**ESKİŞEHİR TEKNİK ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ**
**B- TEORİK BİLİMLER**

Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences

**2023, 11(2), pp. 109-115, DOI10.20290/estubtdb.1211055**

**RESEARCH ARTICLE**

# ON THE KLEIN-4 INVARIANTS

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## ABSTRACT

Let $K[X_4] = K[x_1, x_2, x_3, x_4]$ be the polynomial algebra with 4 algebraically independent commuting variables over a field $K$ of characteristic zero. The symmetric group $S_4$ acts on $K[X_4]$ naturally by the action of permutations exchanging the indices of variables with respect to the corresponding permutation. It is well known that the algebra $K[X_4]^{S_4}$ of all polynomials preserved under the action of $S_4$ is generated by 4 algebraically independent elements called the elementary symmetric polynomials. In this study, we consider the subalgebra $G$ of $S_4$ generated by the transpositions $(13)$ and $(24)$ which is isomorphic to the Klein-4 group, and find a free generating set for the algebra $K[X_4]^G$ of $G$-invariants.

**Keywords:** Action, Invariants, Symmetric group

## 1. INTRODUCTION

The initiation of study of $G$-invariants, where $G$ is a subgroup of the general linear group $GL_n(K)$ for a field $K$ of characteristic zero, dates back to the beginning of the twentieth century. The fourteenth of twenty three problems given by David Hilbert [1] is related to the algebra $K[X_n]^G$ of $G$-invariants of the polynomial algebra $K[X_n]$ in $n$ commuting variables $x_1, \dots, x_n$ over the field $K$. Nagata [2] showed that $K[X_n]^G$ is not finitely generated in general, while it is finitely generated for finite subgroups $G$ of $GL_n(K)$ via Noether [3].

The most interesting group in this theory is the symmetric group $S_n$. The algebra $K[X_n]^{S_n}$ of $S_n$-invariants is called the algebra of symmetric polynomials, and each polynomial in this algebra is called a symmetric polynomial. The action of each permutation $\pi \in S_n$ on a monomial is defined as follows.

$$\pi\left(x_{i_1} \cdots x_{i_k}\right) = x_{\pi(i_1)} \cdots x_{\pi(i_k)}.$$

It is well known by Cayley's Theorem that every group is a subgroup of $S_n$ (see e.g., [4]). In this study, we realize the Klein-4 group $G$ as a subgroup of $S_4$ generated by two transpositions $(13)$ and $(24)$, and describe the algebra $K[X_4]^G$ by providing its generators.

## 2. THE KLEIN-4 INVARIANTS

In this section, we investigate the algebra

$$K[X_4]^G = \{p \in K[X_4]: p(x_1, x_2, x_3, x_4) = p(x_3, x_2, x_1, x_4) = p(x_1, x_4, x_3, x_2)\},$$

where

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Received: 28.11.2022 Published: 28.08.2023

$$G = \langle (13),(24) \rangle = \{(1),(13),(24),(13)(24)\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\},$$

that is isomorphic to the Klein-4 group. Then, we give a finite generating set for $K[X_4]^G$.

**Lemma 1.** $K[X_4]^G = K[\alpha_{ab}, \beta_{ab}: 0 \le a, b]$ , such that

$$\alpha_{ab} = x_1^a x_3^b + x_1^b x_3^a \,, \qquad \beta_{ab} = x_2^a x_4^b + x_2^b x_4^a \,,$$

where $0 \le a, b$.

