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Research Article

The Frenet Vectors and the Curvatures of Curves $\mathbf{N}-\mathbf{T}^{*}\mathbf{B}^{*}$ in $\mathbf{E}^{3}$

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Abstract
In this paper, we describe a new pair of curves where the principal normal vector of a curve β and an vector R^{*} lying in the rectifian plane of a curve β^{*} are linearly dependent. We name them the curves $N-T^{*}B^{*}$. And we express the Frenet vectors and the curvatures of the curve β^{*} in terms of the Frenet vectors and the curvatures of the curve β.

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1. Introduction

One of the most studied topics in differential geometry is the theory of curves and surfaces, [1–3]. Producing one curve and another curve in conjunction with each other is a widely used method in differential geometry. Such curves are called associated curves. The studies on the associated curves are available in [4–12]. There are many special curve pairs produced this way. Some examples of these are Mannheim curves, Bertrand curves, involute and evolute curves. Mannheim curves were first described by A. Mannheim in 1878. If the principal normal direction of one curve is linearly dependent on the binormal direction of the other curve, these curves are called Mannheim curves, [13]. Bertrand curves were studied by Bertrand in 1850. If the principal normal of one curve is linearly dependent on the principal normal of another curve, these curves are called Bertrand curves, [14]. Involute and evolute curves are curves whose tangent directions intersect orthogonally, so the principal normal direction of the first curve and the tangent direction of the second curve are linearly dependent. There are many studies on these special curve pairs, [15–24]. Just like these special curves, new curve pairs can be obtained under different conditions. Various frames can be installed on the curves such as Frenet, Darboux, Sannia, etc. Of these, the Frenet frame consists of orthonormal vectors (tangent, principal normal and binormal vector) obtained by taking the derivatives of the curve. Curvatures, which play an important role in determining the character of the curve, are obtained from these Frenet vectors. Of these, the first curvature (curvature) measures how much the curve is curved, and the second curvature (torsion) measures how much the curve deviates from the plane. The studies on curves in general can be accessed from the resources [11, 25–39].

In this paper, we describe a new pair of curves under a new condition. The condition is that the principal normal vector of a curve β and an vector R^{*} lying in the rectifian plane of a curve β^{*} are linearly dependent. We name them the curves $N-T^{*}B^{*}$. So we calculate the Frenet vectors and the curvatures of the curve β^{*} in terms of the Frenet vectors and the curvatures of the curve β.

2. The Frenet vectors and the curvatures of curves $\mathbf{N-T^{*}B^{*}}$ in $\mathbf{E^3}$

Let $\beta(s)$ and $\beta^{*}(s^{*})$ be two differentiable unit speed curves in E^3. The Frenet frames and the curvatures of these curves are $\{T,N,B\}$, κ ,τ and $\{T^{*},N^{*},B^{*}\}$, κ^{*}, τ^{*} respectively.

Definition 1. *If the principal normal vector N of the curve $\beta(s)$ and the vector*

$$R^{*}(s)=\frac{aT^{*}(s)+bB^{*}(s)}{\sqrt{a^2+b^2}} \tag{1}$$

of the curve β^{*} are linearly dependent, then the curve $\beta(s)$ is called the curve $N-T^{*}B^{*}$ and the curve β^{*} is called the partner curve $N-T^{*}B^{*}$. So, these new curves are called as the curves $N-T^{*}B^{*}$ with the parametrizations $\beta^{*}(s)=\beta(s)+cR^{*}(s)$, where a,b is constant but c is not constant (Fig. 1).

