Journal of New Results in Science 12 (1) (2023) 22-26

Journal of New Results in Science
https://dergipark.org.tr/en/pub/jnrs
Research Article
Open Access

E-ISSN: 1304-7981 https://doi.org/10.54187/jnrs.1219721

Mersenne version of Brocard-Ramanujan equation

Seyran Ibrahimov[1], **Ayşe Nalli**[2]

Keywords
Brocard-Ramanujan equation, Mersenne numbers, Diophantine equations

Abstract — In this study, we deal with a special form of the Brocard-Ramanujan equation, which is one of the interesting and still open problems of Diophantine analysis. We search for the positive integer solutions of the Brocard-Ramanujan equation for the case where the right-hand side is Mersenne numbers. By using the definition of Mersenne numbers, appropriate inequalities for the parameters of the equation, and the prime factorization of $n!$ we show that there is no positive integer solution to this equation. Thus, we obtain this interesting result demonstrating that the square of any Mersenne number can not be expressed as $n!+1$.

Subject Classification (2020): 11D72, 11D61

1. Introduction

Diophantine equations are one of the important subjects of number theory. If there are two or more unknowns in an equation, this equation is called the Diophantine equation [1]. Usually, it is necessary to find all the integer solutions of these equations. However, there is no general method with finite numerical calculations for solving these equations in the set of integers. This theorem is the 10th problem among illustrious German mathematician Hilbert's 23 famous problems, which Matiyasevich proved in 1970. Matiyasevich demonstrated that there is no general method for showing whether an integer solution to an arbitrary Diophantine equation exists or not. Since there is no general method for solving Diophantine equations, methods are being developed for solving a particular class of these equations. For instance, we can mention some of the existing methods by factoring method, parametric method, modular arithmetic method, solving Diophantine equations using inequalities, and other methods [2]. In this field, there are interesting works by prominent mathematicians, such as Fermat, Euler, Ramanujan, Puancare, and Erdös. One of the famous Diophantine equations is the Brocard-Ramanujan equation. This equation is still an open problem in number theory. The only known solutions to are $(n,m)\in\{(4,5),(5,11),(7,71)\}$. This problem was formulated by Brocard and Ramanujan [3] in 1876 and 1911, respectively, as follows:

$$n!+1=m^2$$

[1]seyranibrahimov1@gmail.com (Corresponding Author); [2]aysenalli@karabuk.edu.tr
[1,2]Department of Mathematics, Karabük University, Karabük, Türkiye
Article History: Received: 15 Dec 2022 - Accepted: 24 Apr 2023 - Published: 30 Apr 2023

Although more than a century has passed, this problem has not been fully resolved. The solutions of this equation under some special conditions have been investigated by different mathematicians. Gerardin [4] put forward the idea that when $m > 71$, m must be a number with at least 20 digits. Lately, Berndt and Galway [5] proved that there are no other solutions up to $n = 10^9$. In addition, Marques [6] studied the form of the equation for Fibonacci numbers. Faco and Marques [7] studied the case where the right-hand side of the equation is Tribonacci numbers. Dabrowski and Ulas [8] investigated some varieties of the Brocard-Ramanujan equation. There are many studies [9–13] related to the famous number sequence and Diophantine equations in the literature.

2. Preliminaries

This section provides several basic notions to be needed for the following section.

Definition 2.1. Mersenne numbers are defined by $M_n = 2^n - 1$, $n \geq 1$.

Lemma 2.2. If $n \in N^*$, then

$$\frac{2}{n+1} \leq (n!)^{\frac{-1}{n}} < \frac{e}{n+1} \tag{2.1}$$

holds.

Proof.

