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Research Article

A Note on the Theory of Gamma and Beta Functions

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Abstract

Physics and engineering problems require a detailed knowledge of applied mathematics and an understanding of special functions such as gamma and beta functions. The topic of special functions is very important and it is constantly expanding with the existence of new problems in the applied sciences. In this article, we describe the basic theory of gamma and beta functions, their connections with each other and their applicability to engineering problems.

Keywords: Beta Function, Gamma Function, Applied Mathematics, Engineering.

Gamma ve Beta Fonksiyonlarının Teorisi Üzerine Bir Not

Öz

Fizik ve mühendislik problemleri detaylı bir uygulamalı matematik bilgisini ve gamma ve beta fonksiyonları gibi özel fonksiyonların anlaşılmasını gerektirir. Özel fonksiyonlar konusu çok önemlidir ve uygulamalı bilimlerdeki yeni problemlerin varlığı ile sürekli genişlemektedir. Biz bu makalede, gamma ve beta fonksiyonlarının temel teorisini, birbirleriyle olan bağlantılarını ve mühendislik problemlerine uygulanabilirliklerini açıklıyoruz.

Anahtar Kelimeler: Beta Fonksiyonu, Gamma Fonksiyonu, Uygulamalı Matematik, Mühendislik.

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1. Introduction

This research includes the definition and the theory of classical special functions. Euler, Gauss, Fourier, Bessel, Legendre spent much time on this topic (Jaabar and Hussain, 2021). Besides applied fields such as fluid dynamics, mathematical physics, engineering and other applied sciences special functions have been a wide range of application areas in pure mathematics. Knowledge of the properties of gamma and beta functions, which are among the simplest and most important functions, is essential for understanding of many other functions, especially hypergeometric functions. In recent years, there have been important studies on the extensions of those functions (Naresh et al., 2021; Rahul et al. 2022; Chaudry et al., 1997).

2. The Gamma Function

Definition 2.1:

The gamma function is defined as follows:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx \qquad (1)$$

where $Re(a) > 0$ and $a \in \mathbb{C}$. This formula was found by Euler (Euler, 1729) and the notation $\Gamma(a)$ was introduced by Legendre (Legendre, 1814). The literature on the gamma function consists of thousands of pages and includes almost 300 years of researches in English, Latin, German and other languages (Rıcardo, 2021).

The gamma function provides

$$\Gamma(a+1) = a\int_0^\infty e^{-x}\, x^{a-1} dx = a\Gamma(a) \qquad (2)$$

the recurrence relation. This relation is called the Euler's functional equation, discovered by Euler in 1729 (Euler, 1729) and this equality gives us the basic property of factorial.

From equation (2) we can write

$$\Gamma(a+1) = a(a-1)\Gamma(a-1),$$

$$\Gamma(a+1) = a(a-1)(a-2)\Gamma(a-2),$$

If we continue in this way, from $\Gamma(1) = 1$ then we reach the following result:

$$\Gamma(a+1) = a! \qquad (3)$$

This result is called the Euler's functional equation, which was discovered by Euler in 1729 (Euler, 1729).

Hovewer, the gamma function we have defined for positive values of a can also be defined for negative values of a.

$(-1 < a < 0)$ $\Gamma(a)$ can be found since $\Gamma(a+1)$ is known.

$$\Gamma(a) = \frac{\Gamma(a+1)}{a} \qquad (0 < a+1 < 1)$$

$(-2 < a < -1)$ $\Gamma(a)$ can be found since $\Gamma(a+2)$ is known.

$$\Gamma(a) = \frac{\Gamma(a+2)}{a(a+1)} \qquad (0 < a+2 < 1)$$

$(-3 < a < -2)$ $\Gamma(a)$ can be found since $\Gamma(a+3)$ is known.

$$\Gamma(a) = \frac{\Gamma(a+3)}{a(a+1)(a+2)} \qquad (0 < a+3 < 1)$$

Similarly, for $-n < a < -n+1$, $\Gamma(a)$ can be found since $\Gamma(a+n)$ is known.

$$\Gamma(a) = \frac{\Gamma(a+n)}{a(a+1)(a+2)\ldots\ldots(a+n-1)} \qquad (0 < a+n < 1)$$

The above equations show that gamma function is unbounded for zero and negative integers and it is finite for all other values of a.