**Proof.** Let $p \in K[X_4]$ be an arbitrary polynomial. One may express

$$p(x_1,x_2,x_3,x_4) = \sum_{0 \le a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d = \sum_{0 \le a,b,c,d} \varepsilon_{abcd} X^{abcd} \ , \varepsilon_{abcd} \in K,$$

as $= p_1 + p_2 + p_3 + p_4$ , where $p_1, p_2, p_3, p_4$ are of the form

$$p_1(x_1,x_2,x_3,x_4) = \sum_{0 \le a} \varepsilon_{aaaa} X^{aaaa}$$

$$\begin{aligned} p_2(x_1,x_2,x_3,x_4) &= \sum_{0 \le a < b} \left(\varepsilon_{abbb} X^{abbb} + \varepsilon_{bbab} X^{bbab}\right) + \left(\varepsilon_{babb} X^{babb} + \varepsilon_{bbba} X^{bbba}\right) \\ &+ \left(\varepsilon_{baaa} X^{baaa} + \varepsilon_{aaba} X^{aaba}\right) + \left(\varepsilon_{abaa} X^{abaa} + \varepsilon_{aaab} X^{aaab}\right) \\ &+ \left(\varepsilon_{aabb} X^{aabb} + \varepsilon_{baab} X^{baab} + \varepsilon_{bbaa} X^{bbaa} + \varepsilon_{abba} X^{abba}\right) + \varepsilon_{abab} X^{abab} \\ &+ \varepsilon_{baba} X^{baba} = \sum_{0 \le a < b} p_{2,1} + \cdots + p_{2,7} \end{aligned}$$

$$\begin{aligned} p_3(x_1,x_2,x_3,x_4) &= \sum_{0 \le a < b < c} \left(\varepsilon_{aabc} X^{aabc} + \varepsilon_{baac} X^{baac} + \varepsilon_{bcaa} X^{bcaa} + \varepsilon_{acba} X^{acba}\right) \\ &+ \left(\varepsilon_{aacb} X^{aacb} + \varepsilon_{caab} X^{caab} + \varepsilon_{cbaa} X^{cbaa} + \varepsilon_{abca} X^{abca}\right) \\ &+ \left(\varepsilon_{abac} X^{abac} + \varepsilon_{acab} X^{acab}\right) + \left(\varepsilon_{baca} X^{baca} + \varepsilon_{caba} X^{caba}\right) \\ &+ \left(\varepsilon_{bbac} X^{bbac} + \varepsilon_{abbc} X^{abbc} + \varepsilon_{acbb} X^{acbb} + \varepsilon_{bcab} X^{bcab}\right) \\ &+ \left(\varepsilon_{bbca} X^{bbca} + \varepsilon_{cbba} X^{cbba} + \varepsilon_{cabb} X^{cabb} + \varepsilon_{bacb} X^{bacb}\right) \\ &+ \left(\varepsilon_{babc} X^{babc} + \varepsilon_{bcba} X^{bcba}\right) + \left(\varepsilon_{abcb} X^{abcb} + \varepsilon_{cbab} X^{cbab}\right) \\ &+ \left(\varepsilon_{ccab} X^{ccab} + \varepsilon_{accb} X^{accb} + \varepsilon_{abcc} X^{abcc} + \varepsilon_{cbac} X^{cbac}\right) \\ &+ \left(\varepsilon_{ccba} X^{ccba} + \varepsilon_{bcca} X^{bcca} + \varepsilon_{bacc} X^{bacc} + \varepsilon_{cabc} X^{cabc}\right) \\ &+ \left(\varepsilon_{cacb} X^{cacb} + \varepsilon_{cbca} X^{cbca}\right) + \left(\varepsilon_{acbc} X^{acbc} + \varepsilon_{bcac} X^{bcac}\right) \\ &= \sum_{0 \le a < b < c} p_{3,1} + \cdots + p_{3,12} \end{aligned}$$

