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Research Article

Characterizations of Lorentzian Para-Sasakian Manifolds with respect to the Schouten-van Kampen Connection

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Abstract
The object of the present paper is to study a Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection.

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1. Introduction

The semi-Riemannian geometry attracts researchers because of its capabilities to resolve the many issues of science, technology, and medical, and their allied areas. A differentiable manifold M of dimension n equipped with a semi-Riemannian metric g, whose signature is (p,q), $(p+q=n)$, known as an n-dimensional semi-Riemannian manifold. In particular, if we take $p=1$, $q=n-1$, or $p=n-1$, $q=1$, then the semi-Riemannian manifold M converts into the well-known Lorentzian manifold. To start the study of Lorentzian manifold M, the causal character of the vectors play a significant role and hence it becomes the convenient choice for the researchers to study the general theory of relativity and cosmology. Space-time is the stage of the present modeling of the physical world: a torsionless, time-oriented Lorentzian manifold. In describing the gravity of the space-time, the Riemannian curvature R, the Ricci tensor S, and the scalar curvature τ play a crucial role.

In [1], K. Matsumoto introduced the notion of Lorentzian para-Sasakian manifolds. In [2], the authors defined the same notion independently and they obtained many results about this type of manifolds (see also [3], and [4]). Several authors have studied Lorentzian para-Sasakian manifolds such as [5–7], and many others.

A Lorentzian para-Sasakian manifold M^n is said to be an η-*Einstein manifold* if the following condition

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{1}$$

holds on M^n, where a,b are smooth functions.

By definition, the conformal curvature tensor C, the projective curvature tensor P, and the conharmonical curvature tensor K are given by [8]

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z-\frac{1}{n-2}\left[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\right] \\ &\quad +\frac{\tau}{(n-1)(n-2)}\left[g(Y,Z)X-g(X,Z)Y\right], \end{aligned} \tag{2}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[S(Y,Z)X - S(X,Z)Y\right], \tag{3}$$

$$K(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right], \tag{4}$$

where R, S, Q, and τ denote the curvature tensor, Ricci tensor, Ricci operator and scalar curvature of M, respectively. For $\dim M > 3$, if $C = 0$, then the manifold is called *conformally flat* manifold.

In the present paper, we study Lorentzian para-Sasakian manifolds with respect to the Schouten-van Kampen connection. The paper is organized as follows: After the introduction, in section 2, firstly we give Lorentzian para-Sasakian manifolds and the Schouten-van Kampen connection. Then we adapt the Schouten-van Kampen connection on Lorentzian para-Sasakian manifolds. In section 3, we study conformally flat, projectively flat, and conharmonically flat Lorentzian para-Sasakian manifolds with respect to the Schouten-van Kampen connection. Also, we investigate Lorentzian para-Sasakian manifolds satisfying the conditions $\breve{R}\cdot\breve{Q} = 0$, $\breve{Q}\cdot\breve{R} = 0$ and $\breve{R}\cdot\breve{S} = 0$ with respect to the Schouten-van Kampen connection, respectively. In the last section, we give an example of a 3-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection which verifies our some corollaries.

2. Preliminaries

Let M^n be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form on M^n such that

$$\eta(\xi) = -1, \tag{5}$$

$$\phi^2 = I + \eta\otimes\xi, \tag{6}$$

which implies

$$i.\ \phi\xi = 0, \quad ii.\ \eta(\phi) = 0, \quad iii.\ rank(\phi) = n-1. \tag{7}$$

Then M^n admits a Lorentzian metric g, such that

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{8}$$

and M^n is said to admit a *Lorentzian almost paracontact structure* (ϕ, ξ, η, g). In this case, we have

$$g(X,\xi) = \eta(X), \quad \nabla_X\xi = \phi X, \tag{9}$$

$$\Omega(X,Y) = g(X,\phi Y) = g(\phi X,Y) = \Omega(Y,X).$$

In equations (5) and (6) if we replace ξ with $-\xi$, then the triple (ϕ, ξ, η) is an almost paracontact structure on M^n defined by Sato ([9]). The Lorentzian metric given by equation (9) stands analogous to the almost paracontact Riemannian metric for any almost paracontact manifold (see [9, 10]).

