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Research Article

New Associated Curves $k-$Principle Direction Curves and N_k-Slant Helix

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Abstract
In this study, we present an alternative orthonormal frame system for spatial curves defined by principal directions in $3-$dimensional Euclidean space. The new curve characterization called as N_k-slant helix, which is an improved version of existing helices, is obtained as a fundamental outcome.

Keywords and 2020 Mathematics Subject Classification
Keywords: Slant helix — Clad helix — direction curve — curve of constant precession
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1. Introduction

From ancient times to the present, the relevance of curves has evolved. We may encounter their actions in the practical world, such as the decorative in art, the path of particles in physics, the profile of technical object in medicine, engineering etc. The definition of a curve as a continuous mapping does not always meet the demands. Hence, increasing the efficiency of curves is based on the examination of their geometric properties in particular, which makes the differentiable structures on them necessary. The geometric approach to the one dimensional manifold depending on the vector sets can be thought by traces out along them in view of common sense. The Frenet trihedron $T(s)$, $N(s)$, $B(s)$ is a well known material to give us information about the behavior of curve $\gamma(s)$ in a neighborhood of s [1,2]. Since the moving frame can be uniquely defined by the tangential vector field T, the principal normal vector field N is referred to as the reference vector of the trihedron. The following equation,

$$\gamma(s) = \int T(s)ds$$

denotes the family of curves with the tangent vector T, then the reference vector N points out which of the curve has the trihedron belongs to.

On the other hand, spatial curves are only distinguished by the way how they bend and twist. These distinctive features are quantitatively measured by the invariants the curvature $\kappa(s)$ and the torsion $\tau(s)$, respectively. The fundamental theorem of curves declares that the existence of a unique regular parametrized curve γ occurs via specific curvature and torsion [3]. This leads to the following form of the curve:

$$\gamma(s) = \gamma(0) + \left(s - \frac{s^3\kappa^2}{6}\right)T + \left(\frac{s^2\kappa}{2} + \frac{s^3\kappa'}{6}\right)N + \frac{s^3\kappa\tau}{6}B + o(s^3). \tag{1}$$

The remarkable side of curvatures appears through the characterizations of some specific curves called helices. A helix whose tangent straight-lines make a constant angle with a fixed direction is made sense by the characterization such as the constant ratio of curvature to torsion which is stated by M. A. Lancret in 1802 and first proved by B. S. Venant in 1845 [4,5]. If the normal lines or the Darboux vector take the same role instead of the tangents, then we obtain other characterizations of helix types called slant helix and constant precession curve, respectively [6,7]. Recently, new kinds of helices, called clad helices and g-clad helices, are defined by Takahashi and Takeuchi [8].

In the theory of the orthonormal frame along curves we can see there are several frame structures that can be built for a spatial curve [9–11]. One of the orthonormal frames is introduced and named as $\{N,C,W\}$ by Uzunoğlu et al. [12]. They also give a definition of $C-$slant helix whose vector field C makes a constant angle along a fix direction. This proves that the characteristic features of the $C-$slant helices and clad helices coincided. This theory also generalized in Minkowski space by using direction curves [13].

In this paper, we concentrate the emphasis on the reference vector N of the curve γ. The integral curves of N called as the principal direction curve of γ motivate us to develop a curve characterization in terms of its Frenet frame by moving on the main curve. Moreover, we establish an algorithm by generalizing $k-$principal direction curves and yielding a set of orthonormal frame which has the same parameter of the main curve γ. Hence, we get the authority to define the main curve γ with different orthonormal frames and γ has a new curve characterization called a N_k-slant helix whose $k-$principal direction curve is a slant helix. This classification provides us to make a connection between helix types such as slant helix, $C-$slant helix, curves of constant precession, g-clad helix and N_k-slant helices for $k=0,1$ and 2. As a consequence, we gather all helix types under a single generalized helix definition and are able to acquire a different orthonormal frame creation method for spatial curves whose character cannot be determined.

2. Preliminaries

In this section, we will call some basic concepts on the geometric features of helical type curves in Euclidean $3-$space.