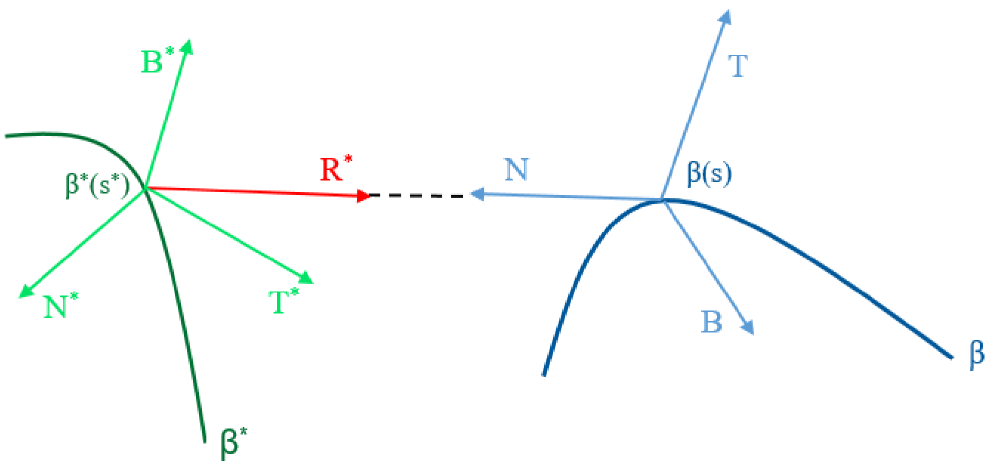

Fig. 1. $N-T^{*}B^{*}$ curves

According to this definition, we have

$$\langle R^{*},N^{*}\rangle=\langle N^{*},N\rangle=\langle R^{*},T\rangle=\langle R^{*},B\rangle=0. \tag{2}$$

And we have also the following equations:

$$\langle N,R^{*}\rangle = \left\langle N,\frac{aT^{*}+bB^{*}}{\sqrt{a^2+b^2}}\right\rangle=\frac{a\langle N,T^{*}\rangle+b\langle N,B^{*}\rangle}{\sqrt{a^2+b^2}}=1, \tag{3}$$

$$\langle T^{*},N\rangle = \langle R^{*},T^{*}\rangle=\left\langle \frac{aT^{*}+bB^{*}}{\sqrt{a^2+b^2}},T^{*}\right\rangle=\frac{a}{\sqrt{a^2+b^2}}, \tag{4}$$

$$\langle B^{*},N\rangle = \langle R^{*},B^{*}\rangle=\left\langle \frac{aT^{*}+bB^{*}}{\sqrt{a^2+b^2}},T^{*}\right\rangle=\frac{b}{\sqrt{a^2+b^2}}. \tag{5}$$

Theorem 2. *The tangent vector T^{*} of $N-T^{*}B^{*}$ partner curve based on the Frenet vectors and the curvatures of the curve β is*

$$T^{*}=\frac{a}{c'\sqrt{a^2+b^2}}\left((1-c\kappa)T+c'N+c\tau B\right). \tag{6}$$

Proof. Since if we take the derivative the equation $\beta^{*}=\beta+cR^{*}$ with respect to s^{*}, we have

$$\frac{d\beta^{*}}{ds^{*}}=\frac{d\beta^{*}}{ds}\frac{ds}{ds^{*}},$$

$$\frac{d\beta^{*}}{ds^{*}}=\left((1-\kappa c)T+c'N+c\tau B\right)\frac{ds}{ds^{*}}, \tag{7}$$

and

$$\left\| \frac{d\beta^*}{ds} \right\| = \sqrt{(1-\kappa c)^2 + c'^2 + \tau^2 c^2} = \sqrt{\delta}, \tag{8}$$

where $\delta = c'^2 + (1-\kappa c)^2 + \tau^2 c^2$. We know that tangent vector field is

$$T^* = \frac{d\beta^*}{ds^*} \left\| \frac{ds^*}{d\beta^*} \right\| = \frac{d\beta^*}{ds} \left\| \frac{ds}{d\beta^*} \right\|,$$

so from the expressions (7) and (8), we have

$$T^* = \frac{(1-\kappa c)\,T + c'N + \tau c B}{\sqrt{(1-\kappa c)^2 + c'^2 + \tau^2 c^2}} = \frac{(1-\kappa c)\,T + c'N + \tau c B}{\sqrt{\delta}}. \tag{9}$$

Also for the curve β^*, we write

$$\left\langle \frac{d\beta^*}{ds^*}, \frac{d\beta^*}{ds^*} \right\rangle = 1.$$

From the expression (7), we get

$$\left\langle \left((1-\kappa c)\,T + c'N + c\tau B\right) \frac{ds}{ds^*}, \left((1-\kappa c)\,T + c'N + c\tau B\right) \frac{ds}{ds^*} \right\rangle = 1,$$

$$\frac{1}{(1-\kappa c)^2 + c'^2 + (c\tau)^2} = \left(\frac{ds}{ds^*} \right)^2.$$