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called *Lorentzian paracontact manifold* [1] if

$$\Omega(X,Y) = \frac{1}{2}((\nabla_X\eta)Y + (\nabla_Y\eta)X).$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called *Lorentzian para-Sasakian manifold* [1] if

$$(\nabla_X\phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In a Lorentzian para-Sasakian manifold the 1-form η is closed. Also in [1], it is proved that if an n-dimensional Lorentzian para-Sasakian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X\nabla_Y\eta)W = g(X,Y)\eta(W) + g(X,W)\eta(Y) + 2\eta(X)\eta(Y)\eta(W),$$

then M^n admits a Lorentzian para-Sasakian structure. It is noticed that the n-dimensional Lorentzian para-Sasakian manifold M satisfies the following relations:

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, &(10)\\ R(\xi,X)Y &= g(X,Y)\xi - \eta(Y)X, &(11)\\ S(X,\xi) &= (n-1)\eta(X), &(12)\\ \eta(R(X,Y)Z) &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y), &(13)\end{aligned}$$

for all $X,Y,Z \in \chi(M)$, where R and S denote the curvature tensor and the Ricci tensor of M, respectively.

On the other hand, we have two naturally defined distributions in the tangent bundle TM of M as follows:

$$H = \ker\eta, \quad V = \text{span}\{\xi\}. \tag{14}$$

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$, and $H \perp V$. For any $X \in TM$, by X^h and X^v we denote the projections of X onto H and V, respectively. Thus, we have $X = X^h + X^v$ with

$$X^h = X + \eta(X)\xi, \quad X^v = -\eta(X)\xi. \tag{15}$$

The Schouten-van Kampen connection $\breve{\nabla}$ associated with the Levi-Civita connection ∇ and adapted to the pair of the distributions (H,V) is defined by [11]

$$\breve{\nabla}_X Y = (\nabla_X Y^h)^h + (\nabla_X Y^v)^v, \tag{16}$$

and the corresponding second fundamental form B is defined by $B = \nabla - \breve{\nabla}$. Note that condition (16) implies the parallelism of the distributions H and V with respect to the Schouten-van Kampen connection $\breve{\nabla}$.

From equation (15), one can compute

$$\begin{aligned} (\nabla_X Y^h)^h &= \nabla_X Y + \eta(\nabla_X Y)\xi + \eta(Y)\nabla_X \xi,\\ (\nabla_X Y^v)^v &= -(\nabla_X \eta)(Y)\xi - \eta(\nabla_X Y)\xi,\end{aligned}$$

which enable us to express the Schouten-van Kampen connection with help of the Levi-Civita connection in the following way [12]. This decomposition allows one to define the Schouten-van Kampen connection $\breve{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\breve{\nabla}$ on an almost (para) contact metric manifold with respect to Levi-Civita connection ∇ is defined by [12]

$$\breve{\nabla}_X Y = \nabla_X Y + \eta(Y)\nabla_X \xi - (\nabla_X \eta)(Y)\xi. \tag{17}$$

Thus with the help of the Schouten-van Kampen connection (17), many properties of some geometric objects connected with the distributions H, V can be characterized [12–15]. For example g, ξ and η are parallel with respect to $\breve{\nabla}$, that is, $\breve{\nabla}\xi = 0$, $\breve{\nabla}g = 0$, $\breve{\nabla}\eta = 0$. Also the torsion $\breve{T}$ of $\breve{\nabla}$ is defined by

$$\breve{T}(X,Y) = \eta(Y)\nabla_X \xi - \eta(X)\nabla_Y \xi - 2d\eta(X,Y)\xi. \tag{18}$$

Now we consider a Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Firstly, using equations (9) and (3) in (17), we get

$$\breve{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{19}$$

Theorem 1. *Let (M,ϕ,ξ,η,g) be a Lorentzian para-Sasakian manifold. The Schouten-van Kampen connection $\breve{\nabla}$ associated to the Levi-Civita connection ∇ of M and adapted to the pair (14) is just the only one affine connection, which is metric and its torsion has the form (18).*

Proof. It is well-known that a metric connection can be stated with the help of its torsion tensor field as follow:

$$g(\breve{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}g(\breve{T}(X,Y),Z) - \frac{1}{2}g(\breve{T}(X,Z),Y) - \frac{1}{2}g(\breve{T}(Y,Z),X).$$