Let $\gamma(s)$ be a unit-parametrized curve and the triple $\{T(s),N(s),B(s)\}$ be the Frenet-Serret frame on γ, then we have the following:

$$T=\gamma',\ N=\frac{\gamma''}{\|\gamma''\|},\ B=T\times N \tag{2}$$

and the Frenet formulas hold

$$\kappa=<T',N>\ \text{ and }\ \tau=<N',B>, \tag{3}$$

where the functions κ, τ are called the curvature and the torsion of γ, respectively. In sense of the curve theory, κ and τ describe completely how the frame evolves in time along the curve. Thus, the geometry of a unit-speed curve that depends only on the values of curvature and torsion can be given as follows:

- If $\kappa=0$, we obtain a uniformly-parametrized straight line,
- If $\tau=0$, γ be a planar curve,
- If γ has constant curvature and zero torsion, then γ becomes part of a circle of radius $1/\kappa$,
- γ is a helix which has the property that the tangent line at any point makes a constant angle with a fixed line called the axis if and only if the rate $\dfrac{\kappa}{\tau}$ is constant,
- If the principal normal vector at any point of γ makes a constant angle with a fixed line, γ is called slant helix and this leads that the geodesic curvature of N, $\sigma=\dfrac{\kappa^2\left(\frac{\tau}{\kappa}\right)'}{(\kappa^2+\tau^2)^{3/2}}$ is a constant function,
- γ is a curve of constant precession which has the property that its Darboux vector revolves about a fixed direction with constant angle and constant speed, then this yields $\kappa^2+\tau^2$ is a constant function.

When some point p moves along the regular curve γ, the Frenet frame $\{T,N,B\}$ changes, so the spherical indicatrix occur, the Frenet frame makes an instant helical motion at each s moment along an axis called Darboux axis of $\gamma(s)$. The vector, indicates the direction of axis, is called Darboux vector of γ and expressed as

$$W=\tau T+\kappa B. \tag{4}$$

The Darboux vector W has also the following symmetrical properties:

$$\begin{aligned} W \times T &= T', \\ W \times N &= N', \\ W \times B &= B'. \end{aligned} \tag{5}$$

Moreover, the Darboux vector W provides a concise way of interpreting curvature κ and torsion τ geometrically: Curvature is the measure of the rotation of the Frenet frame about the binormal unit vector B, whereas torsion is the measure of the rotation of the Frenet frame about the tangent vector T.

The axes of helices and slant helices can be explained respectively, as follows:

$$\vec{u} = \cos\theta T + \sin\theta B \text{ or } \vec{u} = \sin\psi W + \cos\psi N, \tag{6}$$

where θ, ψ are the constant angles with the tangent and the normal vector field of γ when γ is a helix or slant helix, respectively, (more details can be seen in [14]).

Another orthonormal frame is determined by using the reference vector N and the Darboux vector W as follows:

$$\{N,\, C = \frac{N'}{\| N \|}, \tilde{W} = N \times C\}, \tag{7}$$

where $\tilde{W}$ is unit Darboux vector of γ. There is a close relationship between Frenet curvatures and the curvatures of equation (7), such as $f = \sqrt{\kappa^2 + \tau^2}$ and $g = \sigma f$. If the vector field C makes a constant angle with a line, then γ is called as a $C-$slant helix [12].

If the functions are constants

$$\Phi(s) = \frac{\sigma'}{f(1+\sigma^2)^{3/2}},\ \Psi(s) = \frac{\Phi'}{\mathscr{F}(1+\Phi^2)^{3/2}},$$

where $\mathscr{F} = \sqrt{f^2 + g^2}$, then the spatial curve γ are called clad helix and $g-$clad helix, respectively [8].

When we compare the consequences of the characterizations $C-$slant helix and clad helix, they denote the same characterization of a spatial curve in terms of different orthonormal frames. Hence we can deduce that a clad helix is a slant helix while Φ is zero and a $g-$clad helix is a clad helix while Ψ is zero.

3. $k-$principal direction curve and N_k-slant helix

In this section, we create an algorithm to produce orthonormal frames for a spatial curve and obtain the helical characterization for the curve. Firstly, we introduce the generalization of the principal direction curves which are members of a subfamily of integral curves of $\gamma \in C^{n+2}$ in terms of $\{T, N, B\}$. The fact that Frenet frames of principal direction curves have the same parameter of the base curve, thus the algorithm works by moving them above the base curve and in the next step, it checks if the second vector field makes a constant angle with a fixed direction. Finally, we can determine the suitable orthonormal frame for the spatial curve which yields the helical feature and calls N_k-slant helix.

