated by X and Y. Thus, $p_2(t,u,v) \in K[t,u,v]^\delta$,

$$\delta^2(p_1(t,u,v)) = \delta\left(\delta(p_1(t,u,v))\right) = 0,$$

and that $p_2(t,u,v) = -\delta(p_1(t,u,v))$. Direct computations give that

$$K[t,u,v]^{\delta^2} = \{w \in K[t,u,v]: \delta^2(w) = 0\} = K[t,u,v]^\delta \oplus vK[t,u,v]^\delta,$$

and hence, there exist $q_1(t,u,v), q_2(t,u,v) \in K[t,u,v]^\delta$ such that

$$p_1(t,u,v) = q_1(t,u,v) + vq_2(t,u,v).$$

Now,

$$\delta(p_1(t,u,v)) = tq_2(t,u,v) = -p_2(t,u,v)$$

and

$$p(X,Y) = \left(q_1(t,u,v) + vq_2(t,u,v)\right)X - tq_2(t,u,v)(t,u,v)Y$$

or

$$p(X,Y) = q_1(t,u,v)X + q_2(t,u,v)(vX - tY).$$

This yields that

$$(K[t,u,v]X \oplus K[t,u,v]Y)^\delta = K[t,u,v]^\delta X \oplus K[t,u,v]^\delta (vX - tY),$$

which completes the proof.

The following theorem is our main result, which descibes the free $K[t,u,v]^\delta$-module structure of the algebra W^δ.

Theorem 2. W^δ is the free $K[t,u,v]^\delta$-module generated by $I, X, vX - tY, [X,Y]$.

Proof. Since, $K[t,u,v]I$ and $K[t,u,v][X,Y]$ are δ-invariant submodules, then by Lemma 1, it is straightforward to see that

$$\begin{aligned} W^\delta &= (K[t,u,v]I \oplus K[t,u,v]X \oplus K[t,u,v]Y \oplus K[t,u,v][X,Y])^\delta \\ &= (K[t,u,v]I)^\delta \oplus (K[t,u,v]X \oplus K[t,u,v]Y)^\delta \oplus (K[t,u,v][X,Y])^\delta \\ &= K[t,u,v]^\delta I \oplus K[t,u,v]^\delta X \oplus K[t,u,v]^\delta (vX - tY) \oplus K[t,u,v]^\delta [X,Y] \end{aligned}$$

which means that $I, X, vX - tY, [X,Y]$ generate the $K[t,u,v]^\delta$-module W^δ. It sufficies to show that these are free generators. Let

$$aI + bX + c(vX - tY) + d[X,Y] = 0$$

for some $a, b, c, d \in K[t,u,v]^\delta$. Then,

$$aI + (b + cv)X - ctY + d[X, Y] = 0$$

or

$$a = b + cv = -ct = d = 0$$

in the free module generated by $I, X, Y, [X, Y]$. Consequently $a = b = b = d = 0$.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

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