By using equation (18), we get

$$\begin{aligned} g(\breve{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2}\eta(Y)g(\phi X, Z) - \frac{1}{2}\eta(X)g(\phi Y, Z) - \frac{1}{2}\eta(Z)g(\phi X, Y)\\ &\quad + \frac{1}{2}\eta(X)g(\phi Z, Y) - \frac{1}{2}\eta(Z)g(\phi Y, X) + \frac{1}{2}\eta(Y)g(\phi Z, X),\end{aligned}$$

which implies

$$g(\breve{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y),$$

that is, equation (19) is satisfied. ■

Let R and $\breve{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\breve{\nabla}$ given by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \breve{R}(X,Y) = [\breve{\nabla}_X, \breve{\nabla}_Y] - \breve{\nabla}_{[X,Y]},$$

respectively. If we substitute equation (19) in the definition of the Riemannian curvature tensor, we have

$$\breve{R}(X,Y)Z = \breve{\nabla}_X \breve{\nabla}_Y Z - \breve{\nabla}_Y \breve{\nabla}_X Z - \breve{\nabla}_{[X,Y]} Z. \tag{20}$$

Using equation (17) in equation (20), we have

$$\begin{aligned} \breve{R}(X,Y)Z &= R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y. \end{aligned} \tag{21}$$

Now taking the inner product in equation (21) with a vector field W, we have

$$\begin{aligned} g(\breve{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\ &\quad + g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ &\quad + g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z). \end{aligned} \tag{22}$$

If we take $X = W = e_i$, $\{i = 1,...,n\}$, in equation (22), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$\breve{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z), \tag{23}$$

where $\breve{S}$ and S denote the Ricci tensor of the connections $\breve{\nabla}$ and ∇, respectively. As a consequence of equation (23), we obtain

$$\breve{Q}Y = QY + (n-1)\eta(Y)\xi. \tag{24}$$

Also if we take $Y = Z = e_i$, $\{i = 1,...,n\}$, in equation (23), we have

$$\breve{r} = r + n - 1. \tag{25}$$

3. Main results

In this section, we give the main results of the paper.

Let M^n be a Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Then using equations (2)-(4) and equations (22)-(25), we can write the followings:

$$\begin{aligned} \breve{C}(X,Y)Z &= C(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + \frac{1}{n-2}[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi \\ &\quad + g(X,Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y], \end{aligned} \tag{26}$$

$$\breve{P}(X,Y)Z = P(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \tag{27}$$

$$\begin{aligned} \breve{K}(X,Y)Z &= K(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad - \frac{1}{n-2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{28}$$

Now let M be a conformally flat manifold with respect to the Schouten-van Kampen connection. Thus, from equation (26) we have

$$\begin{aligned} C(X,Y)Z &= g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y - \frac{1}{n-2}[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \frac{1}{n-2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{29}$$

Putting $X=\xi$ in equation (29), we obtain

$$C(\xi,Y)Z+\frac{2}{n-2}[g(Y,Z)\xi-\eta(Z)Y]=0, \tag{30}$$

that is,

$$R(\xi,Y)Z-\frac{1}{n-2}[S(Y,Z)\xi-S(\xi,Z)Y+g(Y,Z)Q\xi-\eta(Z)QY]+\{\frac{\tau+2(n-1)}{(n-1)(n-2)}\}[g(Y,Z)\xi-\eta(Z)Y]=0. \tag{31}$$

Using equations (11) and (12) in equation (31), we get

$$\left(\frac{n(n-1)+\tau}{(n-1)(n-2)}\right)[g(Y,Z)\xi-\eta(Z)Y]-\frac{1}{n-2}[S(Y,Z)\xi-(n-1)\eta(Z)Y+(n-1)g(Y,Z)\xi-\eta(Z)QY]=0. \tag{32}$$

Multiplying equation (32) with ξ, we obtain

$$\left(\frac{n(n-1)+\tau}{(n-1)(n-2)}\right)[g(Y,Z)+\eta(Z)\eta(Y)]-\frac{1}{n-2}[S(Y,Z)+2(n-1)\eta(Z)\eta(Y)+(n-1)g(Y,Z)]=0.$$

i.e.,

$$S(Y,Z)=(1+\frac{\tau}{n-1})g(Y,Z)-(n-2-\frac{\tau}{n-1})\eta(Y)\eta(Z). \tag{33}$$

Hence the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection. Now using equation (33) in equation (23), we get

$$\breve{S}(Y,Z)=(1+\frac{\tau}{n-1})[g(Y,Z)+\eta(Y)\eta(Z)]. \tag{34}$$

Thus the manifold M is also an η-Einstein manifold with respect to the Schouten-van Kampen connection.

