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GENERALIZED TOPOLOGICAL OPERATOR THEORY IN GENERALIZED TOPOLOGICAL SPACES

PART II. GENERALIZED INTERIOR AND GENERALIZED CLOSURE

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Abstract. In a recent paper (Cf. [19]), we have presented the definitions and the essential properties of the generalized topological operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior* and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closure operators*) in a generalized topological space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$*-space*). Principally, we have shown that $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g}},\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\right) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is $(\Omega, \emptyset)$-*grounded*, (*expansive, non-expansive*), (*idempotent, idempotent*) and $(\cap, \cup)$-*additive*. We have also shown that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\text{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\text{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. In this paper, we study the commutativity of $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}$-sets having some $\left(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\right)$-based properties ($\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$*-properties*) in $\mathscr{T}_{\mathfrak{g}}$-spaces. The main results of the study are: The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *duals* and $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under their $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operations*. A $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$*-property* is *equivalent* to the $\mathfrak{T}_{\mathfrak{g}}$-set or its complement having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$*-property*. The $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\cup$*-operation* and $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\left\{\cup, \cap, \complement\right\}$*-operations*. Finally, a $\mathfrak{T}_{\mathfrak{g}}$-set having $\left\{\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}\right\}$*-property* also has $\left\{\mathfrak{P}_{\mathfrak{g}}, \mathfrak{Q}_{\mathfrak{g}}\right\}$*-property*.

1. Introduction

Many mathematicians have studied several kinds of ordinary and generalized topological operators ($\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-operators*) in ordinary ($\mathfrak{a} = \mathfrak{o}$) and generalized ($\mathfrak{a} = \mathfrak{g}$) topological spaces ($\mathscr{T}_{\mathfrak{a}}$*-spaces*) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Jung and Nam [3] have used the $\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{T}_{\mathfrak{o}}$-closure operators $(\cdot)^{\circ}$, $\bar{(\cdot)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ to establish several necessary and sufficient conditions related

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to openness and closeness properties of sets in a $\mathscr{T}_{\mathfrak{o}}$-space. Lei and Zhang [4] have considered the $\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{T}_{\mathfrak{o}}$-closure operators **Int**, **Cl** : $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in studying some topological characterizations axiomatically in $\mathscr{T}_{\mathfrak{o}}$-spaces. Gupta and Sarma [5] have established a variety of generalized sets ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets*) under the possible compositions of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators i_γ, $c_\gamma : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (γ*-interior* and γ*-closure operators*), respectively, where $\gamma \in \{\alpha, \beta, \pi, \sigma\}$, in $\mathscr{T}_{\mathfrak{g}}$-spaces. Rajendiran and Thamilselvan [6] have studied the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-closure operators $\mathrm{g}^*\mathrm{s}^*\mathrm{Int}$, $\mathrm{g}^*\mathrm{s}^*\mathrm{Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (g^*s^*-*interior* and g^*s^*-*closure operators*), respectively, in $\mathscr{T}_{\mathfrak{o}}$-spaces. In $\mathscr{T}_{\mathfrak{g}}$-spaces, Tyagi and Choudhary [7] have study stronger forms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $I_{(\cdot)}$, $C_{(\cdot)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ while Pankajam, V. [9] has presented some properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators i_δ, $\mathrm{c}_\delta : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (δ*-interior* and δ*-closure operators*), respectively, to mention but a few references.

Despite these references, in regard to the study of the commutativity of $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators in $\mathscr{T}_{\mathfrak{a}}$-spaces ($\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$), the literature is, to our knowledge, almost void of studies in this direction [17, 16]. Levine, N. [17] has studied the commutativity of the $\mathfrak{T}_{\mathfrak{o}}$-interior and $\mathfrak{T}_{\mathfrak{o}}$-closure operators $\mathrm{int}_{\mathfrak{o}}$, $\mathrm{cl}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{a}}$-space. Staley, D. H. [16] has presented some characterizations of ordinary sets ($\mathfrak{T}_{\mathfrak{o}}$*-sets*) for which the $\mathfrak{T}_{\mathfrak{o}}$-interior operator $\mathrm{int}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ commutes with the $\mathfrak{T}_{\mathfrak{o}}$-boundary operator $\mathrm{bd}_{\mathfrak{o}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{o}}$-space. In general, since $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}}) \neq (\Omega, \mathscr{T}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}}$ by virtue of $\mathscr{T}_{\mathfrak{o}} \neq \mathscr{T}_{\mathfrak{g}}$ and, $\left(\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}\right) \neq \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a}}\right)$ for each $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, so it seems reasonable to expect the existence of nice and interesting results in a $\mathscr{T}_{\mathfrak{g}}$-space with respect to those established by Levine, N. [17] and Staley, D. H. [16] in a $\mathscr{T}_{\mathfrak{o}}$-space.

Having made the study of the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces one subject of inquiry (Cf. [19]), the study of the commutativity properties of these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators in $\mathscr{T}_{\mathfrak{g}}$-spaces may be made another subject of inquiry. In this paper, we endeavor to undertake such inquiry.

The rest of the paper is structured as thus: In Sect. 2, necessary and sufficient preliminary notions are described in Subsects 2.1, 2.2 and the main results are reported in Sect. 3. In Sect. 4, the establishment of the various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed in Sects 4.1. To support the work, a nice application of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, the work is concluded in Sect. 5.

2. Theory

2.1. **Necessary Preliminaries.** As in Part I. (Cf. [19]), the standard reference for notations and concepts is the Ph.D. Thesis of Khodabocus, M. I. [2].

Herein, $\mathfrak{U}$ symbolizes the *universe* of discourse, fixed within the framework of $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator theory in $\mathscr{T}_{\mathfrak{a}}$-spaces, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, and containing *underlying sets*, *underlying subsets*, and so forth. By convention, the relation $(\alpha_1, \alpha_2, \ldots)\,\mathrm{R}\,\mathscr{A}_1 \times \mathscr{A}_2 \times \cdots$ means $\alpha_1\,\mathrm{R}\,\mathscr{A}_1$, $\alpha_2\,\mathrm{R}\,\mathscr{A}_2$, $\ldots$ where $\mathrm{R} = \in, \subset, \supset, \ldots$. The pairs $\left(I_n^0, I_n^*\right) \subset \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ and $\left(I_\infty^0, I_\infty^*\right) \sim \mathbb{Z}_+^0 \times \mathbb{Z}_+^*$ are pairs of *finite* and *infinite index sets* [1, 2].

Definition 2.1 ($\mathscr{T}_{\mathfrak{a}}$-Space [1, 2])**.** *A* $\mathscr{T}_{\mathfrak{a}}$*-space is a topological structure* $\mathfrak{T}_{\mathfrak{a}} \stackrel{\mathrm{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{a}})$ *in which* $\Omega \subset \mathfrak{U}$ *is an underlying set and* $\begin{array}{rccc} \mathscr{T}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{O}_{\mathfrak{a}} & \longmapsto & \mathscr{T}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}}) \end{array}$ *is* *an* $\mathfrak{a}$*-topology satisfying the compound* $\mathscr{T}_{\mathfrak{a}}$*-axiom:*

$$\mathrm{Ax}\left(\mathscr{T}_{\mathfrak{a}}\right) \stackrel{\text{def}}{\longleftrightarrow} \begin{cases} \left(\mathscr{T}_{\mathfrak{o}}\left(\emptyset\right) = \emptyset\right) \wedge \left(\mathscr{T}_{\mathfrak{o}}\left(\mathscr{O}_{\mathfrak{o},\nu}\right) \subseteq \mathscr{O}_{\mathfrak{o},\nu}\right) \\ \qquad \wedge \left(\mathscr{T}_{\mathfrak{o}}\left(\bigcap_{\nu \in I_n^*} \mathscr{O}_{\mathfrak{o},\nu}\right) = \bigcap_{\nu \in I_n^*} \mathscr{T}_{\mathfrak{o}}\left(\mathscr{O}_{\mathfrak{o},\nu}\right)\right) \\ \qquad \wedge \left(\mathscr{T}_{\mathfrak{o}}\left(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{o},\nu}\right) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{o}}\left(\mathscr{O}_{\mathfrak{o},\nu}\right)\right) \quad (\mathfrak{a} = \mathfrak{o}), \\ \left(\mathscr{T}_{\mathfrak{g}}\left(\emptyset\right) = \emptyset\right) \wedge \left(\mathscr{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right) \subseteq \mathscr{O}_{\mathfrak{g},\nu}\right) \\ \qquad \wedge \left(\mathscr{T}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)\right) \quad (\mathfrak{a} = \mathfrak{g}). \end{cases}$$

By assumption, the $\mathscr{T}_{\mathfrak{a}}$-space is void of any $\mathfrak{T}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separation axioms (*ordinary* and *generalized separation axioms*) unless otherwise stated [1, 2, 20]. If $\mathfrak{a} = \mathfrak{o}$ (*ordinary*), then $\mathrm{Ax}\left(\mathscr{T}_{\mathfrak{o}}\right)$ stands for an $\mathfrak{o}$-topology (*ordinary topology*) and $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}}) = (\Omega, \mathscr{T}) = \mathfrak{T}$ is called a $\mathscr{T}_{\mathfrak{o}}$-space (*ordinary topological space*) and if $\mathfrak{a} = \mathfrak{g}$ (*generalized*), then $\mathrm{Ax}\left(\mathscr{T}_{\mathfrak{g}}\right)$ stands for a $\mathfrak{g}$-topology (*generalized topology*) and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is called a $\mathscr{T}_{\mathfrak{g}}$-space (*generalized topological space*). If $\Omega \in \mathscr{T}_{\mathfrak{g}}$, then $\mathfrak{T}_{\mathfrak{a}}$ is a *strong* $\mathscr{T}_{\mathfrak{a}}$-space [2, 21, 22] and if $\mathscr{T}_{\mathfrak{g}}\left(\bigcap_{\nu \in I_n^*} \mathscr{O}_{\mathfrak{g},\nu}\right) = \bigcap_{\nu \in I_n^*} \mathscr{T}_{\mathfrak{g}}\left(\mathscr{O}_{\mathfrak{g},\nu}\right)$ for any $I_n^* \subset I_\infty^*$, then $\mathfrak{T}_{\mathfrak{g}}$ is a *quasi* $\mathscr{T}_{\mathfrak{g}}$-space [2, 23]. The notations $\Gamma \subset \Omega$, $\mathscr{O}_{\mathfrak{a}} \in \mathscr{T}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} \stackrel{\text{def}}{=} \left\{\mathscr{K}_{\mathfrak{a}} : \complement_{\Omega}\left(\mathscr{K}_{\mathfrak{a}}\right) \in \mathscr{T}_{\mathfrak{a}}\right\}$ and $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ state that Γ, $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a}}$ and $\mathscr{S}_{\mathfrak{a}}$ are a Ω*-subset*, $\mathscr{T}_{\mathfrak{a}}$*-open set*, $\mathscr{T}_{\mathfrak{a}}$*-closed set* and $\mathfrak{T}_{\mathfrak{a}}$*-set*, respectively [1, 2]. The operators $\begin{array}{rcl} \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathrm{int}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right),\ \mathrm{cl}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \end{array}$ are the $\mathfrak{T}_{\mathfrak{a}}$-*interior* and $\mathfrak{T}_{\mathfrak{a}}$*-closure operators*, respectively [1, 2]. For convenience of notation, let $\left(\mathscr{P}^*, \mathscr{T}_{\mathfrak{a}}^*, \neg\mathscr{T}_{\mathfrak{a}}^*\right)(\Omega) = \left(\mathscr{P} \setminus \{\emptyset\}, \mathscr{T}_{\mathfrak{a}} \setminus \{\emptyset\}, \neg\mathscr{T}_{\mathfrak{a}} \setminus \{\emptyset\}\right)(\Omega)$.