Theorem 2.1:

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \qquad (4)$$

This property of the gamma function was found by Euler (Euler, 1729) and discussed by Luke (Luke, 1969) and Bell (Bell, 1968).

Theorem 2.2:

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi z)}, \qquad (5)$$

where $Re(a) > 0$ and $a \in \mathbb{C}$. This formula is called Euler's completion formula (Euler, 1771).

Theorem 2.3:

$$\Gamma(2a)\,\Gamma\left(\tfrac{1}{2}\right) = 2^{2a-1}\,\Gamma(a)\,\Gamma\left(a+\tfrac{1}{2}\right), \qquad (6)$$

where $a \in \mathbb{C}\backslash Z_0^-$. This formula is called Legendre's duplication formula (Legendre, 1814).

Theorem 2.4:

If $Re(a) > 0$, $Re(b) > 0$ and $a \in \mathbb{C}$ and $b \in \mathbb{C}$ then we write

$$\int_0^{\pi/2} cos^{a-1}t\, sin^{b-1}t\, dt = \frac{1}{2}\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)} \;. \qquad (7)$$

This property was first defined by Whittaker (Whittaker, 1902).

3. The Pochhammer symbol

Definition 3.1:

If $z \in R$ or $z \in \mathbb{C}$ and r is zero or a positive integer then we write

$$(z)_r = z(z+1)(z+2)\cdots(z+r-1). \tag{8}$$

The above expression is known as the Pochhammer symbol and it was first defined by Pochhammer (Pochhammer, 1870).

From the known properties of gamma function, the following features of Pochhammer symbol can be written (Temme, 2011).

$$\Gamma(z+r) = (z+r-1)\Gamma(z+r-1)$$

$$\Gamma(z+r) = (z+r-1)(z+r-2)\Gamma(z+r-2)$$

$$\cdots$$

$$\Gamma(z+r) = (z+r-1)(z+r-2)\ldots(z+1)z\,\Gamma(z)$$

$$\Gamma(z+r) = (z)_r\,\Gamma(z),$$

$$(z)_r = \frac{\Gamma(z+r)}{\Gamma(z)}, \tag{9}$$

and

$$(z)_{r+1} = \frac{\Gamma(z+r+1)}{\Gamma(z)} = \frac{z\Gamma(z+r+1)}{z\Gamma(z)} = z\frac{\Gamma((z+1)+r)}{\Gamma(z+1)} = z(z+1)_r\,. \tag{10}$$

Specially, if we take $r = 0$ in equation (9), it is seen that

$$(z)_0 = 1.$$

4. The Beta function

Definition 4.1:

The Beta function is defined as follows:

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \tag{11}$$

where $Re(a) > 0$, $Re(b) > 0$. This formula was found by Euler (Euler, 1771) and by Legendre (Legendre, 1814).

If we define a new integration variable $x = k/(1+k)$, then (11) becomes

$$\mathrm{B}(a,b) = \int_0^\infty \frac{k^{a-1}}{(1+k)^{a+b}}\,dk. \tag{12}$$

The expression of beta function in terms of the gamma function is as follows:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,. \tag{13}$$

If we choose $a + b = 1$ in equation (12), then

$$\mathrm{B}(a,1-a) = \int_0^\infty \frac{k^{a-1}}{1+k}\,dk = \frac{\pi}{\sin \pi a}\,. \qquad 0 < a < 1$$

Using the above property, we can see that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ such that for $a = \frac{1}{2}$,

$$B\left(\frac{1}{2}, 1-\frac{1}{2}\right) = B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \frac{\pi}{\sin\frac{\pi}{2}} = \pi\,,$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \quad , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,.$$

Also, beta function is symmetric to its variables (Temme, 2011):

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \frac{\Gamma(b)\Gamma(a)}{\Gamma(b+a)} = B(b,a). \tag{14}$$

Theorem 4.1:

$$\mathrm{B}(a,b) = \int_0^\infty x^{a-1}(1+x)^{-(a+b)}\,dx$$

and

$$\mathrm{B}(a,b) = \mathrm{B}(a+1,b) + \mathrm{B}(a,b+1)\,.$$

This results were found by Watson and Whittaker (Whittaker et al. 1927).

5. Conclusion

In this paper, we introduced some important and fundamental properties and the theory of Gamma function, Pochammer symbols, Beta function and we discussed the relation with other definitions. We believe that these analyzes will be useful for researchers to understand the theory of gamma and beta functions.

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