Now, let us arrange the mathematical steps for the characterization of N_k-slant helices.

Definition 1. *Let $\gamma(s)$ be a regular unit-speed curve in terms of $\{T, N, B\}$. The integral curves of $T(s), N(s)$ and $B(s)$ are called the tangent direction curve, principal direction curve and binormal direction curve of γ, respectively [15].*

According to the sense of Frenet frame, the tangent direction curve $\gamma_0 = \int T(s)ds = \gamma + \vec{c}$ which is the translation form of γ along the vector $\vec{c}$ has the same frame with γ as $\{T_0 = T,\ N_0 = N,\ B_0 = B\}$. Also the binormal direction curve has the permutation frame such as $\{B, N, T\}$. On the other hand the principal direction curve of γ, $\gamma_1 = \int N(s)ds$ has a new frame as follows:

$$\begin{aligned} T_1 &= N, \\ N_1 &= \frac{N'}{\| N' \|} = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}, \\ B_1 &= T_1 \times N_1 = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}, \end{aligned} \tag{8}$$

where the tangent vector and the binormal vector of γ_1 are the principal normal vector and the unit Darboux vector of γ, respectively. Also curvatures of γ_1 are

$$\kappa_1 = \sqrt{\kappa^2+\tau^2},\ \ \tau_1 = \sigma\kappa_1, \tag{9}$$

where σ is the geodesic curvature of N_1.

Remark 2. *The Frenet frame of* γ_1*, the principal direction curve of* γ*, coincides with the frame* $\{N, C, W\}$ *along* γ*.*

If the principal direction curve of γ is a regular curve, then its principal-direction curve can be defined and the process repeated again for itself. So if γ is a $(n+2)-$differentiable curve, then the principal direction curves generated as follows:
The first principal direction curve is

$$\gamma_1 = \int N(s)ds \text{ with the frame } \left\{T_1 = N, N_1 = \frac{N'}{\| N' \|}, B_1 = T_1 \times N_1\right\}.$$

The second principal direction curve is

$$\gamma_2 = \int N_1(s)ds \text{ with the frame } \left\{T_2 = N_1, N_2 = \frac{N_1'}{\| N_1' \|}, B_2 = T_2 \times N_2\right\}.$$

$$\vdots$$

The (n)th principal direction curve is

$$\gamma_n = \int N_{n-1}(s)ds \text{ with the frame } \left\{T_n = N_{n-1}, N_n = \frac{N_{n-1}'}{\| N_{n-1}' \|}, B_n = T_n \times N_n\right\}.$$

Definition 3. *Let* $\gamma(s)$ *be a unit-parametrized and* $(n+2)-$*differentiable curve in terms of* $\{T, N, B\}$ *in* $\mathbb{E}^3$ *and* γ_0 *be the tangent direction curve of* γ*. The* $k-$*principal direction curve of* γ *is defined as*

$$\gamma_k(s) = \int N_{k-1}ds, \quad 1 \leq k \leq n, \tag{10}$$

where N_{k-1} *is the principal normal vector of* γ_{k-1}*.* γ *is called the main curve of* γ_k*.*

The Frenet frame and curvatures of γ_k are

$$T_k = N_{k-1},\ N_k = \frac{N_{k-1}'}{\| N_{k-1}' \|},\ B_k = T_k \times N_k, \tag{11}$$

$$\kappa_k = \sqrt{\kappa_{k-1}^2 + \tau_{k-1}^2},\ \ \tau_k = \sigma_{k-1}\kappa_k, \tag{12}$$

where $N_{k-1}, \kappa_{k-1}, \tau_{k-1}, \sigma_{k-1}$ are the principal normal vector, curvature, torsion and geodesic curvature of N_{k-1}, respectively. Besides, B_k is the unit Darboux vector of γ_{k-1}.

$\{T_k, N_k, B_k\}$ the Frenet frame and $W_k = \tau_k T_k + \kappa_k B_k$ the Darboux vector of γ_k satisfy the equations as follows:

$$\begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & -\tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}, \tag{13}$$

$$\begin{aligned} W_k \times T_k &= T_k', \\ W_k \times N_k &= N_k', \\ W_k \times B_k &= B_k'. \end{aligned} \tag{14}$$

With the help of helix theory, we can talk about relation between $k-$principal direction curve and its main curve. Therefore, the following definition are given to keep the generality.