Definition 2.2 ($\mathfrak{g}$-Operation [1, 2])**.** *A mapping* $\begin{array}{rcl} \mathrm{op}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \end{array}$ *is called a* $\mathfrak{g}$*-operation if and only if the following statements hold:*

$$\begin{aligned} \left(\forall \mathscr{S}_{\mathfrak{a}} \in \mathscr{P}^*(\Omega)\right)\left(\exists \left(\mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}\right) \in \mathscr{T}_{\mathfrak{a}}^* \times \neg\mathscr{T}_{\mathfrak{a}}^*\right)\big[\left(\mathrm{op}_{\mathfrak{a}}(\emptyset) = \emptyset\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{a}}(\emptyset) = \emptyset\right) \\ \vee \left(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right) \vee \left(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)\big], \quad (2.1) \end{aligned}$$

where $\begin{array}{rcl} \neg\,\mathrm{op}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \end{array}$ *is called its complementary* $\mathfrak{g}$*-operation, and for all* $\mathfrak{T}_{\mathfrak{a}}$*-sets* $\mathscr{S}_{\mathfrak{a}}$, $\mathscr{S}_{\mathfrak{a},\nu}$, $\mathscr{S}_{\mathfrak{a},\mu} \in \mathscr{P}^*(\Omega)$*, the following axioms are satisfied:*

- Ax. I. $\left(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right) \vee \left(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)$,
- Ax. II. $\left(\mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \subseteq \mathrm{op}_{\mathfrak{a}} \circ \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a}}\right)\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \supseteq \neg\,\mathrm{op}_{\mathfrak{a}} \circ \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a}}\right)\right)$,
- Ax. III. $\left(\mathscr{S}_{\mathfrak{a},\nu} \subseteq \mathscr{S}_{\mathfrak{a},\mu} \longrightarrow \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a},\nu}\right) \subseteq \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a},\mu}\right)\right)$
  $\vee\left(\mathscr{S}_{\mathfrak{a},\mu} \subseteq \mathscr{S}_{\mathfrak{a},\nu} \longleftarrow \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a},\mu}\right) \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a},\nu}\right)\right)$,
- Ax. IV. $\left(\mathrm{op}_{\mathfrak{a}}\left(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{a},\sigma}\right) \subseteq \bigcup_{\sigma=\nu,\mu} \mathrm{op}_{\mathfrak{a}}\left(\mathscr{O}_{\mathfrak{a},\sigma}\right)\right)$
  $\vee\left(\neg\,\mathrm{op}_{\mathfrak{a}}\left(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{a},\sigma}\right) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\,\mathrm{op}_{\mathfrak{a}}\left(\mathscr{K}_{\mathfrak{a},\sigma}\right)\right)$,

for some $\mathscr{T}_{\mathfrak{a}}$*-sets* $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{O}_{\mathfrak{a},\nu}$, $\mathscr{O}_{\mathfrak{a},\mu} \in \mathscr{T}_{\mathfrak{a}}^*$ *and* $\mathscr{K}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a},\nu}$, $\mathscr{K}_{\mathfrak{a},\mu} \in \neg\mathscr{T}_{\mathfrak{a}}^*$.

The class $\mathscr{L}_{\mathfrak{a}}[\Omega] \stackrel{\text{def}}{=} \left\{\mathbf{op}_{\mathfrak{a},\nu} = \left(\mathrm{op}_{\mathfrak{a},\nu}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}\right) : \nu \in I_3^0\right\} \subseteq \mathscr{L}_{\mathfrak{a}}^{\omega}[\Omega] \times \mathscr{L}_{\mathfrak{a}}^{\kappa}[\Omega] = \left\{\mathrm{op}_{\mathfrak{a},\nu} : \nu \in I_3^0\right\} \times \left\{\neg\,\mathrm{op}_{\mathfrak{a},\nu} : \nu \in I_3^0\right\}$, where

$$\begin{aligned} \left\langle \mathrm{op}_{\mathfrak{a},\nu} : \nu \in I_3^0 \right\rangle &= \left\langle \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} \right\rangle, \\ \left\langle \neg\,\mathrm{op}_{\mathfrak{a},\nu} : \nu \in I_3^0 \right\rangle &= \left\langle \mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}} \right\rangle, \end{aligned}$$

is the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$.

Definition 2.3 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Sets [1, 2])**.** *Let* $\left(\mathscr{S}_{\mathfrak{a}},\mathscr{O}_{\mathfrak{a}},\mathscr{K}_{\mathfrak{a}},\mathbf{op}_{\mathfrak{a},\nu}\right)\in\mathscr{P}(\Omega)\times\mathscr{T}_{\mathfrak{a}}\times\neg\mathscr{T}_{\mathfrak{a}}\times\mathscr{L}_{\mathfrak{a}}[\Omega]$ *and let the predicates*

$$\begin{aligned}\mathrm{P}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}},\mathscr{O}_{\mathfrak{a}};\mathrm{op}_{\mathfrak{a},\nu};\subseteq\right) &\overset{\text{def}}{=} \left(\exists\left(\mathscr{O}_{\mathfrak{a}},\mathrm{op}_{\mathfrak{a},\nu}\right)\in\mathscr{T}_{\mathfrak{a}}\times\mathscr{L}_{\mathfrak{a}}^{\omega}[\Omega]\right)\left[\mathscr{S}_{\mathfrak{a}}\subseteq\mathrm{op}_{\mathfrak{a},\nu}\left(\mathscr{O}_{\mathfrak{a}}\right)\right],\\ \mathrm{Q}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}},\mathscr{K}_{\mathfrak{a}};\neg\,\mathrm{op}_{\mathfrak{a},\nu};\supseteq\right) &\overset{\text{def}}{=} \left(\exists\left(\mathscr{K}_{\mathfrak{a}},\neg\,\mathrm{op}_{\mathfrak{a},\nu}\right)\in\neg\mathscr{T}_{\mathfrak{a}}\times\mathscr{L}_{\mathfrak{a}}^{\kappa}[\Omega]\right)\\ &\qquad\qquad\left[\mathscr{S}_{\mathfrak{a}}\supseteq\neg\,\mathrm{op}_{\mathfrak{a},\nu}\left(\mathscr{K}_{\mathfrak{a}}\right)\right]\end{aligned}\tag{2.2}$$

be Boolean-valued functions on $\mathscr{P}(\Omega)\times\left(\mathscr{T}_{\mathfrak{a}}\cup\neg\mathscr{T}_{\mathfrak{a}}\right)\times\left(\mathscr{L}_{\mathfrak{a}}^{\omega}\cup\mathscr{L}_{\mathfrak{a}}^{\kappa}\right)[\Omega]\times\{\subseteq,\supseteq\}$, *then* $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]\ \&\overset{\text{def}}{=}\&\ \mathfrak{g}$-$\nu$-$\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]\cup\mathfrak{g}$-$\nu$-$\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$ *is the class of all* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-sets and,*

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] &\overset{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{a}}:\ \mathrm{P}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}},\mathscr{O}_{\mathfrak{a}};\mathrm{op}_{\mathfrak{a},\nu};\subseteq\right)\right\},\\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] &\overset{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{a}}:\ \mathrm{Q}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}},\mathscr{K}_{\mathfrak{a}};\neg\,\mathrm{op}_{\mathfrak{a},\nu};\supseteq\right)\right\},\end{aligned}\tag{2.3}$$

respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-open and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-closed sets of category* ν *in* $\mathfrak{T}_{\mathfrak{a}}$.

Then, $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]=\left\{\mathscr{S}_{\mathfrak{a}}:\ \mathrm{P}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}},\mathscr{S}_{\mathfrak{a}};\mathrm{op}_{\mathfrak{a},0};\subseteq\right)\right\}\cup\left\{\mathscr{S}_{\mathfrak{a}}:\ \mathrm{Q}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}},\mathscr{S}_{\mathfrak{a}};\neg\,\mathrm{op}_{\mathfrak{a},0};\supseteq\right)\right\}=\bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-*open* and $\mathfrak{T}_{\mathfrak{a}}$-*closed sets* in $\mathfrak{T}_{\mathfrak{a}}$ [1, 2]. Further,

$$\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]\overset{\text{def}}{=}\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]=\bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}}\mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]=\bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}}\mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$$

Definition 2.4 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Connected [2])**.** *A* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-separation of two* $\mathfrak{T}_{\mathfrak{a}}$*-sets* $\emptyset\neq\mathscr{R}_{\mathfrak{a}},\ \mathscr{S}_{\mathfrak{a}}\subseteq\mathfrak{T}_{\mathfrak{a}}$ *of a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}}=(\Omega,\mathscr{T}_{\mathfrak{a}})$ *is realised if and only if there exists either* $(\mathscr{O}_{\mathfrak{a},\xi},\mathscr{O}_{\mathfrak{a},\zeta})\in\bigtimes_{\alpha\in I_2^*}\mathfrak{g}$-$\nu$-$\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ *or* $(\mathscr{K}_{\mathfrak{a},\xi},\mathscr{K}_{\mathfrak{a},\zeta})\in\bigtimes_{\alpha\in I_2^*}\mathfrak{g}$-$\nu$-$\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ *such that:*

$$\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{O}_{\mathfrak{a},\lambda}=\mathscr{R}_{\mathfrak{a}}\sqcup\mathscr{S}_{\mathfrak{a}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{K}_{\mathfrak{a},\lambda}=\mathscr{R}_{\mathfrak{a}}\sqcup\mathscr{S}_{\mathfrak{a}}\Big).\tag{2.4}$$

Otherwise, $\mathscr{R}_{\mathfrak{a}},\ \mathscr{S}_{\mathfrak{a}}$ *are said to be* $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$*-connected.*

Thus, $\mathscr{S}_{\mathfrak{a}}\subset\mathfrak{T}_{\mathfrak{a}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connected if and only if $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{a}}]=\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separated if and only if $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{a}}]=\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{a}}]$ where,