Now we can state the following:

Theorem 2. *Let M be a conformally flat n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Then the manifold M is an η-Einstein manifold with respect to the Levi-Civita connection and the Schouten-van Kampen connection.*

Now we consider the manifold M is a projectively flat manifold with respect to the Schouten-van Kampen connection. Thus, we have

$$\breve{R}(X,Y)Z=\frac{1}{n-1}[\breve{S}(Y,Z)X-\breve{S}(X,Z)Y]. \tag{35}$$

Using equations (21) and (23) in equation (35), we get

$$\begin{aligned}&R(X,Y)Z+g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X\\&+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\\=&\frac{1}{n-1}[S(Y,Z)X-S(X,Z)Y]+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X.\end{aligned} \tag{36}$$

Putting $X=\xi$ in equation (36), we obtain

$$R(\xi,Y)Z-g(Y,Z)\xi+\eta(Z)Y=\frac{1}{n-1}[S(Y,Z)\xi-S(\xi,Z)Y]-\eta(Y)\eta(Z)\xi-\eta(Z)Y,$$

i.e.,

$$0=S(Y,Z)\xi-n\eta(Z)Y-\eta(Z)\eta(Y)\xi. \tag{37}$$

Multiplying equation (37) with ξ , we have

$$S(Y,Z)=-(n-1)\eta(Z)\eta(Y). \tag{38}$$

Using equation (38) in equation (23), we get

$$\breve{S}(Y,Z) = 0.$$

Thus the manifold M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection. Hence from equation (35), the manifold M is a flat manifold with respect to the Schouten-van Kampen connection.

Conversely, let M be a flat manifold with respect to the Schouten-van Kampen connection. Then we say that the manifold M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection. Hence from equation (27), we get $\breve{P}(X,Y)Z = 0$, that is, the manifold M is a projectively flat manifold with respect to the Schouten-van Kampen connection.

Thus we have the following:

Theorem 3. *Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Then the following statements are equivalent:*

1. *M is projectively flat with respect to the Schouten-van Kampen connection.*
2. *M is Ricci flat with respect to the Shouten-van Kampen connection.*
3. *M is flat with respect to the Schouten-van Kampen connection.*

Now we consider the manifold M is a conharmonically flat manifold with respect to the Schouten-van Kampen connection. Thus, we can write

$$\breve{R}(X,Y)Z = \frac{1}{n-2}\left[\breve{S}(Y,Z)X - \breve{S}(X,Z)Y + g(Y,Z)\breve{Q}X - g(X,Z)\breve{Q}Y\right]. \tag{39}$$

Using equations (21), (23) and (24) in equation (39), we get

$$R(X,Y)Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X - \frac{1}{n-2}\begin{bmatrix} S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ -g(X,Z)QY + g(Y,Z)\eta(X)\xi \\ -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \end{bmatrix} = 0. \tag{40}$$

Putting $X = \xi$ in equation (40), we obtain

$$R(\xi,Y)Z - \frac{1}{n-2}\begin{bmatrix} S(Y,Z)\xi - S(\xi,Z)Y + g(Y,Z)Q\xi \\ -g(\xi,Z)QY - g(Y,Z)\xi + \eta(Z)Y \end{bmatrix} = 0. \tag{41}$$

Using equations (11) and (12) in equation (41), we get

$$g(Y,Z)\xi - \eta(Z)Y - \frac{1}{n-2}\begin{bmatrix} S(Y,Z)\xi - (n-1)\eta(Z)Y + (n-1)g(Y,Z)\xi \\ -\eta(Z)QY - g(Y,Z)\xi + \eta(Z)Y \end{bmatrix} = 0. \tag{42}$$

Multiplying equation (42) with ξ, we have

$$g(Y,Z) + \eta(Z)\eta(Y) + \frac{1}{n-2}\begin{bmatrix} -S(Y,Z) - (n-1)\eta(Y)\eta(Z) - (n-1)g(Y,Z) \\ -(n-1)\eta(Y)\eta(Z) + g(Y,Z) + \eta(Y)\eta(Z) \end{bmatrix} = 0,$$

i.e.,

$$S(Y,Z) = -(n-1)\eta(Y)\eta(Z). \tag{43}$$

Thus using equations (43) in equation (23), we get

$$\breve{S}(Y,Z) = 0.$$

which implies M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection. Thus from equation (39) the manifold M is a flat manifold with respect to the Schouten-van Kampen connection.