And so, from the expression (8), we have

$$\frac{ds}{ds^*} = \frac{1}{\sqrt{\delta}}. \tag{10}$$

From the expression (9), we get

$$\langle T^*, N \rangle = \left\langle \frac{(1-\kappa c)\,T + c'N + \tau c B}{\sqrt{(1-\kappa c)^2 + c'^2 + \tau^2 c^2}}, N \right\rangle = \frac{c'}{\sqrt{\delta}}. \tag{11}$$

From the equations of the expressions (4) and (11), we obtain

$$\frac{c'}{\sqrt{\delta}} = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{or} \quad \delta = \frac{c'^2 \left(a^2 + b^2\right)}{a^2}. \tag{12}$$

If we substitute the expression (12) in the expression (9), we get

$$T^* = \frac{a\,(1-\kappa c)}{c'\sqrt{a^2 + b^2}} T + \frac{a}{\sqrt{a^2 + b^2}} N + \frac{ac\tau}{c'\sqrt{a^2 + b^2}} B.$$

So the proof is completed. ■

Corollary 3. *There is the following relationship:*

$$\frac{\delta'}{\delta} = \frac{2c''}{c'}. \tag{13}$$

Proof. If we take the derivative the expression (12), since $\frac{\left(a^2+b^2\right)}{a^2}$ is constant, we get

$$\delta' = \frac{2c'c'' \left(a^2 + b^2\right)}{a^2}.$$

■

Corollary 4. *There is the relationship between the curvatures κ, τ, and the values a, b, c for the partner curve $N - T^*B^*$:*

$$c' = \frac{a}{b}\sqrt{(1-\kappa c)^2 + \tau^2 c^2}. \tag{14}$$

Proof. From the equation of the expressions (8) and (12), we have

$$\begin{aligned} \frac{c'^2\left(a^2+b^2\right)}{a^2} &= c'^2 + (1-\kappa c)^2 + \tau^2 c^2, \\ c'^2 &= \frac{a^2}{b^2}(1-\kappa c)^2 + \tau^2 c^2, \\ c' &= \frac{a}{b}\sqrt{(1-\kappa c)^2 + \tau^2 c^2}. \end{aligned}$$

■

Theorem 5. *There is the relationship between curvatures κ, τ and the values a, b, c of the partner curve $N - T^*B^*$:*

$$\frac{c'\delta'}{2\delta} = c'' - c\left(\kappa^2 + \tau^2\right) + \kappa. \tag{15}$$

Proof. Since $\dfrac{dT^*}{ds^*} = \kappa^* N^*$, we write

$$N^* = \frac{1}{\kappa^*}\frac{dT^*}{ds}\frac{ds}{ds^*}.$$

From the expression (2), we know that $\langle N^*, R^*\rangle = \langle N^*, N\rangle = 0$. So

$$\left\langle \frac{1}{\kappa^*}\frac{dT^*}{ds}\frac{ds}{ds^*}, N\right\rangle = 0.$$

Since $\dfrac{1}{\kappa^*} \neq 0$, $\dfrac{ds}{ds^*} \neq 0$, then we have

$$\left\langle \frac{dT^*}{ds}, N\right\rangle = 0. \tag{16}$$

If we take the derivative of the expression (2) with respect to s, we get

$$\frac{dT^*}{ds} = \frac{\left[(1-\kappa c)T + c'N + \tau cB\right]'\sqrt{\delta} - \left[(1-\kappa c)T + c'N + \tau c\right]\sqrt{\delta}'}{\delta},$$

$$\frac{dT^*}{ds} = \frac{1}{\delta}\left(\left[(1-\kappa c)T + c'N + \tau cB\right]'\sqrt{\delta} - \left[(1-\kappa c)T + c'N + \tau cB\right]\sqrt{\delta}'\right). \tag{17}$$

Here $\dfrac{1}{\delta} \neq 0$. If we substitute the expression (17) in the expression (16), we have

$$\left\langle N, \left[(1-\kappa c)T + c'N + \tau cB\right]'\sqrt{\delta}\right\rangle - c'\sqrt{\delta}' = 0. \tag{18}$$