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}[\mathfrak{T}_{\mathfrak{a}}]\overset{\text{def}}{=}\Bigg\{&\mathscr{S}_{\mathfrak{a}}\subset\mathfrak{T}_{\mathfrak{a}}:\left(\forall\left(\mathscr{O}_{\mathfrak{a},\lambda},\mathscr{K}_{\mathfrak{a},\lambda}\right)_{\lambda=\xi,\zeta}\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right]\times\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right]\right)\\ &\Bigg[\neg\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{O}_{\mathfrak{a},\lambda}=\mathscr{S}_{\mathfrak{a}}\Big)\bigwedge\neg\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{O}_{\mathfrak{a},\lambda}=\mathscr{S}_{\mathfrak{a}}\Big)\Bigg]\Bigg\};\end{aligned}\tag{2.5}$$

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{D}[\mathfrak{T}_{\mathfrak{a}}]\overset{\text{def}}{=}\Bigg\{&\mathscr{S}_{\mathfrak{a}}\subset\mathfrak{T}_{\mathfrak{a}}:\left(\exists\left(\mathscr{O}_{\mathfrak{a},\lambda},\mathscr{K}_{\mathfrak{a},\lambda}\right)_{\lambda=\xi,\zeta}\in\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right]\times\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right]\right)\\ &\Bigg[\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{O}_{\mathfrak{a},\lambda}=\mathscr{S}_{\mathfrak{a}}\Big)\bigvee\Big(\bigsqcup_{\lambda=\xi,\zeta}\mathscr{K}_{\mathfrak{a},\lambda}=\mathscr{S}_{\mathfrak{a}}\Big)\Bigg]\Bigg\}.\end{aligned}\tag{2.6}$$

Definition 2.5 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Interior, $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Closure Operators [19]). *In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the one-valued maps*

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}} \end{aligned} \tag{2.8}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] : \ \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ *and* $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] : \ \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$ *are called $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Then,* $\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu} : \ \nu \in I^0_3\}$ *and* $\mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} : \nu \in I^0_3\}$ *are the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively.*

Definition 2.6 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-Vector Operator [19]). *In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$, the two-valued map*

$$\begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto (\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})) \end{aligned} \tag{2.9}$$

is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{a}}$-vector operator. Then, $\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}) : \nu \in I^0_3\}$ *is the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.*

Remark. *For every $\nu \in I^0_3$, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = \mathbf{ic}_{\mathfrak{a}} \stackrel{\mathrm{def}}{=} (\mathrm{int}_{\mathfrak{a}}, \mathrm{cl}_{\mathfrak{a}})$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. Then,* $\begin{aligned} \mathbf{ic}_{\mathfrak{a}} : \ \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto (\mathrm{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})) \end{aligned}$ *is a $\mathfrak{T}_{\mathfrak{a}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.*

2.2. **Sufficient Preliminaries.** The notions of $\mathfrak{T}_{\mathfrak{a}}$-sets having $\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$-properties and $\mathfrak{Q}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-properties in $\mathscr{T}_{\mathfrak{a}}$-spaces are now presented.

Definition 2.7 (Complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Operator). *Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map*

$$\begin{aligned} \mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \complement_{\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}), \end{aligned} \tag{2.10}$$

where $\complement_{\mathscr{R}_{\mathfrak{a}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ denotes the relative complement of its operand with respect to $\mathscr{R}_{\mathfrak{a}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{a}}]$, is called a natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

For clarity, $\mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}} = \mathfrak{g}\text{-Op}_{\mathfrak{a}}$ whenever $\mathscr{R}_{\mathfrak{a}} = \Omega$ and $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}} = \mathrm{Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}$ (*natural complement $\mathfrak{T}_{\mathfrak{a}}$-operator*) whenever $\mathscr{R}_{\mathfrak{a}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{a}}]$.

Definition 2.8 (Symmetric Difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Operator). *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, the one-valued map*

$$\mathfrak{g}\text{-Sd}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \tag{2.11}$$

$$(\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) \,\&\! \longmapsto \& \ \mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{a}}}(\mathscr{S}_{\mathfrak{a}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{a},\mathscr{S}_{\mathfrak{a}}}(\mathscr{R}_{\mathfrak{a}})$$

is called the symmetric difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operator on $\mathscr{P}(\Omega)$.

If $\mathfrak{g}\text{-Sd}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is based on $\text{Op}_{\mathfrak{a},\mathscr{R}_{\mathfrak{g}}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, the concept of *symmetric difference* $\mathfrak{T}_{\mathfrak{a}}$*-operator* $\text{Sd}_{\mathfrak{a}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ presents itself.

Definition 2.9 ($\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{a}}$-Property)**.** *A* $\mathfrak{T}_{\mathfrak{a}}$*-set* $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ *in a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is said to have* $\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$ *if and only if it belongs to:*

$$\mathfrak{g}\text{-}\nu\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{a}} :\ \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu}\left(\mathscr{S}_{\mathfrak{a}}\right) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} \circ \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu}\left(\mathscr{S}_{\mathfrak{a}}\right)\right\}, \quad (2.12)$$

called the class of all $\mathfrak{T}_{\mathfrak{a}}$*-sets having* $\mathfrak{g}$-ν-$\mathfrak{P}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$*.*

Then, $\text{P}\left[\mathfrak{T}_{\mathfrak{a}}\right] \& \stackrel{\text{def}}{=} \& \left\{\mathscr{S}_{\mathfrak{a}} :\ \text{int}_{\mathfrak{a}} \circ \text{cl}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right) \longleftrightarrow \text{cl}_{\mathfrak{a}} \circ \text{int}_{\mathfrak{a}}\left(\mathscr{S}_{\mathfrak{a}}\right)\right\}$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-sets having $\mathfrak{P}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$. By $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ is meant a $\mathfrak{T}_{\mathfrak{a}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$.

Definition 2.10 ($\mathfrak{g}$-ν-$\mathfrak{Q}_{\mathfrak{a}}$-Property)**.** *A* $\mathfrak{T}_{\mathfrak{a}}$*-set* $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ *in a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is said to have* $\mathfrak{g}$-ν-$\mathfrak{Q}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$ *if and only if it belongs to:*

$$\mathfrak{g}\text{-}\nu\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \left\{\mathscr{S}_{\mathfrak{a}} :\ \mathfrak{g}\text{-Int}_{\mathfrak{a},\nu} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{a},\nu} : \mathscr{S}_{\mathfrak{a}} \longmapsto \emptyset\right\}, \quad (2.13)$$

called the class of all $\mathfrak{T}_{\mathfrak{a}}$*-set having* $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$*-property in* $\mathfrak{T}_{\mathfrak{a}}$*.*

Then, $\text{Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right] \& \stackrel{\text{def}}{=} \& \left\{\mathscr{S}_{\mathfrak{a}} :\ \text{int}_{\mathfrak{a}} \circ \text{cl}_{\mathfrak{a}} : \mathscr{S}_{\mathfrak{a}} \longmapsto \emptyset\right\}$ is the class of all $\mathfrak{T}_{\mathfrak{a}}$-sets having $\mathfrak{Q}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$. By $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ is meant a $\mathfrak{T}_{\mathfrak{a}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-property in $\mathfrak{T}_{\mathfrak{a}}$.

3. Main Results

The main results relative to the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators, and $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-properties in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented.

Lemma 3.1. *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the natural complement* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator of its components in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{aligned}\left(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\right)\Big[&\left(\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right) \\ &\wedge\left(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\right)\Big]. \quad (3.1)\end{aligned}$$

Proof. Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a given and, let $\mathfrak{g}\text{-Op}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the natural complement $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator of its components in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, for a $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ taken arbitrarily, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\text{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \mathscr{O}_{\mathfrak{g}}\Bigg); \\ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) &\longmapsto \mathfrak{g}\text{-Op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\text{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right]} \mathscr{K}_{\mathfrak{g}}\Bigg).\end{aligned}$$

Let $\{\mathscr{O}_{\mathfrak{g},\nu} : (\forall\nu \in I^{*}_{\infty})[\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathscr{S}_{\mathfrak{g}}]\}$ and $\{\mathscr{K}_{\mathfrak{g},\nu} : (\forall\nu \in I^{*}_{\infty})[\mathscr{K}_{\mathfrak{g},\nu} \supseteq \mathscr{S}_{\mathfrak{g}}]\}$ stand for $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}\Bigg) &= \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu\in I^{*}_{\infty}} \left(\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\Bigg)\\
&= \complement_{\Omega}\Bigg(\bigcup_{\nu\in I^{*}_{\infty}} \left(\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\Bigg)\\
&= \bigcap_{\nu\in I^{*}_{\infty}} \left(\complement_{\Omega}(\mathscr{O}_{\mathfrak{g},\nu}) \supseteq \complement_{\Omega}\left(\complement_{\Omega}(\mathscr{S}_{\mathfrak{g}})\right)\right)\\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}};\\
\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}\Bigg) &= \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\Bigg(\bigcup_{\nu\in I^{*}_{\infty}} \left(\mathscr{O}_{\mathfrak{g},\nu} \subseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\Bigg)\\
&= \complement_{\Omega}\Bigg(\bigcap_{\nu\in I^{*}_{\infty}} \left(\mathscr{K}_{\mathfrak{g},\nu} \supseteq \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right)\Bigg)\\
&= \bigcup_{\nu\in I^{*}_{\infty}} \left(\complement_{\Omega}(\mathscr{K}_{\mathfrak{g},\nu}) \subseteq \complement_{\Omega}\left(\complement_{\Omega}(\mathscr{S}_{\mathfrak{g}})\right)\right)\\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}.
\end{aligned}$$

Since $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ is arbitrary, it follows that, for every $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$, the relations

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\\
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})
\end{aligned}$$

hold. The proof of the lemma is complete. □

Theorem 3.2. *A $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is said to have $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$ if and only if:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]. \tag{3.2}$$

Proof. Necessity. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \quad & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ = \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ = \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ = \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ = \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ = \quad & \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ = \quad & \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \end{aligned}$$

Thus, it follows that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big),$$

and hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The condition of the theorem is, therefore, necessary.

Sufficiency. Conversely, suppose $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Set $\mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then,

$$\mathscr{S}_{\mathfrak{g}} \;\longleftrightarrow\; \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \;\longleftrightarrow\; \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}).$$

But $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and it in turn implies $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Hence, it follows that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The condition of the theorem is, therefore, sufficient. □

Proposition 3.3. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$,
- II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

Proof. I. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned} & \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \\ & \quad \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \end{aligned}$$

Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of ITEM I. of the proposition is complete.

II. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \\
&\longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \longleftrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big)
\end{aligned}$$

Hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of ITEM II. of the proposition is complete. □

Theorem 3.4. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$, then $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.*

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ such that $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ or $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then:

CASE I. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it follows that $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. But $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \supseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows, furthermore, that $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) = \emptyset = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and, hence, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

CASE II. Suppose $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ in $\mathfrak{T}_{\mathfrak{g}}$. Then, by virtue of the above case, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and by virtue of the fact that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ is equivalent to $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it results that $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ implies $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of the theorem is complete. □

Theorem 3.5. *Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$, where $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma : \ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$. Then:*

- I. $\Gamma \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *implies* $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$,
- II. $\Gamma \in \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *implies* $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$.

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and let $\big(\mathfrak{g}\text{-Int}_{\mathfrak{g},\Lambda}, \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Lambda}\big) \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right] \times \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$ be a pair of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Lambda} : \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda)$, respectively, where $\Lambda \in \{\Omega, \Gamma\}$. Then:

I. Suppose $\Gamma \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\Gamma \cap \mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\Gamma]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\Gamma) = \Gamma.
\end{aligned}$$

Thus, $\Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. On the other hand,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
&\longleftrightarrow \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{O}_{\mathfrak{g}} \cap \Gamma) \\
&\longleftrightarrow \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} (\mathscr{O}_{\mathfrak{g}} \cap \Gamma) \\
&\longleftrightarrow \Gamma \cap \left(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \right) = \Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

But $\Gamma \cap \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$.

II. Suppose $\Gamma \in \mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \\
&\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\Gamma]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\Gamma) = \Gamma.
\end{aligned}$$

Consequently, $\Gamma\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. On the other hand,

$$\begin{aligned}\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Gamma}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\\ &\longleftrightarrow \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\Gamma)\\ &\longleftrightarrow \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\Gamma)\\ &\longleftrightarrow \Gamma\cap\left(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\right)=\Gamma\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

But $\Gamma\cap\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$ and hence, $\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Gamma}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Cl}_{\mathfrak{g},\Omega}(\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. ☐

Theorem 3.6. *Let $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set and let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$, then:*

$$\big(\forall\,\mathfrak{g}\text{-Int}_{\mathfrak{g}}\in\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}]\big)\left[\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big)=\displaystyle\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\sigma})\right].\qquad(3.3)$$

Proof. Let $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set, let $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\mathfrak{T}_{\mathfrak{g}}\times\mathfrak{T}_{\mathfrak{g}}$ be a pair of $\mathfrak{T}_{\mathfrak{g}}$-sets in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ and, suppose $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$. Then, for every $\mathscr{S}_{\mathfrak{g}}\in\{\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta}\}$,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\\ &\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\alpha}\cup\mathscr{S}_{\mathfrak{g},\beta}]}\mathscr{O}_{\mathfrak{g}}=\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big).\end{aligned}$$

Consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big)\supseteq\bigcup_{\sigma=\alpha,\beta}\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\sigma})$. Set $\hat{\mathscr{S}}_{\mathfrak{g},\alpha}=\mathscr{S}_{\mathfrak{g},\alpha}\cap\mathscr{Q}_{\mathfrak{g}}$ and $\hat{\mathscr{S}}_{\mathfrak{g},\beta}=\mathscr{S}_{\mathfrak{g},\beta}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$. Then, since $(\mathscr{S}_{\mathfrak{g},\alpha},\mathscr{S}_{\mathfrak{g},\beta})\subseteq\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned}
\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big] &= \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\textstyle\bigcup_{\sigma=\alpha,\beta}\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\big] \\
&= \Big\{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] :\ \mathscr{O}_{\mathfrak{g}} \subseteq \bigcup_{\sigma=\alpha,\beta} \hat{\mathscr{S}}_{\mathfrak{g},\sigma}\Big\} \\
&= \Big\{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] :\ \bigvee_{\sigma=\alpha,\beta} \big(\mathscr{O}_{\mathfrak{g}} \subseteq \hat{\mathscr{S}}_{\mathfrak{g},\sigma}\big)\Big\} \\
&= \bigcup_{\sigma=\alpha,\beta} \big\{\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] :\ \mathscr{O}_{\mathfrak{g}} \subseteq \hat{\mathscr{S}}_{\mathfrak{g},\sigma}\big\} \\
&= \bigcup_{\sigma=\alpha,\beta} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\hat{\mathscr{S}}_{\mathfrak{g},\sigma}\big] = \bigcup_{\sigma=\alpha,\beta} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g},\sigma}\big].
\end{aligned}$$

Therefore, $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big] = \bigcup_{\sigma=\alpha,\beta} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g},\sigma}\big]$, as a consequence of the condition $(\mathscr{S}_{\mathfrak{g},\alpha}, \mathscr{S}_{\mathfrak{g},\beta}) \subseteq \big(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}\left(\mathscr{Q}_{\mathfrak{g}}\right)\big)$. Taking this fact into account, it follows, moreover, that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \textstyle\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\alpha}\cup\mathscr{S}_{\mathfrak{g},\beta}]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \bigcup_{\sigma=\alpha,\beta} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\sigma}]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \bigcup_{\sigma=\alpha,\beta} \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\sigma}]} \mathscr{O}_{\mathfrak{g}} \Bigg) = \bigcup_{\sigma=\alpha,\beta} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\sigma}\right).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\bigcup_{\sigma=\alpha,\beta}\mathscr{S}_{\mathfrak{g},\sigma}\big) \subseteq \bigcup_{\sigma=\alpha,\beta} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\sigma}\right)$. The proof of the theorem is complete. □

Theorem 3.7. *Let* $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ *be a* $\mathscr{T}_{\mathfrak{g}}$*-subspace of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$*, where* $\mathscr{T}_{\mathfrak{g},\Gamma} : \mathscr{P}(\Gamma) \longmapsto \mathscr{T}_{\mathfrak{g},\Gamma} = \big\{\mathscr{O}_{\mathfrak{g}} \cap \Gamma :\ \mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\Omega}\big\}$*. If* $\Gamma \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ *and* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$*, then* $\mathscr{S}_{\mathfrak{g}} \cap \Gamma \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]$*.*

Proof. Let $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathscr{T}_{\mathfrak{g},\Gamma})$ be a $\mathscr{T}_{\mathfrak{g}}$-subspace of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathscr{T}_{\mathfrak{g},\Omega})$ and, suppose $\Gamma \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ and $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$. Then, since $\Gamma \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right] \cap \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ implies $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma}\left(\mathscr{S}_{\mathfrak{g}}\right) = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right)$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma} : \mathscr{S}_{\mathfrak{g}} \cap \Gamma &\longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}} \cap \Gamma\right) \\
&\subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega}\left(\mathscr{S}_{\mathfrak{g}}\right).
\end{aligned}$$

Since $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$, it follows, moreover, that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Omega} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Omega} : \mathscr{S}_{\mathfrak{g}} \longmapsto \emptyset$. Consequently, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\Gamma} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\Gamma} : \mathscr{S}_{\mathfrak{g}} \cap \Gamma \longmapsto \emptyset$ and hence, $\mathscr{S}_{\mathfrak{g}} \cap \Gamma \in \mathfrak{g}\text{-}\mathrm{Nd}\left[\mathfrak{T}_{\mathfrak{g},\Gamma}\right]$. The proof of the theorem is complete. □

Theorem 3.8. *In order that a* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *satisfies the condition* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$*, it is necessary and sufficient that there exist a*

$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property such that it be expressible as:

$$\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}). \tag{3.4}$$

Proof. Sufficiency. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and let there exist $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ such that the relation $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$ holds. Clearly, $(\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}) \subseteq \big(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$, implying

$$\begin{aligned} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\big[(\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})\big] &= \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}\big] \\ &\cup \ \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}\big]. \end{aligned}$$

Set $\mathscr{S}_{\mathfrak{g},(q,r)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(r,q)} = \mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)}\big) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)$. Since $\big(\mathscr{S}_{\mathfrak{g},(q,r)}, \mathscr{S}_{\mathfrak{g},(r,q)}\big) \subseteq \big(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big)$ and $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big), \\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big), \\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big), \\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big) &= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \\ &= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})\right]} \mathscr{O}_{\mathfrak{g}} \\ &= \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)})]} \mathscr{O}_{\mathfrak{g}} \Bigg) \\ &\cup \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})\right]} \mathscr{O}_{\mathfrak{g}} \Bigg) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)\big) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) \\ &\cup \ \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big) \\ &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) \\ &\cup \ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big) \\ &\cup \ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big). \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)})\cup \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})\right]} \mathscr{K}_{\mathfrak{g}} \\
&= \Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)})]} \mathscr{K}_{\mathfrak{g}}\Bigg) \\
&\cup \Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\sup}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})]} \mathscr{K}_{\mathfrak{g}}\Bigg) \\
&= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \cup \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)})\big) \\
&= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \\
&\cup \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}) \\
&= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \\
&\cup \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}).
\end{aligned}$$

Hence, it results that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) \\
&\cup \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}).
\end{aligned}$$

By virtue of the relation $(\mathscr{S}_{\mathfrak{g},(q,r)}, \mathscr{S}_{\mathfrak{g},(r,q)}) \subseteq (\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))$, it is plain that $\mathscr{S}_{\mathfrak{g},(q,r)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(r,q)} = \mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$. But $\mathscr{S}_{\mathfrak{g},(q,r)} = \complement_{\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathscr{R}_{\mathfrak{g}}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$ and hence,

$$\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}) = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g},(q,r)}).$$

On the other hand, the statement that $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathscr{R}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$ implies that $\mathscr{S}_{\mathfrak{g},(r,q)}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})$ and therefore,

$$\begin{aligned}
&\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}) \\
&\qquad = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\mathfrak{g}\text{-}\mathrm{Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}(\mathscr{S}_{\mathfrak{g},(r,q)}).
\end{aligned}$$

When all the foregoing set-theoretic expressions are taken into account, it results that

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\\
&\cup\ \mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)\\
&= \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}\big(\mathscr{S}_{\mathfrak{g},(q,r)}\big)\\
&\cup\ \mathfrak{g}\text{-Cl}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g},\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})}\big(\mathscr{S}_{\mathfrak{g},(r,q)}\big)\\
&= \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The condition of the theorem is, therefore, sufficient.

Necessity. Conversely, suppose that $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. Then, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Set $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Then, $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, meaning that $\mathscr{Q}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set in $\mathfrak{T}_{\mathfrak{g}}$. Set $\mathscr{S}_{\mathfrak{g},(s,q)} = \mathscr{S}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g},(q,s)} = \mathscr{Q}_{\mathfrak{g}} - \mathscr{S}_{\mathfrak{g}}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(s,q)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}};\\
\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(s,q)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}).
\end{aligned}$$

But $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$ and consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g},(s,q)}\longmapsto\emptyset$, meaning that $\mathscr{Q}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{S}_{\mathfrak{g}}$. On the other hand,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}};\\
\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\big) &\subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\\
&= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\
&= \mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}).
\end{aligned}$$

Since $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$ it follows, consequently, that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g},(q,s)}\longmapsto\emptyset$, meaning that $\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathscr{Q}_{\mathfrak{g}}$. Set $\mathscr{R}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g},(q,s)}\cup\mathscr{S}_{\mathfrak{g},(s,q)}$. Then,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} &\longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\cup\mathscr{S}_{\mathfrak{g},(s,q)}\big)\\
&= \mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(q,s)}\big)\cup\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(s,q)}\big)\\
&= \emptyset\cup\emptyset = \emptyset,
\end{aligned}$$

implying that $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Having evidenced the existence of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property, it only remains to show that $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ is expressible as $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}})\cup(\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$.