$$
\begin{aligned}
&p_4(x_1,x_2,x_3,x_4)\\
&\quad = \sum_{0\le a<b<c<d} \left(\varepsilon_{abcd}X^{abcd} + \varepsilon_{adcb}X^{adcb} + \varepsilon_{cbad}X^{cbad} + \varepsilon_{cdab}X^{cdab}\right)\\
&\quad + \left(\varepsilon_{abdc}X^{abdc} + \varepsilon_{acdb}X^{acdb} + \varepsilon_{dbac}X^{dbac} + \varepsilon_{dcab}X^{dcab}\right)\\
&\quad + \left(\varepsilon_{acbd}X^{acbd} + \varepsilon_{adbc}X^{adbc} + \varepsilon_{bcad}X^{bcad} + \varepsilon_{bdac}X^{bdac}\right)\\
&\quad + \left(\varepsilon_{bacd}X^{bacd} + \varepsilon_{bdca}X^{bdca} + \varepsilon_{cabd}X^{cabd} + \varepsilon_{cdba}X^{cdba}\right)\\
&\quad + \left(\varepsilon_{badc}X^{badc} + \varepsilon_{bcda}X^{bcda} + \varepsilon_{dabc}X^{dabc} + \varepsilon_{dcba}X^{dcba}\right)\\
&\quad + \left(\varepsilon_{cadb}X^{cadb} + \varepsilon_{cbda}X^{cbda} + \varepsilon_{dacb}X^{dacb} + \varepsilon_{dbca}X^{dbca}\right)\\
&\quad = \sum_{0\le a<b<c} p_{4,1} + \cdots + p_{4,6}
\end{aligned}
$$

such that $p_{i,j}$ counts the sum in the paranthesis indicated as a sum in the expression of $p_i$ , $i = 1,2,3,4$.

Now let $p \in K[X_4]^G$. Then, clearly $p = \pi(p)$ gives that

$$p_1 + p_2 + p_3 + p_4 = \pi(p_1 + p_2 + p_3 + p_4) = \pi(p_1) + \pi(p_2) + \pi(p_2) + \pi(p_2)$$

and that $\pi(p_1) = p_1$, $\pi(p_2) = p_2, \pi(p_3) = p_3, \pi(p_4) = p_4$, $\pi \in G$, since the elements of the form $p_i$ are $G$-invariants for each $i = 1,2,3,4$, due to the number of distinct powers of the variables in the monomials of corresponding summands.

Initially,

$$X^{aaaa} = x_1^a x_2^a x_3^a x_4^a = \left(\frac{\alpha_{11}\beta_{11}}{4}\right)^a = \sigma_4^a$$

that means

$$p_1(x_1,x_2,x_3,x_4) = \sum_{0\le a} \varepsilon_{aaaa}\sigma_4^a \in K[\sigma_4].$$

Secondly, let us consider $p_2 = \sum_{0\le a<b}(p_{2,1} + \cdots + p_{2,7})$. Recall that

$$p_{2,1}(x_1,x_2,x_3,x_4) = \sum_{0\le a<b} \left(\varepsilon_{abbb}X^{abbb} + \varepsilon_{bbab}X^{bbab}\right).$$

The orbit of the monomial $X^{abbb}$ is

$$X^{abbb}, X^{bbab}, X^{abbb}, X^{bbab}$$

with respect to the group $G$. Similarly the orbit of the monomial $X^{bbab}$ is

$$X^{bbab}, X^{abbb}, X^{bbab}, X^{abbb}.$$

Hence,

$$\pi(p_{2,1}) \in \operatorname{span}_K\{X^{abbb}, X^{bbab}\},$$

or $\pi(p_{2,1}) = p_{2,1}$, for $\pi = (13), (24)$. This implies that

$$\varepsilon_{abbb}X^{abbb} + \varepsilon_{bbab}X^{bbab} = (13)\left(\varepsilon_{abbb}X^{abbb} + \varepsilon_{bbab}X^{bbab}\right) = \varepsilon_{abbb}X^{bbab} + \varepsilon_{bbab}X^{abbb},$$

or

$$(\varepsilon_{abbb}-\varepsilon_{bbab})X^{abbb}+(\varepsilon_{bbab}-\varepsilon_{abbb})X^{bbab}=0,$$

for each pair $(a,b)$. Thus, $\varepsilon_{abbb}=\varepsilon_{bbab}$, $0\le a<b$. Therefore,