Definition 4. *Let* γ *be a* $(n+2)-$*differentiable spatial curve,* γ_0 *and* γ_k *be the tangent direction curve and* $k-$*principal direction curves of* γ*,* $1 \leq k \leq n$*, respectively. The curve* γ *is called* N_k-*slant helix which has the property that the principal normal vector of* γ_k *makes a constant angle with a fixed line called the axis.*

Remark 5. *If the normal vector of any curve makes a constant angle with a fixed line, then we called the curve as a slant helix in the preliminaries section of the paper. In this theory, being slant helix is the case of* $k = 0$*.*

After giving all materials which are the above-mentioned definitions, now let give the following theorems and results to complete the theory.

Theorem 6. *Let γ be a $(n+2)-$differentiable curve in $\mathbb{E}^3$ and γ_k be the $k-$principal direction curves of γ, $1 \leq k \leq n$. The curve γ is a N_k-slant helix if and only if γ_k is a slant helix.*

Proof. From Definition 4, if γ is a N_k-slant helix, then he principal normal vector of γ_k makes a constant angle with a fixed direction $\vec{u}$

$$\langle N_k, \vec{u} \rangle = \cos\theta.$$

So, this definition coincides with being the slant helix of γ_k.

This completes the proof. ■

Corollary 7. *The main curve γ is a N_k-slant helix if and only if $\sigma_k = \dfrac{\kappa_k^2\left(\frac{\tau_k}{\kappa_k}\right)'}{\left(\kappa_k^2+\tau_k^2\right)^{3/2}}$ which is the geodesic curvature of N_k is a constant function.*

Corollary 8. *If γ is a N_1-slant helix if and only if γ is a $C-$slant helix or a clad helix and the first principle direction curve of γ is a slant helix.*

Corollary 9. *If γ is a N_2-slant helix if and only if γ is a $g-$clad helix and the first principle direction curve of γ is a slant helix.*

Theorem 10. *Let γ be a $(n+2)-$differentiable spatial curve, γ_k be the $k-$principal direction curve of γ, $1 \leq k \leq n$. The curve γ is a N_k-slant helix if and only if the tangent indicatrix of γ_k is a helix.*

Proof. Let γ be a N_k-slant helix, then from Theorem 6 the $k-$principal direction curve γ_k is a slant helix. So, its tangent indicatrix T_k is a helix.

Conversely, if T_k is a helix, then the rate its curvature κ_T and torsion τ_T is constant. This means that,

$$\frac{\tau_T}{\kappa_T} = \frac{\kappa_k^2\left(\frac{\tau_k}{\kappa_k}\right)'}{\left(\kappa_k^2+\tau_k^2\right)^{3/2}} = \text{constant},$$

where τ_k, κ_k are the torsion and curvature of γ_k, respectively.

This completes the proof. ■

The Frenet frame of γ_k, $\{T_k, N_k, B_k\}$ is an orthonormal basis of $\mathbb{E}^3$ also, parametrized the same parameter of γ. Thus $\{T_k, N_k, B_k\}$ is an alternative orthonormal frame for the main curve γ. With the help of the frame sense, let define the axis of N_k-slant helix.

Theorem 11. *Let γ, the main curve of γ_k, be a N_k-slant helix in $\mathbb{E}^3$. The axis of γ is*

$$\vec{u} = \sin\theta \overline{W}_k + \cos\theta N_k, \tag{15}$$

where $N_k, \overline{W}_k$ are the principal normal vector, the unit Darboux vector of γ_k and θ is the constant angle with N_k.

Proof. Let the main curve γ be a N_k-slant helix with the axis $\vec{u}$, then

$$\langle N_k, \vec{u} \rangle = \cos\theta, \tag{16}$$

where N_k is the principal normal vector of γ_k.