If we write the expression (18) more clearly, we obtain

$$\begin{aligned} \frac{c'\sqrt{\delta}'}{\sqrt{\delta}} &= \left\langle N, \left[(1-\kappa c)T + c'N + \tau cB\right]'\right\rangle \\ &= \left\langle N, (1-\kappa c)'T + (1-\kappa c)T' + c''N + c'N' + (\tau c)'B + \tau cB'\right\rangle, \\ \frac{c'\delta'}{2\delta} &= c'' - c\left(\kappa^2+\tau^2\right) + \kappa. \end{aligned}$$

■

Theorem 6. *The first curvature κ^* of the partner curve $N-T^*B^*$ based on the Frenet vectors and the curvatures of the curve β is*

$$\kappa^* = \frac{1}{c'\delta}\left[\ \left(\left(-2c'^2\kappa - cc'\kappa'\right) - (1-\kappa c)\,c''\right)^2 + \left(c'\kappa - cc'\left(\kappa^2+\tau^2\right)\right)^2 + \left(\left(2c'^2\tau + cc'\tau'\right) - cc''\tau\right)^2\ \right]^{\frac{1}{2}}. \tag{19}$$

Proof. Since the equation $\kappa^* N^* = \dfrac{dT^*}{ds^*}$, from the expression (18), we have

$$\kappa^* N^* = \frac{dT^*}{ds}\frac{ds}{ds^*} = \frac{dT^*}{ds}\frac{1}{\sqrt{\delta}}.$$

And from the expression (17), we write

$$\kappa^* N^* = \frac{1}{\delta}\left[\ \left(\left(-2c'\kappa - c\kappa'\right) - (1-\kappa c)\frac{\delta'}{2\delta}\right)T + \left(\left(c''+\kappa-\kappa^2 c - c\tau^2\right) - c'\frac{\delta'}{2\delta}\right)N + \left(\left(2c'\tau + c\tau'\right) - c\tau\frac{\delta'}{2\delta}\right)B\ \right]. \tag{20}$$

Also from the equation $\kappa^{*2} = \langle \kappa^* N^*, \kappa^* N^*\rangle$ and the expression (13), we get the expression (19). ■

Theorem 7. *The normal vector field of the partner curve $N-T^*B^*$ based on the Frenet vectors and the curvatures of the curve β is*

$$N^* = \frac{1}{\Delta}\left[\ \left(-2c'^2\kappa - cc'\kappa' - (1-\kappa c)\,c''\right)T + \left(\left(\kappa - cc'\left(\kappa^2+\tau^2\right)\right)\right)N + \left(2c'^2\tau + cc'\tau' - cc''\tau\right)B\ \right], \tag{21}$$

where $\Delta = c'\delta\kappa^*$.

Proof. From the expressions (13) and (20), we get

$$\kappa^* N^* = \frac{1}{\delta}\left(\ \left(\left(-2c'\kappa - c\kappa'\right) - (1-\kappa c)\frac{c''}{c'}\right)T + \left(\left(c''+\kappa-\kappa^2 c - c\tau^2\right) - c'\frac{c''}{c'}\right)N + \left(\left(2c'\tau + c\tau'\right) - c\tau\frac{c''}{c'}\right)B\ \right).$$

Hence

$$N^* = \frac{1}{\delta\kappa^*}\left(\ \left(\left(-2c'\kappa - c\kappa'\right) - (1-\kappa c)\frac{c''}{c'}\right)T + \left(\left(c''+\frac{\kappa}{c'}-\kappa^2 c - c\tau^2\right) - c'\frac{c''}{c'}\right)N + \left(\left(2c'\tau + c\tau'\right) - c\tau\frac{c''}{c'}\right)B\ \right).$$