$$\begin{aligned}\varepsilon_{abbb}X^{abbb}+\varepsilon_{bbab}X^{bbab}&=\varepsilon_{abbb}\left(X^{abbb}+X^{bbab}\right)\\&=\varepsilon_{abbb}X^{aaaa}\left(X^{0(b-a)(b-a)(b-a)}+X^{(b-a)(b-a)0(b-a)}\right)\\&=\varepsilon_{abbb}\sigma_4^aX^{0(b-a)0(b-a)}\left(X^{00(b-a)0}+X^{(b-a)000}\right)=\varepsilon_{abbb}\sigma_4^a\frac{\beta_{(b-a)(b-a)}}{2}\alpha_{(b-a)0}\end{aligned}$$

and thus

$$p_{2,1}(x_1,x_2,x_3,x_4)=\sum_{0\le a<b}\varepsilon_{abbb}\sigma_4^a\frac{\beta_{(b-a)(b-a)}}{2}\alpha_{(b-a)0}\in K[\sigma_4,\alpha_{a0},\beta_{aa}:0\le a].$$

Similar arguments gives that $\pi(p_{2,i})=p_{2,i}$, $i=2,\dots,7$, $\pi(p_{3,j})=p_{3,j}$, $j=1,\dots,12$, $\pi(p_{4,k})=p_{4,k}$, $k=1,\dots,6$, for $\pi=(13),(24)$, and that

$$p_{2,2}=\sum_{0\le a<b}\varepsilon_{babb}\left(X^{babb}+X^{bbba}\right)=\sum_{0\le a<b}\varepsilon_{babb}\sigma_4^a\frac{\alpha_{(b-a)(b-a)}}{2}\beta_{(b-a)0}\in K[\sigma_4,\alpha_{aa},\beta_{a0}:0\le a],$$

$$p_{2,3}=\sum_{0\le a<b}\varepsilon_{baaa}\left(X^{baaa}+X^{aaba}\right)=\sum_{0\le a<b}\varepsilon_{baaa}\sigma_4^a\alpha_{(b-a)0}\in K[\sigma_4,\alpha_{a0}:0\le a],$$

$$p_{2,4}=\sum_{0\le a<b}\varepsilon_{abaa}\left(X^{abaa}+X^{aaab}\right)=\sum_{0\le a<b}\varepsilon_{baaa}\sigma_4^a\beta_{(b-a)0}\in K[\sigma_4,\beta_{a0}:0\le a],$$

$$p_{2,5}=\sum_{0\le a<b}\varepsilon_{aabb}\left(X^{aabb}+X^{baab}+X^{bbaa}+X^{abba}\right)=\sum_{0\le a<b}\varepsilon_{aabb}\sigma_4^a\alpha_{(b-a)0}\beta_{(b-a)0}$$
$$\in K[\sigma_4,\alpha_{a0},\beta_{a0}:0\le a],$$

$$p_{2,6}=\sum_{0\le a<b}\varepsilon_{abab}X^{abab}=\sum_{0\le a<b}\varepsilon_{aabb}\sigma_4^a\frac{\beta_{(b-a)(b-a)}}{2}\in K[\sigma_4,\beta_{aa}:0\le a],$$

$$p_{2,7}=\sum_{0\le a<b}\varepsilon_{baba}X^{baba}=\sum_{0\le a<b}\varepsilon_{baba}\sigma_4^a\frac{\alpha_{(b-a)(b-a)}}{2}\in K[\sigma_4,\alpha_{aa}:0\le a],$$

$$p_{3,1}=\sum_{0\le a<b<c}\varepsilon_{aabc}\left(X^{aabc}+X^{baac}+X^{bcaa}+X^{acba}\right)=\sum_{0\le a<b<c}\varepsilon_{aabc}\sigma_4^a\alpha_{(b-a)0}\beta_{(c-a)0}$$
$$\in K[\sigma_4,\alpha_{a0},\beta_{a0}:0\le a],$$