Observe that

$$
\begin{aligned}
&\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} \\
&\qquad = \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\} \cup \{\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\qquad = \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}[(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))]\} \\
&\qquad \cup\{[(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))] \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\qquad = \{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}))\} \\
&\qquad \cup\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\qquad = \{\mathscr{Q}_{\mathfrak{g}} \cap (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cup \mathscr{S}_{\mathfrak{g}}) \cap (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathscr{Q}_{\mathfrak{g}})\} \\
&\qquad \cup\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\qquad = \{(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \cap (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cup \mathscr{Q}_{\mathfrak{g}})\} \cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\} \\
&\qquad = (\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) \cup (\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})).
\end{aligned}
$$

But since $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{Q}_{\mathfrak{g}} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and the latter in turn implies $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}))$, it follows that $\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}} = \mathscr{S}_{\mathfrak{g}}$ and $\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) = \emptyset$. Consequently, $\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} = \mathscr{S}_{\mathfrak{g}}$. But, $\mathscr{S}_{\mathfrak{g},(q,r)} \cup \mathscr{S}_{\mathfrak{g},(r,q)} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$ and hence, $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$. The condition of the theorem is, therefore, necessary. □

Observe that $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}})$. Thus, an immediate consequence of the above theorem is the following corollary.

Corollary 3.9. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$ if and only if:*

$$(\exists \mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}])(\exists \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])[\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}})]. \tag{3.5}$$

Proposition 3.10. *If $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property, then $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \Omega$:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \Omega. \tag{3.6}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows that $\Omega \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\Omega) = \Omega$. But, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset$. Thus, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \emptyset \neq \Omega = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega)$, implying $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \Omega$. The proof of the proposition is complete. □

Proposition 3.11. *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and $\mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, then:*

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longleftrightarrow (\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]) \vee (\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]). \tag{3.7}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ and $\mathfrak{T}_{\mathfrak{g}}$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected. Suppose $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$. Then, there exist a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set $\mathscr{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property such that $\mathscr{S}_{\mathfrak{g}}$ be expressible as $\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})$. Since the strong $\mathscr{T}_{\mathfrak{g}}$-space

$\mathfrak{T}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connected, the only $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open-closed set are the improper $\mathfrak{T}_{\mathfrak{g}}$-sets $\emptyset$, $\Omega \subset \mathfrak{T}_{\mathfrak{g}}$. Consequently,

$$\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right] \longleftrightarrow \big(\mathscr{Q}_{\mathfrak{g}} \in \{\emptyset, \Omega\}\big)\big[\mathscr{S}_{\mathfrak{g}} = (\mathscr{Q}_{\mathfrak{g}} - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \mathscr{Q}_{\mathfrak{g}})\big].$$

Case I. Suppose $\mathscr{Q}_{\mathfrak{g}} = \emptyset$. Then $\mathscr{S}_{\mathfrak{g}} = (\emptyset - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \emptyset)$. But $\emptyset - \mathscr{R}_{\mathfrak{g}} = \emptyset$ and $\mathscr{R}_{\mathfrak{g}} - \emptyset = \mathscr{R}_{\mathfrak{g}}$. Therefore, $\mathscr{S}_{\mathfrak{g}} = \emptyset \cup \mathscr{R}_{\mathfrak{g}} = \mathscr{R}_{\mathfrak{g}}$. Thus, $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

Case II. Suppose $\mathscr{Q}_{\mathfrak{g}} = \Omega$. Then $\mathscr{S}_{\mathfrak{g}} = (\Omega - \mathscr{R}_{\mathfrak{g}}) \cup (\mathscr{R}_{\mathfrak{g}} - \Omega)$. But $\Omega - \mathscr{R}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and $\mathscr{R}_{\mathfrak{g}} - \Omega = \emptyset$. Consequently, $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \emptyset = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$ and therefore, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}}$. Hence, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of the proposition is complete. ☐

Lemma 3.12. *If* $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ *be a triple of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets and* $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be the symmetric difference* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operator in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\mathscr{Q}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,
- II. $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big), \mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big)\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$,
- III. $\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\big)$.

Proof. Let $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ and, let $\mathfrak{g}\text{-Sd}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ be the symmetric difference $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. The proof that $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}\mathscr{Q}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ holds for any $(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ is first supplied. It is evident that

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \\ &= \big(\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big) \cup \big(\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}})\big) \subseteq \mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}, \end{aligned}$$

implying $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) \subseteq \mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}}$. Since $\mathscr{Q}_{\mathfrak{g}} \cup \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it follows that $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Items I., II. and III. are now proved.

I. Since the order of the operands under the $\cup$-operation does not change, it follows that

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{Q}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}\big). \end{aligned}$$

Hence, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

II. For any $(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it is plain that $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}}) = \mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore,

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big), \mathscr{S}_{\mathfrak{g}}\big) &= \big\{\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\ &\cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big)\big)\big\} \\ &= \big\{\mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\ &\cup \big\{\mathscr{R}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big\} \\ &\cup \big\{\mathscr{S}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big\} \\ &\cup \{\mathscr{S}_{\mathfrak{g}} \cap \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}\}. \end{aligned}$$

If $\mathrm{P}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \mathscr{Q}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then

$$\begin{aligned} \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}\big), \mathscr{S}_{\mathfrak{g}}\big) &= \mathrm{P}(\mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \cup \mathrm{P}(\mathscr{R}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \\ &\cup \mathrm{P}(\mathscr{S}_{\mathfrak{g}}, \mathscr{Q}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \cup \big(\mathscr{S}_{\mathfrak{g}} \cap \mathscr{Q}_{\mathfrak{g}} \cap \mathscr{R}_{\mathfrak{g}}\big). \end{aligned}$$

Since $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\big) &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g}}=\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}}=\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}=\mathscr{S}_{\mathfrak{g}})\big)\\ &= \mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})\cup\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})\\ &\cup\ \mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\cup\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big).\end{aligned}$$

But by virtue of the associativity and distributive properties of the $\cap$, $\cup$-operations, the relations $\mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})=\mathrm{P}(\mathscr{Q}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})$, $\mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})=\mathrm{P}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})$, $\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})=\mathrm{P}(\mathscr{S}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}})$, and $\mathscr{S}_{\mathfrak{g}}\cap\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}=\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}$ hold. Thus, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{Q}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}}),\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\big)\in\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.

III. Since the relation $\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})=\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ holds for any $(\mathscr{S}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]\times\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, it results that

$$\begin{aligned}\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\big) &= \mathscr{Q}_{\mathfrak{g}}\cap\big(\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &= \big(\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\big)\cup\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &= \big(\mathscr{Q}_{\mathfrak{g}}\cap\big(\mathscr{R}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big)\cup\big(\mathscr{Q}_{\mathfrak{g}}\cap\big(\mathscr{S}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\big)\\ &= \big((\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\cup\big((\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\big)\\ &= \mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}}}(\mathscr{S}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Op}_{\mathfrak{g},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}}(\mathscr{R}_{\mathfrak{g}})\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big).\end{aligned}$$

Hence, $\mathscr{Q}_{\mathfrak{g}}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}}\big)=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}},\mathscr{Q}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}\big)\in\mathfrak{g}\text{-S}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The proof of the lemma is complete. □

Theorem 3.13. *If $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, …, $\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ are $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then $\bigcap_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$.*

Proof. Let $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, …, $\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g},1}$, $\mathscr{S}_{\mathfrak{g},2}$, …, $\mathscr{S}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g}}\big]$, there exist $\sigma\geq 1$ $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-open-closed sets $\mathscr{Q}_{\mathfrak{g},1}$, $\mathscr{Q}_{\mathfrak{g},2}$, …, $\mathscr{Q}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathscr{R}_{\mathfrak{g},1}$, $\mathscr{R}_{\mathfrak{g},2}$, …, $\mathscr{R}_{\mathfrak{g},\sigma}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property such that

$$\begin{aligned}\mathscr{S}_{\mathfrak{g},1} &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},1},\mathscr{R}_{\mathfrak{g},1}\big),\\ \mathscr{S}_{\mathfrak{g},2} &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},2},\mathscr{R}_{\mathfrak{g},2}\big),\ \ldots,\ \mathscr{S}_{\mathfrak{g},\sigma}=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\sigma},\mathscr{R}_{\mathfrak{g},\sigma}\big).\end{aligned}$$

For an arbitrary pair $(\nu,\mu)\in I_{\sigma}^{*}\times I_{\sigma}^{*}$, set $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)}=\mathscr{Q}_{\mathfrak{g},\nu}\cap\mathscr{Q}_{\mathfrak{g},\mu}$, $\mathscr{W}_{\mathfrak{g},(\nu,\mu)}=\mathscr{Q}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}$, and $\mathscr{R}_{\mathfrak{g},(\nu,\mu)}=\mathscr{R}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}$. Then,

$$\begin{aligned}\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{S}_{\mathfrak{g},\mu} &= \mathscr{S}_{\mathfrak{g},\nu}\cap\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\mu},\mathscr{R}_{\mathfrak{g},\mu}\big)\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{Q}_{\mathfrak{g},\mu},\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{R}_{\mathfrak{g},\mu}\big)\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\nu}\big)\cap\mathscr{Q}_{\mathfrak{g},\mu},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\nu}\big)\cap\mathscr{R}_{\mathfrak{g},\mu}\big]\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\mathscr{W}_{\mathfrak{g},(\mu,\nu)}\big),\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\big]\\ &= \mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big\{\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathscr{W}_{\mathfrak{g},(\mu,\nu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\big]\big\}.\end{aligned}$$

But, $\mathscr{R}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, $(\mathscr{Q}_{\mathfrak{g},\nu},\mathscr{R}_{\mathfrak{g},\mu})\in\big(\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\big)\times\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{W}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and, $\mathscr{Q}_{\mathfrak{g},\nu}$, $\mathscr{Q}_{\mathfrak{g},\mu}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Thus, $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$, implying $\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big[\mathscr{W}_{\mathfrak{g},(\mu,\nu)},\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{W}_{\mathfrak{g},(\nu,\mu)},\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\big]=\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Therefore, $\mathscr{S}_{\mathfrak{g},\nu}\cap\mathscr{S}_{\mathfrak{g},\mu}=\mathfrak{g}\text{-Sd}_{\mathfrak{g}}\big(\mathscr{Q}_{\mathfrak{g},(\nu,\mu)},\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)}\big)$, where $\mathscr{Q}_{\mathfrak{g},(\nu,\mu)}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\cap$