Conversely, let M be a flat manifold with respect to the Schouten-van Kampen connection. Then we say that the manifold M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection. Hence from equation (28), we get $\breve{K}(X,Y)Z = 0$, that is, the manifold M is a conharmonically flat manifold with respect to the Schouten-van Kampen connection.

Now we have the following:

Theorem 4. *Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Then the following statements are equivalent:*

1. *M is conharmonically flat with respect to the Schouten-van Kampen connection.*
2. *M is Ricci flat with respect to the Shouten-van Kampen connection.*
3. *M is flat with respect to the Schouten-van Kampen connection.*

Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection satisfying the condition $\breve{R}\cdot\breve{Q}=0$. Then we can write

$$(\breve{R}(X,Y)\breve{Q})Z=\breve{R}(X,Y)\breve{Q}Z-\breve{Q}\breve{R}(X,Y)Z=0. \tag{44}$$

Using equation (24) in equation (44), we have

$$\breve{R}(X,Y)QZ-Q\breve{R}(X,Y)Z=0. \tag{45}$$

Now using equation (21) in equation (45), we obtain

$$\begin{aligned}
&R(X,Y)QZ-QR(X,Y)Z+g(\phi X,QZ)\phi Y\\
&-g(\phi Y,QZ)\phi X+g(Y,QZ)\eta(X)\xi-g(X,QZ)\eta(Y)\xi\\
&+\eta(Y)\eta(QZ)X-\eta(X)\eta(QZ)Y-g(X,\phi Z)Q\phi Y\\
&+g(Y,\phi Z)Q\phi X-g(Y,Z)\eta(X)Q\xi+g(X,Z)\eta(Y)Q\xi\\
&-\eta(Y)\eta(Z)QX+\eta(X)\eta(Z)QY=0.
\end{aligned} \tag{46}$$

Suppose that the manifold M is satisfying the condition $R\cdot Q=0$. Then equation (46) turns to

$$\begin{aligned}
&g(\phi X,QZ)\phi Y-g(\phi Y,QZ)\phi X+g(Y,QZ)\eta(X)\xi\\
&-g(X,QZ)\eta(Y)\xi+\eta(Y)\eta(QZ)X-\eta(X)\eta(QZ)Y\\
&-g(X,\phi Z)Q\phi Y+g(Y,\phi Z)Q\phi X-g(Y,Z)\eta(X)Q\xi\\
&+g(X,Z)\eta(Y)Q\xi-\eta(Y)\eta(Z)QX+\eta(X)\eta(Z)QY=0.
\end{aligned} \tag{47}$$

Multiplying equation (47) with W, we obtain

$$\begin{aligned}
&g(\phi X,QZ)g(\phi Y,W)-g(\phi Y,QZ)g(\phi X,W)+g(Y,QZ)\eta(X)\eta(W)\\
&-g(X,QZ)\eta(Y)\eta(W)+\eta(Y)\eta(QZ)g(X,W)-\eta(X)\eta(QZ)g(Y,W)\\
&-g(X,\phi Z)g(Q\phi Y,W)+g(Y,\phi Z)g(Q\phi X,W)-g(Y,Z)\eta(X)g(Q\xi,W)\\
&+g(X,Z)\eta(Y)g(Q\xi,W)-\eta(Y)\eta(Z)g(QX,W)+\eta(X)\eta(Z)g(QY,W)=0.
\end{aligned} \tag{48}$$

Taking $X=\xi$ in equation (48), we have

$$S(Y,Z)\eta(W)-S(Z,\xi)g(Y,W)-g(Y,Z)S(\xi,W)+\eta(Z)S(Y,W)=0. \tag{49}$$

Again taking $W=\xi$ in equation (49), we get

$$S(Y,Z)=(n-1)g(Y,Z). \tag{50}$$

Hence the manifold M is an Einstein manifold with respect to the Levi-Civita connection. Using equation (50) in equation (23), we get

$$\breve{S}(Y,Z)=(n-1)g(Y,Z)+(n-1)\eta(Y)\eta(Z). \tag{51}$$

Thus the manifold M is an η-Einstein manifold with respect to the Schouten-van Kampen connection.