$$p_{3,2}=\sum_{0\le a<b<c}\varepsilon_{aacb}\left(X^{aacb}+X^{caab}+X^{cbaa}+X^{abca}\right)=\sum_{0\le a<b<c}\varepsilon_{aabc}\sigma_4^a\alpha_{(c-a)0}\beta_{(b-a)0}$$
$$\in K[\sigma_4,\alpha_{a0},\beta_{a0}:0\le a],$$

$$p_{3,3}=\sum_{0\le a<b<c}\varepsilon_{abac}\left(X^{abac}+X^{acab}\right)=\sum_{0\le a<b<c}\varepsilon_{abac}\sigma_4^a\beta_{(c-a)(b-a)}\in K[\sigma_4,\beta_{ab}:0\le a,b],$$

$$p_{3,4} = \sum_{0\le a<b<c} \varepsilon_{baca}\left(X^{baca} + X^{caba}\right) = \sum_{0\le a<b<c} \varepsilon_{abac}\sigma_4^a \alpha_{(c-b)(b-a)} \in K[\sigma_4, \alpha_{ab}: 0 \le a, b],$$

$$p_{3,5} = \sum_{0\le a<b<c} \varepsilon_{bbac}\left(X^{bbac} + X^{abbc} + X^{acbb} + X^{bcab}\right) = \sum_{0\le a<b<c} \varepsilon_{bbac}\sigma_4^a \alpha_{(b-a)0}\beta_{(c-a)(b-a)} \in K[\sigma_4, \alpha_{a0}, \beta_{ab}: 0 \le a, b],$$

$$p_{3,6} = \sum_{0\le a<b<c} \varepsilon_{bbca}\left(X^{bbca} + X^{cbba} + X^{cabb} + X^{bacb}\right) = \sum_{0\le a<b<c} \varepsilon_{bbca}\sigma_4^a \alpha_{(c-a)(b-a)}\beta_{(b-a)0} \in K[\sigma_4, \alpha_{ab}, \beta_{a0}: 0 \le a, b],$$

$$p_{3,7} = \sum_{0\le a<b<c} \varepsilon_{babc}\left(X^{babc} + X^{bcba}\right) = \sum_{0\le a<b<c} \varepsilon_{babc}\sigma_4^a \frac{\alpha_{(b-a)(b-a)}}{2}\beta_{(c-a)0} \in K[\sigma_4, \alpha_{aa}, \beta_{a0}: 0 \le a],$$

$$p_{3,8} = \sum_{0\le a<b<c} \varepsilon_{abcb}\left(X^{abcb} + X^{cbab}\right) = \sum_{0\le a<b<c} \varepsilon_{abcb}\sigma_4^a \alpha_{(c-a)0}\frac{\beta_{(b-a)(b-a)}}{2} \in K[\sigma_4, \alpha_{a0}, \beta_{aa}: 0 \le a],$$

$$p_{3,9} = \sum_{0\le a<b<c} \varepsilon_{ccab}\left(X^{ccab} + X^{accb} + X^{abcc} + X^{cbac}\right) = \sum_{0\le a<b<c} \varepsilon_{ccab}\sigma_4^a \alpha_{(c-a)0}\beta_{(c-a)(b-a)} \in K[\sigma_4, \alpha_{a0}, \beta_{ab}: 0 \le a, b],$$

$$p_{3,10} = \sum_{0\le a<b<c} \varepsilon_{ccba}\left(X^{ccba} + X^{bcca} + X^{bacc} + X^{cabc}\right) = \sum_{0\le a<b<c} \varepsilon_{ccba}\sigma_4^a \alpha_{(c-a)(b-a)}\beta_{(c-a)0} \in K[\sigma_4, \alpha_{ab}, \beta_{a0}: 0 \le a, b],$$

$$p_{3,11} = \sum_{0\le a<b<c} \varepsilon_{cacb}\left(X^{cacb} + X^{cbca}\right) = \sum_{0\le a<b<c} \varepsilon_{cacb}\sigma_4^a \frac{\alpha_{(c-a)(c-a)}}{2}\beta_{(b-a)0} \in K[\sigma_4, \alpha_{ab}, \beta_{a0}: 0 \le a, b],$$