By arithmetic and geometric means inequality, we have

$$\frac{n+1}{2} \geq (n!)^{\frac{1}{n}}$$

Then, we obtain

$$\frac{2}{n+1} \leq (n!)^{\frac{-1}{n}}$$

Now, we must prove

$$(n!)^{\frac{-1}{n}} < \frac{e}{n+1}$$

We use the induction method for this. If $n = 1$, then we get $1 < \frac{e}{2}$. Suppose that Inequality (2.1) is true for $n = k$ and show that it holds for $n = k + 1$. We know that [14]

$$e^{\frac{1}{n+1}} \leq 1 + \frac{1}{n} \leq e^{\frac{1}{n}}, n \geq 1 \tag{2.2}$$

Thus, by assumption and Inequality (2.2), we obtain

$$\frac{1}{(k+1)!} < \left(\frac{e}{k+1}\right)^k \frac{1}{k+1} < \left(\frac{e}{k+2}\right)^{k+1}$$

□

Lemma 2.3. If $n \in N^*$, then

$$\frac{(n+1)^n}{3} > 2^{2n} - 2^{n+1} \tag{2.3}$$

holds for all $n \geq 11$.

Proof.

Suppose that $n = 11$. The

$$4^{11} > 2^{22} - 2^{12}$$

That is true. Besides, we assume that Inequality (2.3) is satisfied for $n = k$. Then, we must illustrate for $n = k + 1$.

$$\left(\frac{k+2}{3}\right)^{k+1} > 2\left(\frac{k+1}{3}\right)^{k+1} = \frac{2(k+1)}{3}\left(\frac{k+1}{3}\right)^{k} > 2^{2k+3} - 2^{k+4} > 2^{2k+2} - 2^{k+2}$$

□

The prime factorization formula of $n!$ provided in [15] can be summarized as follows: To acquire the prime factorization of $n!$, we must find, for each of these primes p, the exponent g_p of the greatest power of p that divides $n!$. The formula is connected to the relationship between n, p, and g_p. This formula belongs to the illustrious Legendre. In order to explain the formula, we first give the representation of the integer n according to the base p.

$$n = \sum_{i=0}^{k} r_i p^i$$

where $p^{k+1} > n, p^k \leq n$ and $0 \leq r_i \leq p - 1$, for all $i = 1, 2, \ldots, k$. Then, for $1 \leq l \leq k$,

$$\frac{n}{p^l} = \frac{\sum_{i=r}^{k} r_i p^i}{p^l} + \frac{\sum_{i=0}^{r-1} r_i p^i}{p^l}$$

thus

$$\sum_{i=0}^{r-1} r_i p^i \leq (p-1)\sum_{i=0}^{r-1} p^i = p^r - 1 < p^r$$

we obtain that

$$\left[\frac{n}{p^l}\right] = \sum_{i=0}^{k-1} r_{k-i} p^{k-l-i} \tag{2.4}$$

for all $1 \leq l \leq k$, we know that, given for any prime $p \leq n$,

$$g_p = \sum_{i=1}^{k} \left[\frac{n}{p^k}\right] \tag{2.5}$$

where $p^{k+1} > n, p^k \leq n$. For each $l (1 \leq l \leq k)$, we write Formula (2.4) and add side-by-side and using Formula (2.5), we get

$$g_p = \sum_{i=1}^{k}(r_i \sum_{j=1}^{i-1} p^{j-1}) = \frac{1}{p-1}\sum_{i=1}^{k}(r_i(p^i - 1)) = \frac{n - d_p}{p-1} \tag{2.6}$$

where $d_p = \sum_{i=0}^{k} r_i$.

3. Main Result

Theorem 3.1. In the following equation, there is no solution in N^*

$$n! + 1 = M_k^2 \tag{3.1}$$

Proof.

We investigate the following cases:
i. Suppose that $n = k$. Then, we must demonstrate there is no solution to the following equation:

$$n! = 2^{2n} - 2^{n+1}$$

Then, from Equation (3.1) and Inequality (2.1), we obtain

$$2^{2n} - 2^{n+1} = n! > \left(\frac{n+1}{3}\right)^{n} \tag{3.2}$$

However, according to Inequality (2.3) this is a contradiction for all $n \geq 11$. Moreover, we can demonstrate by simple numerical calculations that there is no solution for all $n \leq 10$.