Conversely, let the manifold M is an Einstein manifold with respect to the Levi-Civita connection and an η-Einstein manifold with respect to the Schouten-van Kampen connection. Then from equations (50) and (51), we have

$$\breve{Q}Y=(n-1)Y+(n-1)\eta(Y)\xi, \tag{52}$$

and

$$QY=(n-1)Y, \tag{53}$$

respectively. Now using equations (52) and (53) in equations (45) and (46), we have $R\cdot Q=0$ and $\breve{R}\cdot\breve{Q}=0$, respectively.

Now we give the following:

Theorem 5. *Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection satisfying the condition $\breve{R}\cdot\breve{Q}=0$. Then the following statements are equivalent:*

1. *If M is satisfying the condition $R\cdot Q=0$, then M is an Einstein manifold with respect to the Levi-Civita connection.*
2. *M is an η-Einstein manifold with respect to the Schouten-van Kampen connection.*

Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection satisfying the condition $\breve{Q}\cdot\breve{R}=0$. Then we can write

$$\breve{Q}\breve{R}(X,Y)Z-\breve{R}(\breve{Q}X,Y)Z-\breve{R}(X,\breve{Q}Y)Z-\breve{R}(X,Y)\breve{Q}Z=0. \tag{54}$$

Using equation (24) in equation (54), we have

$$Q\breve{R}(X,Y)Z-\breve{R}(QX,Y)Z-\breve{R}(X,QY)Z-\breve{R}(X,Y)QZ=0, \tag{55}$$

which

$$\begin{aligned}
&QR(X,Y)Z-R(QX,Y)Z-R(X,QY)Z-R(X,Y)QZ\\
&+g(X,\phi Z)Q\phi Y-g(Y,\phi Z)Q\phi X+g(Y,Z)\eta(X)Q\xi-g(X,Z)\eta(Y)Q\xi\\
&+\eta(Y)\eta(Z)QX-\eta(X)\eta(Z)QY+g(Y,\phi Z)\phi QX-g(X,\phi Z)\phi QY\\
&+g(X,Z)S(Y,\xi)\xi-g(Y,Z)S(X,\xi)\xi+2S(X,Z)\eta(Y)\xi-2S(Y,Z)\eta(X)\xi\\
&+S(X,\xi)\eta(Z)Y-S(Y,\xi)\eta(Z)X+S(Y,\phi Z)\phi X-S(X,\phi Z)\phi Y\\
&+\eta(X)\eta(Z)QY-\eta(Y)\eta(Z)QX+g(\phi Y,QZ)\phi X-g(\phi X,QZ)\phi Y\\
&+S(Z,\xi)\eta(X)Y-S(Z,\xi)\eta(Y)X=0.
\end{aligned} \tag{56}$$

Suppose that the manifold M is satisfying the condition $Q\cdot R=0$. Taking $X=\xi$ in equation (56), then we have

$$2S(Y,Z)\xi-g(Y,Z)Q\xi-g(Y,Z)S(\xi,\xi)\xi+S(\xi,Z)\eta(Y)\xi+S(\xi,\xi)\eta(Z)Y-\eta(Y)\eta(Z)Q\xi-S(Z,\xi)Y=0. \tag{57}$$

Multiplying equation (57) with ξ, we obtain

$$S(Y,Z)=-(n-1)\eta(Z)\eta(Y). \tag{58}$$

Using equation (58) in equation (23), we get

$$\breve{S}(Y,Z)=0.$$

Thus the manifold M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection.

Conversely if the manifold M is a Ricci-flat manifold with the condition $Q\cdot R=0$, then the condition $\breve{Q}\cdot\breve{R}=0$ with respect to the Schouten-van Kampen connection is always satisfied on M.