$$p_{3,12} = \sum_{0\le a<b<c} \varepsilon_{acbc}\left(X^{acbc} + X^{bcac}\right) = \sum_{0\le a<b<c} \varepsilon_{acbc}\sigma_4^a \alpha_{(b-a)0}\frac{\beta_{(c-a)(c-a)}}{2} \in K[\sigma_4, \alpha_{a0}, \beta_{ab}: 0 \le a, b],$$

$$p_{4,1} = \sum_{0\le a<b<c<d} \varepsilon_{abcd}\left(X^{abcd} + X^{adcb} + X^{cbad} + X^{cdab}\right) = \sum_{0\le a<b<c<d} \varepsilon_{abcd}\sigma_4^a \alpha_{(c-a)0}\beta_{(d-a)(b-a)} \in K[\sigma_4, \alpha_{a0}, \beta_{ab}: 0 \le a, b],$$

$$p_{4,2} = \sum_{0\le a<b<c<d} \varepsilon_{abdc}\left(X^{abdc} + X^{acdb} + X^{dbac} + X^{dcab}\right) = \sum_{0\le a<b<c<d} \varepsilon_{abdc}\sigma_4^a \alpha_{(d-a)0}\beta_{(c-a)(b-a)} \in K[\sigma_4, \alpha_{a0}, \beta_{ab}: 0 \le a, b],$$

$$p_{4,3} = \sum_{0\le a<b<c<d} \varepsilon_{acbd}\left(X^{acbd} + X^{adbc} + X^{bcad} + X^{bdac}\right) = \sum_{0\le a<b<c<d} \varepsilon_{acbd}\sigma_4^a \alpha_{(b-a)0}\beta_{(d-a)(c-a)} \in K[\sigma_4, \alpha_{a0}, \beta_{ab}: 0 \le a, b],$$

$$p_{4,4} = \sum_{0\le a<b<c<d} \varepsilon_{bacd}\left(X^{bacd} + X^{bdca} + X^{cabd} + X^{cdba}\right) = \sum_{0\le a<b<c<d} \varepsilon_{bacd}\sigma_4^a \alpha_{(c-a)(b-a)}\beta_{(d-a)0}$$
$$\in K[\sigma_4, \alpha_{ab}, \beta_{a0}: 0 \le a, b],$$

$$p_{4,5} = \sum_{0\le a<b<c<d} \varepsilon_{badc}\left(X^{badc} + X^{bcda} + X^{dabc} + X^{dcba}\right) = \sum_{0\le a<b<c<d} \varepsilon_{badc}\sigma_4^a \alpha_{(d-a)(b-a)}\beta_{(c-a)0}$$
$$\in K[\sigma_4, \alpha_{ab}, \beta_{a0}: 0 \le a, b],$$

$$p_{4,6} = \sum_{0\le a<b<c<d} \varepsilon_{cadb}\left(X^{cadb} + X^{cbda} + X^{dacb} + X^{dbca}\right) = \sum_{0\le a<b<c<d} \varepsilon_{cadb}\sigma_4^a \alpha_{(d-a)(c-a)}\beta_{(b-a)0}$$
$$\in K[\sigma_4, \alpha_{ab}, \beta_{a0}: 0 \le a, b].$$

This yields that $K[X_4]^G \subseteq K[\alpha_{ab}, \beta_{ab}: 0 \le a, b]$. Conversely, it is straightforward to show that the elements $\alpha_{ab}, \beta_{ab}$, $0 \le a, b$, are $G$-invariants, which completes the proof.