$\mathfrak{g}$-K$\,[\mathfrak{T}_{\mathfrak{g}}]$ and $\hat{\mathscr{R}}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}$-Nd$\,[\mathfrak{T}_{\mathfrak{g}}]$, and consequently, $\mathscr{S}_{\mathfrak{g},\nu} \cap \mathscr{S}_{\mathfrak{g},\mu} \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$ for any $(\nu,\mu) \in I_{\sigma}^{*} \times I_{\sigma}^{*}$. Hence, $\bigcap_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}$-P$\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The proof of the theorem is complete. □

Proposition 3.14. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets each of which having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}$ has also $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\textstyle\bigwedge_{\nu \in I_{\sigma}^{*}} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \ \longrightarrow\ \displaystyle\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.8}$$

Proof. Let $\mathscr{S}_{\mathfrak{g},1}, \mathscr{S}_{\mathfrak{g},2}, \ldots, \mathscr{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$ be $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathscr{S}_{\mathfrak{g}} = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ for any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, it follows that $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu}) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu})\big)$ for any arbitrary pair $(\nu,\mu) \in I_{\sigma}^{*} \times I_{\sigma}^{*}$. But, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}), \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu}) \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$ and therefore, $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu}) \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$. Set $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) = \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}) \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu})$. Then, since $\mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$ is equivalent to $\mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big) \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$ and, the relation $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} = \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Op}_{\mathfrak{g}}\big(\hat{\mathscr{S}}_{\mathfrak{g}}\big)$ holds, it follows that $\mathscr{S}_{\mathfrak{g},\nu} \cup \mathscr{S}_{\mathfrak{g},\mu} \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the proposition is complete. □

Theorem 3.15. *Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ has $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, then it has also $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\big(\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}\big)\big[\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}\,[\mathfrak{T}_{\mathfrak{g}}] \ \longrightarrow\ \mathscr{S}_{\mathfrak{g}} \in \text{P}\,[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.9}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}$-P$\,[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, it satisfies the relation $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \longleftrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Since $\big(\text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \subseteq \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}), \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$, it follows that

$$\begin{aligned}
&\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\,,\\
&\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) &= \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\\
&= \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\,,
\end{aligned}$$

implying $\text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But, $\text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \cap \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently, it results that $\text{int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ which, in turn, implies $\text{cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Therefore, $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, meaning that $\mathscr{S}_{\mathfrak{g}}$ has also $\mathfrak{P}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$. Hence, $\mathscr{S}_{\mathfrak{g}} \in$ P$\,[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. □

Proposition 3.16. *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_{\sigma}^{*}\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then $\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}$ has also $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$\textstyle\bigwedge_{\nu \in I_{\sigma}^{*}} \big(\mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\big) \ \longrightarrow\ \displaystyle\bigcup_{\nu \in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Nd}\,[\mathfrak{T}_{\mathfrak{g}}]\,. \tag{3.10}$$

Proof. Let $\{\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]:\ \nu\in I_{\sigma}^{*}\}$ be a collection of $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Suppose $\bigwedge_{\nu\in I_{\sigma}^{*}}(\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])$ implies $\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is an untrue logical statement. Then, $\bigwedge_{\nu\in I_{\sigma}^{*}}(\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}])$ is true and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\longmapsto\emptyset$ is untrue. Thus, to prove the proposition, it suffices to prove that $\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\notin\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is a contradiction. For arbitrary $(\nu,\mu(\nu))\in I_{\sigma}^{*}\times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*}=I_{\sigma}^{*}\setminus\{\nu\}$, set $\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}=\mathscr{S}_{\mathfrak{g},\nu}\cup\mathscr{S}_{\mathfrak{g},\mu(\nu)}$, where $\{\mathscr{S}_{\mathfrak{g},\nu},\mathscr{S}_{\mathfrak{g},\mu(\nu)}\}\subset\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu})\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}&(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})\\&\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})\\&=\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu}).\end{aligned}$$

Thus, for arbitrary $(\nu,\mu(\nu))\in I_{\sigma}^{*}\times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*}=I_{\sigma}^{*}\setminus\{\nu\}$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})\big]&\\\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\nu})=\emptyset.&\end{aligned}$$

Since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it results that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\cap\mathfrak{g}\text{-Op}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})=\emptyset,$$

and therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})$. On the other hand, since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g},\mu(\nu)})=\emptyset,$$

Thus, $\mathscr{S}_{\mathfrak{g},(\nu,\mu(\nu))}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ holds for arbitrary $(\nu,\mu(\nu))\in I_{\sigma}^{*}\times I_{\sigma(\nu)}^{*}$ such that $I_{\sigma(\nu)}^{*}=I_{\sigma}^{*}\setminus\{\nu\}$ and hence, $\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. The relation $\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\notin\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is therefore a contradiction. The proof of the proposition is complete. □

Theorem 3.17. *Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. If $\mathscr{S}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$, then it has also $\mathfrak{Q}_{\mathfrak{g}}$-property in $\mathfrak{T}_{\mathfrak{g}}$:*

$$(\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}})\big[\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]\ \longleftarrow\ \mathscr{S}_{\mathfrak{g}}\in\text{Nd}[\mathfrak{T}_{\mathfrak{g}}]\big].\tag{3.11}$$

Proof. Let $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-property in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Suppose $\mathscr{S}_{\mathfrak{g}}\in\text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is an untrue logical statement. Then, $\mathscr{S}_{\mathfrak{g}}\in\text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\emptyset$ is untrue. Thus, to prove the theorem, it suffices to prove that $\mathscr{S}_{\mathfrak{g}}\notin\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is a contradiction. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it follows that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Consequently,

$$\text{int}_{\mathfrak{g}}\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big]\subseteq\text{int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Since $\mathscr{S}_{\mathfrak{g}}\in\text{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, it follows that $\text{int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\emptyset$ and therefore, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset$. Since $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, it results that

$$\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})=\emptyset,$$

implying $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\circ\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}}\longmapsto\emptyset$. Hence, $\mathscr{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$. The relation $\mathscr{S}_{\mathfrak{g}}\notin\mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is therefore a contradiction. The proof of the theorem is complete. □

The important remark given below ends the present section.

Remark. *In a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, the converse of the following statements with respect to some* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *are in general untrue:*

– I. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$,
– II. $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$,
– III. $\big(\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]\big) \longrightarrow \mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{g}}]$.

Because, in the event that $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}}) = \big(\mathbb{R}, \mathscr{T}_{\mathfrak{g},\mathbb{R}}\big) = \mathfrak{T}_{\mathfrak{g},\mathbb{R}}$ *and* $\mathscr{S}_{\mathfrak{g}} = \mathbb{Q}$ *(*$\mathbb{Q}$ *and* $\mathbb{R}$*, respectively, denote the sets of rational and real numbers, where* $\mathbb{R} \supset \mathbb{Q}$*), the converse of* ITEMS I.,II. *and* III.*, reading*

– IV. $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \longleftarrow \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,
– V. $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \longleftarrow \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,
– VI. $\big(\mathbb{Q} \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \longleftarrow \mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$,

respectively, are all untrue. In fact, every $\mathscr{T}_{\mathfrak{g}}$*-open set* $\mathscr{O}_{\mathfrak{g}} \in \mathscr{T}_{\mathfrak{g},\mathbb{R}}$ *contains both points* $\xi \in \mathbb{Q}$ *and* $\zeta \in \mathbb{R} \setminus \mathbb{Q}$*. Consequently, there are no* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-interior points of* $\mathbb{Q}$*. Therefore,* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) = \emptyset$ *and* $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) = \mathbb{R}$ *and thus,* $\mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \ni \mathbb{R} = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{R}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) \neq \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset) = \emptyset \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$*;* $\big(\mathbb{Q}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q})\big) \notin \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \times \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$*. In* ITEMS IV., V. *and* VI.*, the consequents* $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$*,* $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ *and* $\big(\mathbb{Q} \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big) \vee \big(\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]\big)$ *are all untrue and on the other hand, their antecedents* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$*,* $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathbb{Q}) \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ *and* $\mathbb{Q} \in \mathfrak{g}\text{-P}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ *are all true. Consequently,* ITEMS IV., V. *and* VI. *are all untrue statements and hence, the converse of* ITEMS I., II. *and* III. *are untrue statements. In addition, since* $\big(\mathbb{Q}, \mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathbb{Q})\big) \notin \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big] \times \mathfrak{g}\text{-Nd}\big[\mathfrak{T}_{\mathfrak{g},\mathbb{R}}\big]$ *it follows that, for some* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$*, the condition* $\mathfrak{g}\text{-Op}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ *can be satisfied without the condition* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$ *being satisfied, though* $\mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Op}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \neq \emptyset$ *for every* $\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ *is a consequence of* $\mathscr{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Nd}[\mathfrak{T}_{\mathfrak{g}}]$.

4. DISCUSSION

4.1. **Categorical Classifications.** Having adopted a categorical approach in the classifications of $\mathfrak{T}_{\mathfrak{a}}$-sets with $\{\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}\}$-property, the twofold purposes here are, firstly, to establish the various relationships amongst the classes of $\mathfrak{T}_{\mathfrak{a}}$-sets with $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}$-properties, $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$, in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and secondly, to illustrate them through diagrams.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{g}}$, since $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-P}[\mathfrak{T}_{\mathfrak{a}}]$ implies $\bigvee_{\nu \in I_3^0}\big(\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-P}[\mathfrak{T}_{\mathfrak{a}}]\big)$, it follows that, $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{a}}$ for each $\nu \in I_3^0$. Therefore, $\mathfrak{g}\text{-0-}\mathfrak{P}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-1-}\mathfrak{P}_{\mathfrak{a}} \longrightarrow \mathfrak{g}\text{-3-}\mathfrak{P}_{\mathfrak{a}} \longleftarrow \mathfrak{g}\text{-2-}\mathfrak{P}_{\mathfrak{a}}$. But, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{g}} \longleftarrow \mathfrak{g}\text{-}\nu\text{-}\mathfrak{P}_{\mathfrak{o}}$ for each $\nu \in I_3^0$. Hence, EQ. (4.1) present itself which may well be called $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}$*-property diagram.*

$$
\begin{array}{ccccccc}
\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-0-}\mathfrak{P}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{P}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{P}_{\mathfrak{o}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{P}_{\mathfrak{o}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-0-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-1-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-3-}\mathfrak{P}_{\mathfrak{g}} & \longleftarrow & \mathfrak{g}\text{-2-}\mathfrak{P}_{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}
\end{array}
\tag{4.1}
$$

In terms of the class $\left\{\mathfrak{g}\text{-}\nu\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right]:\ \nu\in I_3^{*}\right\}$, FIG. 1 present itself which may well be called $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}$-*class diagram.*