Now we give the following:

Theorem 6. *Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection satisfying the condition $\breve{Q}\cdot\breve{R}=0$ with the condition $Q\cdot R=0$ if and only if M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection.*

Definition 7. *A semi-Riemannian manifold $(M^n,g), n>3$, is said to be Ricci semisymmetric if*

$$R(X,Y)\cdot S=0,$$

holds on M for all $U,W\in\chi(M)$.

Let M be an n-dimensional Ricci semisymmetric Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Then we can write

$$(\breve{R}(X,Y)\cdot\breve{S})(Z,W)=0,$$

which implies

$$\breve{S}(\breve{R}(X,Y)Z,W)+\breve{S}(Z,\breve{R}(X,Y)W)=0. \tag{59}$$

Using equation (23) in equation (59), we obtain

$$S(\breve{R}(X,Y)Z,W)+S(Z,\breve{R}(X,Y)W)=0. \tag{60}$$

Now using equation (21) in equation (60), we obtain

$$\begin{aligned}
&S(R(X,Y)Z,W)+S(R(X,Y)W,Z)+g(X,\phi Z)S(\phi Y,W)\\
&-g(Y,\phi Z)S(\phi X,W)+g(Y,Z)\eta(X)S(\xi,W)-g(X,Z)\eta(Y)S(\xi,W)\\
&+\eta(Y)\eta(Z)S(X,W)-\eta(X)\eta(Z)S(Y,W)+g(X,\phi W)S(\phi Y,Z)\\
&-g(Y,\phi W)S(\phi X,Z)+g(Y,W)\eta(X)S(\xi,Z)-g(X,W)\eta(Y)S(\xi,Z)\\
&+\eta(Y)\eta(W)S(X,Z)-\eta(X)\eta(W)S(Y,Z)=0.
\end{aligned} \tag{61}$$

Suppose that the manifold M is satisfying the condition $R\cdot S=0$. Taking $X=\xi$ in equation (61) and from equation (11), we have

$$(n-1)g(Y,Z)\eta(W)-\eta(Z)S(Y,W)+(n-1)g(Y,W)\eta(Z)-\eta(W)S(Y,Z)=0. \tag{62}$$

Now taking $W=\xi$ in equation (62), we get

$$S(Y,Z)=(n-1)g(Y,Z). \tag{63}$$

Hence the manifold M is an Einstein manifold with respect to the Levi-Civita connection. Using equation (63) in equation (23), we get

$$\breve{S}(Y,Z)=(n-1)g(Y,Z)+(n-1)\eta(Y)\eta(Z). \tag{64}$$

Thus the manifold M is an η-Einstein manifold with respect to the Schouten-van Kampen connection.

Conversely, we can consider the proof of Theorem 5.

Then we give the following:

Theorem 8. *Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection satisfying the condition $\breve{R}\cdot\breve{S}=0$. Then the following statements are equivalent:*

1. *If M is satisfying the condition $R\cdot S=0$, then M is an Einstein manifold with respect to the Levi-Civita connection.*
2. *M is an η-Einstein manifold with respect to the Schouten-van Kampen connection.*

4. Example

We consider a 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3\}$, where (x,y,z) are the standard coordinates of $\mathbb{R}^3$. Let $\{e_1,e_2,e_3\}$ be linearly independent global frame on M given by

$$e_1=e^z\frac{\partial}{\partial x},\quad e_2=e^z\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z},$$

where a is non-zero constant. Let g be the Lorentzian metric, η be the 1-form and ϕ be the $(1,1)$-tensor field defined by

$$g(e_1,e_3)=g(e_1,e_2)=g(e_2,e_3)=0,\ \ g(e_1,e_1)=g(e_2,e_2)=1,\ \ g(e_3,e_3)=-1,$$

$$\eta(X)=g(X,e_3),$$

$$\phi e_1=-e_1,\ \phi e_2=-e_2,\ \phi e_3=0,$$

for any $X\in\chi(M)$, respectively. We have

$$\eta(e_3)=-1,\quad \phi^2X=X+\eta(X)e_3$$

and

$$g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$. Thus, for $e_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M. Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0, \ [e_1, e_3] = -e_1, \ [e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Lorentzian metric g, we have

$$\nabla_{e_1} e_1 = -e_3, \ \nabla_{e_1} e_2 = 0, \ \nabla_{e_1} e_3 = -e_1, \ \nabla_{e_2} e_1 = 0, \ \nabla_{e_2} e_2 = -e_3, \ \nabla_{e_2} e_3 = -e_2, \ \nabla_{e_3} e_1 = 0, \ \nabla_{e_3} e_2 = 0, \ \nabla_{e_3} e_3 = 0.$$

Hence, it can be easily seen that (ϕ, ξ, η, g) is an Lorentzian para-Sasakian structure on M. So, $M^3(\phi, \xi, \eta, g)$ is a Lorentzian para-Sasakian manifold.