**Remark 2.** Note that $K[X_4]^{S_4} = K[\sigma_1, \sigma_2, \sigma_3, \sigma_4] \subseteq K[X_4]^G = K[\alpha_{ab}, \beta_{ab}: 0 \le a, b]$, where

$$\sigma_1 = x_1 + x_2 + x_3 + x_4\,,$$
$$\sigma_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\,,$$
$$\sigma_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4\,,$$
$$\sigma_4 = x_1x_2x_3x_4\,,$$

which can be verified by the simple computations given below.

$$\sigma_1 = \alpha_{10} + \beta_{10}\,, \qquad \sigma_2 = \alpha_{10}\beta_{10} + \frac{\alpha_{11}}{2} + \frac{\beta_{11}}{2}\,,$$
$$\sigma_3 = \frac{\alpha_{11}\beta_{10} + \alpha_{10}\beta_{11}}{2}\,, \qquad \sigma_4 = \frac{\alpha_{11}\beta_{11}}{4}.$$

The next theorem is our main result.

**Theorem 3.** The algebra $K[X_4]^G$ is freely generated by the set $\{\alpha_{10}, \alpha_{11}, \beta_{10}, \beta_{11}\}$.

**Proof.** Firstly, direct computations give that

$$\alpha_{ab} = \frac{\alpha_{11}\alpha_{(a-1)(b-1)}}{2}\,, \qquad \beta_{ab} = \frac{\beta_{11}\beta_{(a-1)(b-1)}}{2}$$

for $1 \le a, b$. This yields that the elements of the form $\alpha_{ab}, \beta_{ab}, 1 \le a, b$, are included in the algebra generated by $\alpha_{11}, \beta_{11}, \alpha_{n0}, \beta_{n0}, 1 \le n$, by induction.

Let $2 \le n = 2m$ be an even positive integer. Then by binomial expansion, we have that

$$\alpha_{10}^n = \alpha_{n0} + n(x_1^{n-1}x_3 + x_1x_3^{n-1}) + \cdots + \binom{n}{m-1}(x_1^{m+1}x_3^{m-1} + x_1^{m-1}x_3^{m+1}) + \binom{n}{m}\left(\frac{\alpha_{11}}{2}\right)^m$$
$$\alpha_{n0} = \alpha_{10}^n - n\alpha_{(n-2)0}\frac{\alpha_{11}}{2} - \cdots - \binom{n}{m-1}\frac{\alpha_{11}}{2}\alpha_{m(m-2)} - \binom{n}{m}\left(\frac{\alpha_{11}}{2}\right)^m$$

and hence, $\alpha_{n0} = \alpha_{(2m)0}$ is included in the algebra generated by the elements $\alpha_{10}, \alpha_{11}$ by induction.

Now let $3 \le n = 2m + 1$ be an odd positive integer. Then,

$$\alpha_{10}^{n} = \alpha_{n0} + n(x_1^{n-1}x_3 + x_1x_3^{n-1}) + \cdots + \binom{n}{m}(x_1^{m+1}x_3^{m-1} + x_1^{m-1}x_3^{m+1})$$
$$\alpha_{n0} = \alpha_{10}^{n} - n\alpha_{(n-2)0}\frac{\alpha_{11}}{2} - \cdots - \binom{n}{m}\frac{\alpha_{11}}{2}\alpha_{m(m-2)}$$

and thus, $\alpha_{n0} = \alpha_{(2m+1)0}$ is included in $K[\alpha_{10}, \alpha_{11}]$. Similarly one may show that $\beta_{n0} \in K[\beta_{10}, \beta_{11}]$ for all $2 \leq n$.

The rest is to show that the elements $\alpha_{10}, \alpha_{11}, \beta_{10}, \beta_{11}$ are algebraically independent. For this purpose, we apply the Jacobian criterion [5]. The determinant

$$\begin{vmatrix} 1 & z & 0 & 0 \\ 0 & 0 & 1 & t \\ 1 & x & 0 & 0 \\ 0 & 0 & 1 & y \end{vmatrix}$$

filled by the entries with respect to the partial derivatives of the corresponding elements is nonzero, that completes the proof.

**CONFLICT OF INTEREST**

The authors stated that there are no conflicts of interest regarding the publication of this article.

## AUTHORSHIP CONTRIBUTIONS

The authors contributed equally to this work

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