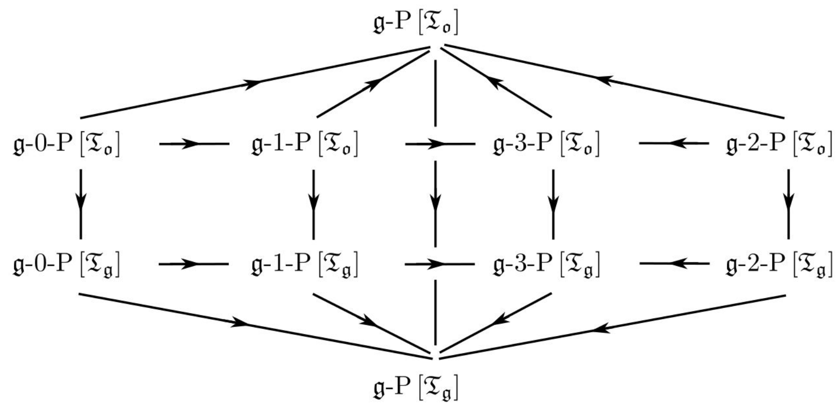

FIGURE 1. Relationships: $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}$-*class diagram* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

In $\mathfrak{T}_{\mathfrak{a}}$, since $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-Q}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ implies $\bigvee_{\nu\in I_3^0}\left(\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-}\nu\text{-Q}\left[\mathfrak{T}_{\mathfrak{a}}\right]\right)$, it follows that, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}\longleftarrow\mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{a}}$ for every $\nu\in I_3^0$. Therefore, $\mathfrak{g}\text{-}0\text{-}\mathfrak{Q}_{\mathfrak{a}}\longrightarrow\mathfrak{g}\text{-}1\text{-}\mathfrak{Q}_{\mathfrak{a}}\longrightarrow\mathfrak{g}\text{-}3\text{-}\mathfrak{Q}_{\mathfrak{a}}\longleftarrow\mathfrak{g}\text{-}2\text{-}\mathfrak{Q}_{\mathfrak{a}}$. But, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{o}}\longrightarrow\mathfrak{g}\text{-}\nu\text{-}\mathfrak{Q}_{\mathfrak{g}}$ for each $\nu\in I_3^0$. Thus, EQ. (4.2) present itself which may well be called $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}$-*property diagram.*

$$\begin{array}{ccccccc}
\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}\text{-}0\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-}1\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-}3\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longleftarrow & \mathfrak{g}\text{-}2\text{-}\mathfrak{Q}_{\mathfrak{o}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}0\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}1\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}3\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftarrow & \mathfrak{g}\text{-}2\text{-}\mathfrak{Q}_{\mathfrak{g}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longleftrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}
\end{array}\tag{4.2}$$

In terms of the class $\left\{\mathfrak{g}\text{-}\nu\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right]:\ \nu\in I_3^{*}\right\}$, FIG. 2 present itself which may well be called $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}$-*class diagram.*

In $\mathfrak{T}_{\mathfrak{a}}$, since $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right]$, $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ and $\mathscr{S}_{\mathfrak{a}}\in\text{Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ imply $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right]$, $\mathscr{S}_{\mathfrak{a}}\in\text{P}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ and $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right]$, respectively, it follows that $\mathfrak{Q}_{\mathfrak{a}}\longrightarrow\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}\longrightarrow\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}\longrightarrow\mathfrak{P}_{\mathfrak{g}}$ in $\mathfrak{T}_{\mathfrak{g}}$. Finally, $\mathscr{S}_{\mathfrak{a}}\in\text{Nd}\left[\mathfrak{T}_{\mathfrak{o}}\right]$ and $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{o}}\right]$ imply $\mathscr{S}_{\mathfrak{a}}\in\text{Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, respectively, and, $\mathscr{S}_{\mathfrak{a}}\in\text{P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ and $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ imply $\mathscr{S}_{\mathfrak{a}}\in\text{P}\left[\mathfrak{T}_{\mathfrak{o}}\right]$ and $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{o}}\right]$, respectively. Altogether, EQ. (4.3) present itself which may well be called $\left(\mathfrak{P}_{\mathfrak{a}},\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}};\mathfrak{Q}_{\mathfrak{a}},\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}\right)$-*properties diagram.*

$$\begin{array}{ccccccc}
\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{o}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{o}} & \longrightarrow & \mathfrak{P}_{\mathfrak{o}} \\
\downarrow & & \downarrow & & \uparrow & & \uparrow \\
\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}} & \longrightarrow & \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}} & \longrightarrow & \mathfrak{P}_{\mathfrak{g}}
\end{array}\tag{4.3}$$

In terms of the class $\left\{\text{Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right],\text{P}\left[\mathfrak{T}_{\mathfrak{a}}\right],\mathfrak{g}\text{-Nd}\left[\mathfrak{T}_{\mathfrak{a}}\right],\mathfrak{g}\text{-P}\left[\mathfrak{T}_{\mathfrak{a}}\right]\right\}$, FIG. 3 present itself which may well be called $\left(\mathfrak{P}_{\mathfrak{a}},\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}};\mathfrak{Q}_{\mathfrak{a}},\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}\right)$-*classes diagram.*

As in our previous works [1, 2, 19, 20], the manner we have positioned the arrows in the $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}$, $\left(\mathfrak{P}_{\mathfrak{a}},\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}};\mathfrak{Q}_{\mathfrak{a}},\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}}\right)$-*properties diagrams* (EQS (4.1),

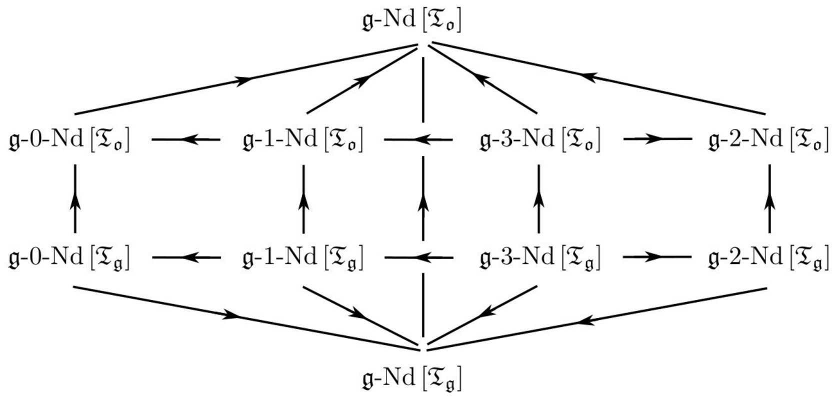

FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$-*property diagram* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

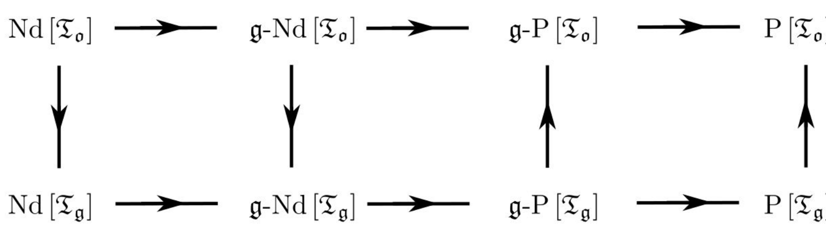

FIGURE 3. Relationships: $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*classes diagram* in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

(4.2), (4.3)) and the $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{a}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{a}}$, $(\mathfrak{P}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{a}}; \mathfrak{Q}_{\mathfrak{a}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{a}})$-*classes diagrams* (FIGS 1, 2, 3) is solely to stress that, in general, the implications in EQS (4.1)–(4.3) and FIGS 1–3 are irreversible.

4.2. **A Nice Application.** It is the purpose of this section to reveal through a nice application some characterizations on the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, and to give some other characterizations associated with $\mathfrak{T}_{\mathfrak{g}}$-sets having $\mathfrak{g}$-$\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{Q}_{\mathfrak{g}}$-properties in a $\mathscr{T}_{\mathfrak{g}}$-space. Consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where $\Omega = \{\zeta_\nu : \nu \in I_5^*\}$ and is topologized by the choice:

$$\mathscr{T}_{\mathfrak{g}}(\Omega) = \{\emptyset, \{\zeta_1\}, \{\zeta_1, \zeta_3, \zeta_5\}, \Omega\} = \{\mathscr{O}_{\mathfrak{g},1}, \mathscr{O}_{\mathfrak{g},2}, \mathscr{O}_{\mathfrak{g},3}, \mathscr{O}_{\mathfrak{g},4}\}; \quad (4.4)$$

$$\neg\mathscr{T}_{\mathfrak{g}}(\Omega) = \{\Omega, \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}, \{\zeta_2, \zeta_4\}, \emptyset\} = \{\mathscr{K}_{\mathfrak{g},1}, \mathscr{K}_{\mathfrak{g},2}, \mathscr{K}_{\mathfrak{g},3}, \mathscr{K}_{\mathfrak{g},4}\}. \quad (4.5)$$

For convenience of notation, let

$$\mathscr{P}(\Omega) = \{\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega) : (\nu, \mu) \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}\}, \quad (4.6)$$

where $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega)$ denotes a $\mathfrak{T}_{\mathfrak{g}}$-set labeled $\nu \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}$ and containing $\mu \in I^0_{\mathrm{card}(\Omega)}$ elements. Then, $\mathscr{R}_{\mathfrak{g},(1,0)} = \emptyset$, ..., $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} = \{\zeta_1, \zeta_2, \ldots, \zeta_\mu\}$, ..., $\mathscr{R}_{\mathfrak{g},(32,5)} = \Omega$.