Now we consider the Schouten-van Kampen connection on M. By direct calculations, we see that the nonzero components of the Schouten-van Kampen connection $\breve{\nabla}$ on M are

$$\breve{\nabla}_{e_1} e_1 = -e_3 + \xi, \quad \breve{\nabla}_{e_2} e_2 = -e_3 + \xi. \tag{65}$$

From equation (65), we can easily see that $\breve{\nabla}_{e_i} e_j = 0$, $(1 \le i, j \le 3)$, for $\xi = e_3$. Thus the manifold M is a flat manifold with respect to the Schouten-van Kampen connection. Since a flat manifold is a Ricci-flat manifold with respect to the Schouten-van Kampen connection, the manifold M is both a projectively flat and a conharmonically flat 3-dimensional Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection. Thus, Theorem 2 and Theorem 3 are verified.

5. Conclusions

The semi-Riemannian geometry attracts researchers because of its capabilities to resolve the many issues of science, technology, and medical, and their allied areas. A differentiable manifold M of dimension n equipped with a semi-Riemannian metric g, whose signature is (p, q), $(p + q = n)$, known as an n-dimensional semi-Riemannian manifold. In particular, if we take $p = 1$, $q = n - 1$, or $p = n - 1$, $q = 1$, then the semi-Riemannian manifold M converts into the well-known Lorentzian manifold. Recently, Schouten-van Kampen connection used by many mathematicians. In this paper we study a Lorentzian para-Sasakian manifold with respect to the Schouten-van Kampen connection.

References

[1] Matsumoto, K. (1989). *On Lorentzian paracontact manifolds.* Bull. of Yamagata Univ. Nat. Sci., 12(2), 151-156.

[2] Mihai, I., & Rosca, R. (1992). On Lorentzian P-Sasakian manifolds. Classical Analysis, World Scientific Publish, Singapore, 155-169.

[3] Matsumoto, K., & Mihai, I. (1988). *On a certain transformation in a Lorentzian para-Sasakian manifold.* Tensor N. S, 47(2), 189-197.

[4] Tripathi, M. M., & De, U. C. (2001) *Lorentzian almost paracontact manifolds and their submanifolds*. J.Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 8(2), 101-105.

[5] Ozgur C. (2003). *ϕ-conformally flat Lorentzian para-Sasakian manifolds.* Radovi Matematicki, 12(1), 99-106.

[6] Shaikh A. A., & Baishya K. K. (2006). *On ϕ-symmetric Lorentzian para-Sasakian manifolds.* Yokohama Math. Journal, 52, 97-112.

[7] Taleshian A., & Asghari N. (2010). *On LP-Sasakian manifolds satisfying certain conditions on the concircular curvature tensor.* Differential Geometry-Dynamical Systems, 12, 228-232.

[8] Yano K., & Kon M. (1985). Structures on manifolds. Series in Pure Math., Vol 3, World Sci.

[9] Sato I. (1976). *On a structure similar to almost contact structure*. Tensor N. S, 30, 219-224.

[10] Sato I. (1977). *On a structure similar to almost contact structure II*. Tensor N. S, 31, 199-205.

[11] Bejancu A., & Faran H. R. (2006). Foliations and geometric structures. Math. and Its Appl., 580, Springer, Dordrecht.

[12] Solov'ev A. F. (1978). *On the curvature of the connection induced on a hyperdistribution in a Riemannian space (in Russian).* Geom. Sb., 19, 12-23.

[13] Solov'ev A. F. (1979). *The bending of hyperdistributions (in Russian)*. Geom. Sb., 20, 101-112.

[14] Solov'ev A. F. (1982). *Second fundamental form of a distribution.* Mat. Zametki, 35, 139-146.

[15] Solov'ev A. F. (1985). *Curvature of a distribution.* Mat. Zametki, 35, 111-124.