ii. Suppose that $n > k$. Then we have to prove there is no solution to the following equation:

$$n! = 2^{2k} - 2^{k+1} \tag{3.3}$$

By Lemma (2.3), we obtain

$$n! > 2^{2n} - 2^{n+1}$$

for all $n \geq 11$. In addition, we know $n > k$. Then, we get

$$n! > 2^{2n} - 2^{n+1} > 2^{2k} - 2^{k+1} \tag{3.4}$$

For all (k, n) pairs satisfying $11 \leq k < n$ or $k < 11 \leq n$. Then, by Inequality (3.4), we obtain that there is no solution of Equation (3.3) for these cases of n and k. Besides, for the remaining case $k < n < 11$, it can be shown by simple mathematical calculations that the equation there is no solution.

iii. Suppose that $n < k$. We see that the right-hand side of the equation is divided by 2^{k+1}, and from Formula (2.6), we obtain that the left-hand side is divided by 2^{n-d_2}. However, we see that $n - d_2 < n < k < k + 1$. This means that in this case, the equation there is no solution.

□

4. Conclusion

The Brocard-Ramanujan equation is one of the still open problems of Diophantine analysis. Although passed a hundred years, the authors solved only some special cases of this equation. In this article, we investigate the positive integer solutions of the Brocard-Ramanujan equation for the case where the right-hand side is Mersenne numbers. By using the definition of Mersenne numbers, appropriate inequalities for the parameters of the equation, and the prime factorization of $n!$, we proved that there is no positive integer solution to this equation. In addition, exciting results can be obtained in future studies by using the properties of number sequences and methods for solving Diophantine equations in the versions of this equation associated with k-Fibonacci, k-Lucas, Pell, Pell-Lucas, and Fermat numbers.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

References

[1] L. J. Mordell, Diophantine equations, Academic press, 1969.

[2] T. Andreescu, D. Andrica, I. Cucurezeanu, An introduction to Diophantine equations, New York: Birkhäuser, 2010.

[3] S. Ramanujan, *Question 294*, Journal of the Indian Mathematical Society 3 (1911) 128.

[4] A. Gérardin, *Contribution à létude de léquation* $1.2.3.4 \ldots z + 1 = y^2$, Nouvelles annales de mathématiques 6 (1906) 222–226.

[5] B. C. Berndt, W. F. Galway, *On the Brocard-Ramanujan Diophantine equation* $n! + 1 = m^2$, The Ramanujan Journal 4 (1) (2000) 41–42.

[6] D. Marques, *The Fibonacci version of the Brocard-Ramanujan Diophantine equation*, Portugaliae Mathematica 68 (2) (2011) 185–189.

[7] V. Facó, D. Marques, *Tribonacci Numbers and the Brocard-Ramanujan Equation*, Journal of Integer Sequences 19 (4) (2016) 1–7.

[8] A. Dabrowski, M. Ulas, *Variations on the Brocard-Ramanujan equation*, Journal of Number Theory 133 (4) (2013) 1168–1185.

[9] B. Demirtürk, R. Keskin, *Integer solutions of some Diophantine equations via Fibonacci and Lucas numbers*, Journal of Integer Sequences 12 (8) (2009) 1–14.

[10] N. Y. Özgür, *On the sequences related to Fibonacci and Lucas numbers*, Journal of the Korean Mathematical Society 42 (1) (2005) 135–151.

[11] J. J. Bravo, F. Luca, K. Yazán, *On a problem of Pillai with Tribonacci numbers and powers of 2*, Bulletin of the Korean Mathematical Society 54 (3) (2017) 1069–1080.

[12] M. Ddamulira, F. Luca, M. Rakotomalala, *On a problem of Pillai with Fibonacci numbers and powers of 2*, Proceedings-Mathematical Sciences 127 (3) (2017) 411–421.

[13] F. Erduvan, R. Keskin, Z. Siar, *Repdigits base b as products of two Lucas numbers*, Quaestiones Mathematicae 44 (10) (2021) 1283–1293.

[14] G. H. Hardy, J. E. Littlewood, G. Pólya, G. Pólya, Inequalities, Cambridge, 1952.

[15] R. J. Friedlander, *Factoring factorials*, The Two-Year College Mathematics Journal 12 (1) (1981) 12–20.