For $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) \in \{0, 5\}$, let $\mathscr{R}_{\mathfrak{g},(1,0)} = \emptyset$ and $\mathscr{R}_{\mathfrak{g},(32,5)} = \Omega$. For $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) \in \{1, 4\}$, let $\mathscr{R}_{\mathfrak{g},(2,1)} = \{\zeta_1\}$, $\mathscr{R}_{\mathfrak{g},(3,1)} = \{\zeta_2\}$, $\mathscr{R}_{\mathfrak{g},(4,1)} = \{\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(5,1)} = \{\zeta_4\}$, and $\mathscr{R}_{\mathfrak{g},(6,1)} = \{\zeta_5\}$; $\mathscr{R}_{\mathfrak{g},(27,4)} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(28,4)} = \{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(29,4)} = \{\zeta_1, \zeta_3, \zeta_4, \zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(30,4)} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$, and $\mathscr{R}_{\mathfrak{g},(31,4)} = \{\zeta_1, \zeta_2, \zeta_4, \zeta_5\}$. For $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{R}_{\mathfrak{g}}) \in \{2, 3\}$, let

$\mathscr{R}_{\mathfrak{g},(7,2)} = \{\zeta_1,\zeta_2\}$, $\mathscr{R}_{\mathfrak{g},(8,2)} = \{\zeta_1,\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(9,2)} = \{\zeta_1,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(10,2)} = \{\zeta_1,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(11,2)} = \{\zeta_2,\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(12,2)} = \{\zeta_2,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(13,2)} = \{\zeta_2,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(14,2)} = \{\zeta_3,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(15,2)} = \{\zeta_3,\zeta_5\}$, and $\mathscr{R}_{\mathfrak{g},(16,2)} = \{\zeta_4,\zeta_5\}$; $\mathscr{R}_{\mathfrak{g},(17,3)} = \{\zeta_1,\zeta_2,\zeta_3\}$, $\mathscr{R}_{\mathfrak{g},(18,3)} = \{\zeta_1,\zeta_3,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(19,3)} = \{\zeta_1,\zeta_4,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(20,3)} = \{\zeta_1,\zeta_2,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(21,3)} = \{\zeta_1,\zeta_2,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(22,3)} = \{\zeta_1,\zeta_3,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(23,3)} = \{\zeta_2,\zeta_3,\zeta_4\}$, $\mathscr{R}_{\mathfrak{g},(24,3)} = \{\zeta_2,\zeta_3,\zeta_5\}$, $\mathscr{R}_{\mathfrak{g},(25,3)} = \{\zeta_3,\zeta_4,\zeta_5\}$, and $\mathscr{R}_{\mathfrak{g},(26,3)} = \{\zeta_2,\zeta_4,\zeta_5\}$. Then,

$$\begin{aligned}\mathrm{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) &= \mathscr{R}_{\mathfrak{g},(\nu,\mu)} \\ &= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big)\end{aligned} \tag{4.7}$$

for every $(\nu,\mu) \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}$. Consequently,

$$\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) = \mathscr{R}_{\mathfrak{g},(\nu,\mu)} = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \tag{4.8}$$

for every $(\nu,\mu) \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}$. Introduce $J^*_{28} = I^*_1 \cup (I^*_7 \setminus I^*_2) \cup (I^*_{16} \setminus I^*_{10}) \cup (I^*_{26} \setminus I^*_{22}) \cup (I^*_{28} \setminus I^*_{27})$. Then,

$$\begin{aligned}\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) = \emptyset = \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big), \\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) = \Omega = \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big)\end{aligned} \tag{4.9}$$

From EQ. (4.8), it follows that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, do *commute*. Thus, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is both *coarser and finer* (or, *smaller and larger*, *weaker and stronger*) than $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Consequently, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$. Furthermore, it is easily checked from EQ. (4.8) that, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue if and only if $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue.

From EQ. (4.9), both $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\nu,\mu) \in J^*_{28} \times I^0_4$ and $\mathscr{R}_{\mathfrak{g},(\delta,\eta)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\delta,\eta) \in \big(I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \setminus J^*_{28}\big) \times I^0_{\mathrm{card}(\Omega)}$ are easily checked. Moreover, it results from EQS (4.8), (4.9) that, $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{R}_{\mathfrak{g},(\nu,\mu)} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue for every $(\nu,\mu) \in \big(J^*_{28} \setminus I^*_1\big) \times I^0_4$. This confirms the statement that, $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}] \longleftarrow \mathscr{R}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue if and only if $\mathscr{R}_{\mathfrak{g}} \in \mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is true and $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{Nd}[\mathfrak{T}_{\mathfrak{g}}]$ is untrue. Observing that, for every $(\nu,\mu) \in J^*_{28} \times I^0_4$ and every $(\delta,\eta) \in \big(I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \setminus J^*_{28}\big) \times I^0_{\mathrm{card}(\Omega)}$, the relations

$$\begin{aligned}\emptyset = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) &\subseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \\ &= \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\nu,\mu)}\big) = \emptyset, \\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) = \Omega &\supseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) \\ &= \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big) \subseteq \Omega = \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\mathscr{R}_{\mathfrak{g},(\delta,\eta)}\big),\end{aligned}$$

respectively, hold, of which the first relation is the dual of the second, and conversely, it follows that the logical statement $\mathscr{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{P}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{R}_{\mathfrak{g}} \in \mathrm{P}[\mathfrak{T}_{\mathfrak{g}}]$ is satisfied for any $\mathscr{R}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$.

5. CONCLUSION

In a recent paper (CF. [19]), we defined and studied the essential properties of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces. We showed in a $\mathscr{T}_{\mathfrak{g}}$-space that $\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\big) : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is $(\Omega,\emptyset)$-*grounded*, (*expansive, non-expansive*), (*idempotent, idempotent*) and $(\cap,\cup)$-*additive*. We also showed in a $\mathscr{T}_{\mathfrak{g}}$-space that $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In this paper, we have studied in $\mathscr{T}_{\mathfrak{g}}$-spaces the commutativity of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\mathfrak{T}_{\mathfrak{g}}$-sets having some $(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}})$-based properties called $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$*-properties*. We have shown that the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are *duals* and $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under their $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-*operations*. We have also shown that a $\mathfrak{T}_{\mathfrak{g}}$-set having $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$*-property* is *equivalent* to the $\mathfrak{T}_{\mathfrak{g}}$-set or its complement having $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$*-property*. The $\mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\cup$*-operation* and $\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}$-property is *preserved* under the set-theoretic $\{\cup, \cap, \complement\}$*-operations*. Finally, a $\mathfrak{T}_{\mathfrak{g}}$-set having $\{\mathfrak{g}\text{-}\mathfrak{P}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{Q}_{\mathfrak{g}}\}$*-property* also has $\{\mathfrak{P}_{\mathfrak{g}}, \mathfrak{Q}_{\mathfrak{g}}\}$*-property*.

An interestingly promising avenue for future research arises if the theorization of $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators of mixed categories in $\mathscr{T}_{\mathfrak{g}}$-spaces be made a new subject of inquiry. For instance, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to study the $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$*-interior* and $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closure operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu\mu}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ respectively, in $\mathscr{T}_{\mathfrak{g}}$-spaces, where $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu\mu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points interior in $\mathscr{S}_{\mathfrak{g}}$ and interiorness are characterized by $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$*-open sets* belonging to the class $\{\mathscr{O}_{\mathfrak{g}} = \mathscr{O}_{\mathfrak{g},\nu} \cup \mathscr{O}_{\mathfrak{g},\mu} : (\mathscr{O}_{\mathfrak{g},\nu}, \mathscr{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\}$; $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu\mu} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g},\nu\mu}(\mathscr{S}_{\mathfrak{g}})$ describes a type of collection of points close to $\mathscr{S}_{\mathfrak{g}}$ and closeness are characterized by $\mathfrak{g}\text{-}(\nu,\mu)\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closed sets* belonging to the class $\{\mathscr{K}_{\mathfrak{g}} = \mathscr{K}_{\mathfrak{g},\nu} \cap \mathscr{K}_{\mathfrak{g},\mu} : (\mathscr{K}_{\mathfrak{g},\nu}, \mathscr{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\}$. Such a study is what we thought would be worth considering, and the discussion of this paper ends here.

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References

[1] M. I. Khodabocus and N. -U. -H. Sookia, *Theory of Generalized Sets in Generalized Topological Spaces*, Journal of New Theory **36** (2021) 18-38.

[2] M. I. Khodabocus, *A Generalized Topological Space endowed with Generalized Topologies*, PhD Dissertation, University of Mauritius, Réduit, Mauritius (2020) 1-311 (i.-xxxvi.).

[3] S. -M. Jung and D. Nam, *Some Properties of Interior and Closure in General Topology*, Mathematics (MDPI Journal) **7**(624) (2019) 1-10.

[4] Y. Lei and J. Zhang, *Generalizing Topological Set Operators*, Electronic Notes in Theoretical Science **345** (2019) 63–76.

[5] A. Gupta and R. D. Sarma, *A Note on some Generalized Closure and Interior Operators in a Topological Space*, Math. Appl. **6** (2017) 11-20.

[6] R. Rajendiran and M. Thamilselvan, *Properties of g^*s^*-Closure, g^*s^*-Interior and g^*s^*-Derived Sets in Topological Spaces*, Applied Mathematical Sciences **8**(140) (2014) 6969–6978.

[7] B. K. Tyagi and R. Choudhary, *On Generalized Closure Operators in Generalized Topological Spaces*, International Journal of Computer Applications **82**(15) (2013) 1-5.

[8] C. Cao and B. Wang and W. Wang, *Generalized Topologies, Generalized Neighborhood Systems, and Generalized Interior Operators*, Acta Math. Hungar. **132**(4) (2011) 310-315.

[9] V. Pankajam, *On the Properties of δ-Interior and δ-Closure in Generalized Topological Spaces*, International Journal for Research in Mathematical Archive **2**(8) (2011) 1321-1332.
[10] B. J. Gardner and M. Jackson, *The Kuratowski Closure-Complement Theorem*, New Zealand Journal of Mathematics **38** (2008) 9-44.
[11] Á. Császár, *Further Remarks on the Formula for γ-Interior*, Acta Math. Hungar. **113**(4) (2006) 325-332.
[12] Á. Császár, *On the γ-Interior and γ-Closure of a Set*, Acta Math. Hungar. **80** (1998) 89-93.
[13] H. Ogata, *Operations on Topological Spaces and Associated Topology*, Math. Japonica **36** (1991) 175-184.
[14] I. Z. Kleiner, *Closure and Boundary Operators in Topological Spaces*, Ukr Math. J. **29** (1977) 295-296.
[15] F. R. Harvey, *The Derived Set Operator*, The American Mathematical Monthly **70**(10) (1963) 1085-1086.
[16] D. H. Staley, *On the Commutivity of the Boundary and Interior Operators in a Topological Space*, The Ohio Journal of Science **68**(2):84 (1968).
[17] N. Levine, *On the Commutivity of the Closure and Interior Operators in Topological Spaces*, Amer. Math. Monthly **68**(5) (1961) 474-477.
[18] C. Kuratowski, *Sur l'Opération $\bar{A}$ de l'Analyse Situs*, Fund. Math. **3** (1922) 182-199.
[19] M. I. Khodabocus and N. -Ul. -H. Sookia, *Generalized Topological Operator Theory in Generalized Topological Spaces: Part I. Generalized Interior and Generalized Closure*, Proceedings of International mathematical Sciences **5**(1) (2023) 6-36.
[20] M. I. Khodabocus and N. -U. -H. Sookia, *Theory of Generalized Separation Axioms in Generalized Topological Spaces*, Journal of Universal Mathematics **5**(1) (2022) 1-23.
[21] Á. Császár, *Generalized Open Sets in Generalized Topologies*, Acta Math. Hungar. **106**(1-2) (2005) 53-66.
[22] V. Pavlović and A. S. Cvetković, *On Generalized Topologies arising from Mappings*, Vesnik **38**(3) (2012) 553-565.
[23] Á. Császár, *Remarks on Quasi-Topologies*, Acta Math. Hungar. **119**(1-2) (2008) 197-200.

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