After the twice derivatives of equation (16), we get

$$\begin{aligned} -\kappa_k\langle T_k, \vec{u}\rangle + \tau_k\langle B_k, \vec{u}\rangle &= 0, \\ -\kappa_k'\langle T_k, \vec{u}\rangle + \tau_k'\langle B_k, \vec{u}\rangle &= \left(\kappa_k^2+\tau_k^2\right)\cos\theta. \end{aligned} \tag{17}$$

Then the solutions of equation (17) are

$$\begin{aligned} \langle T_k, \vec{u}\rangle &= \frac{1}{\sigma_k}\tau_k\cos\theta, \\ \langle B_k, \vec{u}\rangle &= \frac{1}{\sigma_k}\kappa_k\cos\theta. \end{aligned} \tag{18}$$

So, the axis of γ is obtained as

$$\vec{u} = \left(\frac{1}{\sigma_k}\tau_k T_k + N_k + \frac{1}{\sigma_k}\kappa_k B_k\right)\cos\theta = \sin\theta \overline{W}_k + \cos\theta N_k. \tag{19}$$

■

Corollary 12. *If any spatial curve γ is a N_k-slant helix with the axis $\vec{u} = \sin\theta\overline{W}_k + \cos\theta N_k$, then it is obviously seen that $\overline{W}_k$ which is the unit Darboux vector of γ_k makes a constant angle with a fixed direction $\vec{u}$. Thus, the curve γ is also called $\overline{W}_k-$Darboux helix in $3-$dimensional Euclidean space.*

4. Curves of N_k-constant precession

In this section, we define a special subfamily of N_k-slant helix called N_k-constant precession in $\mathbb{E}^3$. Characteristics of N_k-constant procession are researched in terms of $k-$principal direction curves.

Any spatial curve called curve of constant precession has the property that its Darboux vector revolves about a fixed direction $\vec{u}$ called the axis with constant angle and constant speed. As a consequence, its Frenet frame precesses about the axis, while its principal normal revolves about the axis with constant complementary angle and constant speed [7]. So, the family of constant precession is a kind of slant helix which has constant-length Darboux vector. With the help of constant precession theory, the following definition can be given.

Definition 13. *Let $\gamma \in C^{n+2}$, γ_0 and γ_k be the tangent direction curve and $k-$principal direction curves of γ, $1 \leq k \leq n$, respectively. The curve γ is called curve of N_k-constant procession which has the property that the Darboux vector of γ_k makes a constant angle with a fixed direction $\vec{u}$ and constant speed in terms of $\{T_k, N_k, B_k\}$.*

Corollary 14. *If γ is a N_k-slant helix with the Darboux vector $\| W_k \| = \omega$ (constant) in terms of $\{T_k,\ N_k,\ B_k\}$, then γ is a curve of N_k-constant precession.*

In previous section, we showed that if the main curve γ is a N_k-slant helix, then its axis lies in the plane spanned by $\overline{W}_k$, N_k in equation (15). If γ is a curve of N_k-constant procession with $\kappa_k^2 + \tau_k^2 = \omega^2$, then the axis of γ is obtained such as

$$\vec{u} = W_k + \mu N_k. \tag{20}$$

Now, lets specialize the Darboux vector for the curve of N_k-constant procession in the following theorem.

Theorem 15. *Any spatial curve γ is a curve of N_k-constant procession with the axis $\vec{u} = W_k + \mu N_k$, a necessary and sufficient condition is that*

$$\kappa_k = \omega \sin\mu\theta \text{ and } \tau_k = \omega\cos\mu\theta, \tag{21}$$

where ω is the length of Darboux vector W_k.

Proof. Let γ be a curve of N_k-constant procession with the axis $\vec{u} = W_k + \mu N_k$ with $\|W_k\| = \omega$, $\|\vec{u}\| = \left(\omega^2 + \mu^2\right)^{1/2}$ where ω and μ are constant. The differentiate of $\vec{u}$

$$\vec{u}' = (\tau_k' - \mu\kappa_k)T_k + (\kappa_k' + \mu\tau_k)B_k \tag{22}$$

is zero, therefore

$$\tau_k' - \mu\kappa_k = 0 \text{ and } \kappa_k' + \mu\tau_k = 0. \tag{23}$$

The solution of this differential equation is uniquely

$$\kappa_k = \omega\sin\mu\theta \text{ and } \tau_k = \omega\cos\mu\theta. \tag{24}$$