■

Theorem 8. *The binormal vector field of the partner curve $N-T^*B^*$ based on the Frenet vectors and the curvatures of the curve β is*

$$B^* = \frac{a}{c'\Delta\sqrt{a^2+b^2}}\left(\begin{array}{c} \left(\left(\left(2c'\tau + c\tau'\right)c'^2 - cc'c''\tau\right) - \left(c\tau\kappa - c^2c'\tau\left(\kappa^2+\tau^2\right)\right)\right)T \\ +\left(\left(1-c\kappa\right)\left(\left(2c'\tau+c\tau'\right)c'\right) + \left(\left(+2c'\kappa + c\kappa'\right)c'c\tau\right)\right)N \\ +\left(\left(1-c\kappa\right)\left(\left(\kappa - cc'\left(\kappa^2+\tau^2\right)\right) - c''\right) - \left(\left(-2c'\kappa - c\kappa'\right)c'^2\right)\right)B \end{array}\right). \tag{22}$$

Proof. From the equation $B^* = T^*\Lambda N^*$ and the expressions (9) and (21), we have

$$B^* = \frac{a}{\Delta c'\sqrt{a^2+b^2}}\left|\begin{array}{ccc} T & N & B \\ (1-c\kappa) & c' & c\tau \\ \left(\left(-2c'^2\kappa - cc'\kappa'\right) - (1-\kappa c)\,c''\right) & \left(\kappa - cc'\left(\kappa^2+\tau^2\right)\right) & \left(2c'\tau + c\tau'\right)c' - cc''\tau \end{array}\right|.$$

■

Theorem 9. *The second curvature τ^* of the partner curve $N-T^*B^*$ based on the Frenet vectors and the curvatures of the curve β is*

$$\tau^* = \frac{\dfrac{\Delta^2}{\delta} - 2c'\left(\kappa^2+\tau^2\right) - c\left(\kappa'\kappa + \tau'\tau\right) + \dfrac{cc''(\kappa+\tau) - c''}{c'}}{\left(\kappa - cc'\left(\kappa^2+\tau^2\right)\right)\sqrt{\delta}}.$$

Proof. From the expression (2), since $\langle N^*, N\rangle = 0$, it is clear that

$$\left\langle \frac{dN^*}{ds}, N\right\rangle + \left\langle N^*, \frac{dN}{ds}\right\rangle = 0.$$

So, from the expression (10), we write

$$\begin{aligned}
\left\langle \left(-\kappa^* T^* + \tau^* B^*\right)\frac{ds^*}{ds}, N\right\rangle + \left\langle N^*, \frac{dN}{ds}\right\rangle &= 0,\\
\frac{ds^*}{ds}\left\langle \left(-\kappa^* T^* + \tau^* B^*\right), N\right\rangle + \left\langle N^*, \frac{dN}{ds}\right\rangle &= 0,\\
\sqrt{\delta}\left\langle \left(-\kappa^* T^* + \tau^* B^*\right), N\right\rangle + \left\langle N^*, \frac{dN}{ds}\right\rangle &= 0.
\end{aligned}$$

From the last equation, we get

$$\tau^* \langle B^*, N\rangle = \kappa^* \langle T^*, N\rangle - \frac{1}{\sqrt{\delta}}\left\langle N^*, \frac{dN}{ds}\right\rangle,$$

$$\tau^* = \frac{1}{\sqrt{\delta}}\frac{\kappa^* c' + \kappa\langle N^*, T\rangle - \tau\langle N^*, B\rangle}{\langle B^*, N\rangle}. \tag{23}$$

Since $\Delta = c'\delta\kappa^*$ and if substitute the equations $\langle N^*, T\rangle, \langle N^*, B\rangle$ in the expression (23), we obtain

$$\tau^* = \frac{1}{\sqrt{\delta}}\frac{\frac{\Delta}{\delta} + \frac{\kappa}{\Delta}\left(\left(-2c'\kappa - c\kappa'\right) - (1-\kappa c)\frac{c''}{c'}\right) - \frac{\tau}{\Delta}\left(2c'\tau + c\tau' - c\frac{c''}{c'}\tau\right)}{\frac{1}{\Delta}\left(\left(\kappa - cc'\left(\kappa^2 + \tau^2\right)\right)\right)}.$$

■

3. Conclusions

Defining pairs of curves under certain conditions is a much studied topic in differential geometry, such as Bertrand, Mannheim, involute-evolute curves. There are many studies examining the relationships and geometric properties between these pairs of curves. It is possible to define new curve pairs under new conditions. Other curve pairs can be produced, such as the new curve pair discussed in this study, and these studies can also be considered in non-Euclidean spaces.

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