This completes the proof. ■

Example 16. *Let $\gamma(s)$ be a unit speed curve with curvatures $\kappa(s) = \sin s\cos(\sin s)$, $\tau(s) = \sin s\sin(\sin s)$ and the curve γ specifies neither helix nor slant helix in terms of the Frenet frame $\{T(s), N(s), B(s)\}$. Now lets obtain principal direction* *curves of* γ *and characterize itself. Using the equations (12), the curvature and torsion of* $\gamma_1 = \int N(s)ds$ *the first principal direction curve of* γ *are obtained as*

$$\kappa_1 = \sin s \text{ and } \tau_1 = \cos s.$$

So from Theorem 1 in [7], γ_1 *is a curve of constant precession with the equation* $\gamma_1(s) = (a_1(s), a_2(s), a_3(s))$ *is*

$$\begin{aligned} a_1(s) &= \frac{(\sqrt{2}+1)^2}{2\sqrt{2}}\sin(\sqrt{2}-1)s - \frac{(\sqrt{2}-1)^2}{2\sqrt{2}}\sin(\sqrt{2}+1)s, \\ a_2(s) &= -\frac{(\sqrt{2}+1)^2}{2\sqrt{2}}\cos(\sqrt{2}-1)s + \frac{(\sqrt{2}-1)^2}{2\sqrt{2}}\cos(\sqrt{2}+1)s, \\ a_3(s) &= \frac{1}{\sqrt{2}}\sin s, \end{aligned}$$

and the tangent vector field $T_1 = (t_1, t_2, t_3)$ *is*

$$\begin{aligned} t_1 &= \frac{\sqrt{2}+1}{2\sqrt{2}}\cos(\sqrt{2}-1)s - \frac{\sqrt{2}-1}{2\sqrt{2}}\cos(\sqrt{2}+1)s, \\ t_2 &= \frac{\sqrt{2}+1}{2\sqrt{2}}\sin(\sqrt{2}-1)s - \frac{\sqrt{2}-1}{2\sqrt{2}}\sin(\sqrt{2}+1)s, \\ t_3 &= \frac{1}{\sqrt{2}}\cos s, \end{aligned}$$

the principal vector field $N_1 = (n_1, n_2, n_3)$ *is*

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{2}}\cos\sqrt{2}s, \\ n_2 &= \frac{1}{\sqrt{2}}\sin\sqrt{2}s, \\ n_3 &= \frac{-1}{\sqrt{2}}, \end{aligned}$$

the binormal vector field $B_1 = (b_1, b_2, b_3)\}$ *is*

$$\begin{aligned} b_1 &= -\sin\sqrt{2}s\cos s + \frac{1}{\sqrt{2}}\cos\sqrt{2}s\sin s, \\ b_2 &= \cos\sqrt{2}s\cos s + \frac{1}{\sqrt{2}}\sin\sqrt{2}s\sin s, \\ b_3 &= \frac{1}{\sqrt{2}}\sin s, \end{aligned}$$

where $T_1 = N(s)$, B_1 *is unit Darboux vector field of* γ *also the Darboux vector of* γ_1 *is*

$$W_1 = (\frac{1}{\sqrt{2}}\cos\sqrt{2}s, \frac{1}{\sqrt{2}}\sin\sqrt{2}s, \frac{1}{\sqrt{2}}).$$

From Theorem 6, the first principal direction curve of γ *is a slant helix so* γ *is a* N_1-*slant helix. Moreover the principal curve* γ_1 *is a curve of constant precession with constant speed Darboux, then* γ *is called curve of* N_1-*constant procession. Because,* $T_1(s)$ *tangent indicatrix of* γ_1 *is a helix with the curvatures are* $\bar{\kappa} = \frac{1}{\sin s}$, $\bar{\tau} = \frac{-1}{\sin s}$ *and the fixed angle between tangent and the fixed direction is* $\theta = \frac{3\pi}{4}$*. So the axis of* γ *is*

$$\begin{aligned} \vec{u} &= \sin\theta\overline{W}_1 + \cos\theta N_1 \\ &= (0, 0, -1). \end{aligned}$$

The second principal direction curve $\gamma_2 = \int N_1 ds = \left(\frac{1}{2}\sin\sqrt{2}s, \frac{-1}{2}\cos\sqrt{2}s, \frac{-1}{\sqrt{2}}s\right)$ *is helix with curvatures and axis* $\kappa_2 = 1$, $\tau_2 = -1$, $\vec{v} = (0, 0, 1)$, *respectively. On the other hand the set of helices are the special subfamily of slant helices. So the curve* γ *is a* N_2-*slant helix with the axis* $\vec{v}$ *in terms of* $\{T_2, N_2, B_2\}$ *the Frenet frame of* γ_2*. With using the equation (1) and the Mathematica programme, we obtain approximately figures of the curve* γ, γ_1 *and* γ_2.

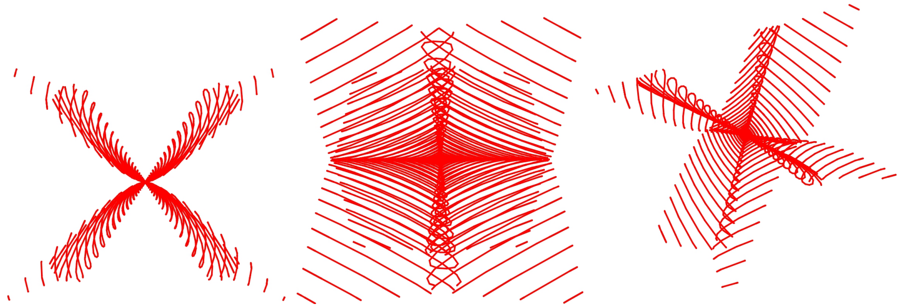

Fig. 1. The main curve $\gamma(s)$

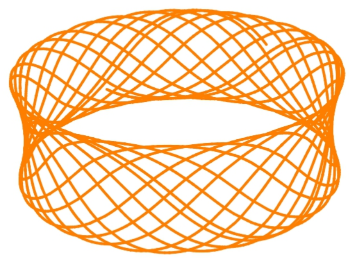

Fig. 2. The first principal direction curve γ_1

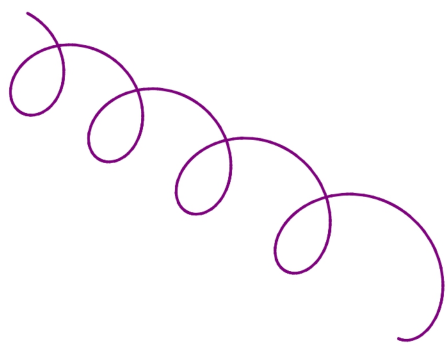

Fig. 3. The second principal direction curve γ_2

5. Conclusions

Obtaining identification of curves is the one of important research area in mathematics. Generally, the Frenet frame and curvatures of curves are used, then the most famous characterizations are given for helix and slant helix by Lancret, Izumiya and Takeuchi [5, 6]. Besides, Scofield calls the slant helices with constant speed Darboux vector field as curves of constant precession [7]. On the other hand, the Frenet frame is not enough to explain the characteristics of curves, new orthonormal frames can be built on the curve to make up the disadvantages of it. In this study, new orthonormal frames which can be positioned on the regular spatial curve γ are obtained to give characterization for γ with the help of principal direction curves of γ. Then we called the curve γ as a N_k-slant helix, if γ_k the $k-$principal direction curve of γ is a slant helix in terms of the frame $\{T_k, N_k, B_k\}$ and showed that the axis of N_k-slant helix lies in the plane $Sp\{\overline{W}_k, N_k\}$ where $\overline{W}_k$ and N_k are the unit Darboux and the principal normal vector of γ_k, thus γ is also named as a $\overline{W}_k-$Darboux helix in $3-$dimensional Euclidean space. When the condition is $k = 1$, the curve γ is a N_1-slant helix or $C-$slant helix according to Uzunoğlu, Gök and Yaylı [12]. Moreover, with the help of the spherical image of tangent indicatrix and its own parametrized Frenet frame, a new special curve family called $k-$slant helix is introduced by Ali [16], but $k-$slant helix and N_k-slant helix totally specify different curve characterizations. Since the arc-length parameter of $k-$principal direction curve of γ is the same as the main curve γ, the impressions and expressions of theorems and definitions are obtained clearly and simply for N_k-slant helix. In addition, if the N_k-slant helix γ has constant speed Darboux vector, then γ is called the curve of N_k-constant precession. It can be easily seen that the curve γ is a slant helix for $k = 0$, if γ has constant speed Darboux then we can obtain the characterization of the curve of constant precession.

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