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Oransal Gecikmeli Uyumlu Zaman-Kesirli Swift-Hohenberg Denkleminin Yeni Yöntemlerle Sayısal Çözümleri

Numerical Solutions of Conformable Time-Fractional Swift-Hohenberg Equation with Proportional Delay by the Novel Methods

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Özet. Uyumlu kesirli q-Shehu homotopi analizi dönüşüm yöntemi ve uyumlu Shehu dönüşümü ayrıştırma yöntemi, oransal gecikmeli uyumlu zaman-kesirli Swift-Hohenberg denklemlerini analiz etmek için kullanılmıştır. Bu problemin sayısal çözümlerinin grafikleri çizdirilmiştir. Önerilen yöntemler, sayısal simülasyonlara göre etkili ve tutarlıdır.

Anahtar Kelimeler: Uyumlu kesirli zaman kesirli mertebeden Swift-Hohenberg denklemi, uyumlu kesirli q-Shehu homotopi analiz dönüşüm metodu, uyumlu kesirli Shehu dönüşümü ayrıştırma metodu.

Abstract. The conformable fractional q-Shehu homotopy analysis transform method and the conformable Shehu transform decomposition method are used to analyze the

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conformable time-fractional Swift-Hohenberg equations with proportional delay. The graphs of the numerical solutions to this problem are drawn. The proposed methods are effective and consistent, according to numerical simulations.

Key words: Conformable time-fractional Swift-Hohenberg equation, conformable fractional q-Shehu homotopy analysis transform method, conformable fractional Shehu transform decomposition method.

1. Introduction

Fractional calculus is a concept that was investigated and defined by many senior academics. They came up with revolutionary definitions of fractional calculus that provided the basis for fractional calculus [2, 6, 22, 25, 29, 31, 34]. Fractional partial differential equations are used a lot in developing nonlinear models and investigating dynamical systems. Fractional-order calculus has been used to evaluate and study many matters, like chaos theory [3], financial models [36], a noisy environment [23], optics [24], and others [5, 32, 38-39]. The solutions to fractional differential equations are important for figuring out which nonlinear problems in nature take a glance like. It is used a variety of analytical and numerical methods because it is hard to find exact solutions to fractional differential equations that represent nonlinear phenomena [10].

The conformable fractional derivative is a basic and helpful tool. It also helps us understand how to describe the behavior of real items. The conformable fractional derivative is a brilliant tool for solving complicated problems. Differential equations with conformable fractional derivatives are easier to solve numerically than those with Riemann-Liouville or Caputo fractional derivatives. This allows the conformable fractional derivative useful for modeling many physical problems. Different fractional order models are used in engineering and applied sciences because they give a more accurate explanation of real-world situations. Several academics have already used conformable fractional derivatives in a wide range of fields [11]. The conformable fractional operator gets around some of the problems with the existing fractional operators. It gives traditional calculus properties like the mean value theorem, the chain rule, the product of two functions, the derivative of the quotient of two functions, and Rolle's theorem [1, 9, 16, 19]. There are several studies on Swift-Hohenberg equation in the last 3 years, some of which contain similar components to this study like proportional delay, conformable derivatives or Shehu transforms [18, 21, 27, 40]. This study included proportional delay, unlike these studies in [18, 21, 27, 40].

Jack Swift and Pierre Hohenberg arose with and collaborated on the Swift–Hohenberg (S–H) equation, which is a universal model of the Rayleigh–Benard convective instability of the fluid with thermal fluctuations for the dynamics of fluid velocity and temperature of convection [13, 37]. The S–H equation is a key part of the theory of pattern formation (specifically, the mechanism of the amplitude of the optical electrical field in the interior of a cavity, the pattern within thin vibrated granular layers, and so on) in fluid layers confined between horizontal well-conducting boundaries. The proposed problem is a model for a lot of interesting localized and non-localized patterns that come from different biological structures [8, 14]. The S–H equation is a very important way to describe many physical phenomena, like lasers, water flow, liquid crystals, flame dynamics, and statistical mechanics [20, 28, 30]. In [17, 33, 41], the following equation has been investigated using various techniques.

$$D_t^{\alpha} w(x,t) + \frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu) w(x,t) + w^3(x,t) = 0, 0 < \alpha \leq 1,$$

$$t > 0. \quad (1)$$

In [33, 42], the fractional Swift-Hohenberg equation with dispersion has been examined using a variety of techniques.

$$D_t^{\alpha} w(x,t) + \frac{\partial^4 w(x,t)}{\partial x^4} - \tau \frac{\partial^3 w(x,t)}{\partial x^3} - \mu w(x,t) - 2w^2(x,t) + w^3(x,t) = 0, \quad (2)$$

where $w(x,t)$ is probability density function, τ and μ are dispersive and bifurcation parameters, respectively.

In the research analysis, we consider the conformable time-fractional nonlinear S–H equation with proportional delay as follows:

$$D_t^{\alpha} w(x,t) + \frac{\partial^4 w(x,t)}{\partial x^4} - \tau \frac{\partial^3 w(x,t)}{\partial x^3} - \mu w(x,t) - 2w^2(x,t) + w^3\left(x, \frac{t}{2}\right) = 0, \quad (3)$$

and moreover, in the presence of the dispersive term as:

$$D_t^{\alpha} w(x,t) + \frac{\partial^4 w(x,t)}{\partial x^4} - \tau \frac{\partial^3 w(x,t)}{\partial x^3} - \mu w(x,t) - 2w^2\left(x, \frac{t}{2}\right) + w^3\left(x, \frac{t}{2}\right) = 0, \quad (4)$$

where, D_t^{α} is conformable time-fractional operator.

In biology, medicine, population ecology, control systems, climate models, and complicated economic macro-dynamics, the partial functional differential equations with proportional delays are a variety of delay partial differential equation [15, 43].

The S-H equation has been solved by numerous techniques, such as homotopy perturbation transform method (HPTM) [42], homotopy analysis method (HAM) [41], residual power series method (RPSM) [33], differential transform method (DTM) [17], variational iteration technique [7, 35]. In this study, however, both conformable fractional and proportional delay versions of the S-H equation were initially solved. In addition, two novel solutions exist for the conformable time-fractional S-H equation with proportional delay. Cq-SATM and CSADM are newly developed techniques.

The remainder of the research is listed below. The second section explains the fundamentals of conformable fractional calculus and the Shehu transform. Conformable q-homotopy analysis transform method and conformable Shehu homotopy perturbation method are introduced in Section 3. Conformable time-fractional S-H equation with proportional delay are illustrated in Section 4. The conclusion is presented in Section 5.

2. Preliminaries

Recall what conformable fractional calculus and the Shehu transform indicate and how they should be utilized in this work.

Definition 2.1 [1, 12, 16] Given a function $f: [0, \infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f order α is given by

$$T_\alpha(f)(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \tag{5}$$

for all $x > 0, \alpha \in (0,1]$.

Theorem 2.1 [1, 12, 16] Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $x > 0$. Then, it is obtained as

i. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$, (6)

ii. $T_\alpha(x^p) = px^{p-1}$, for all $p \in \mathbb{R}$, (7)

iii. $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$, (8)

iv. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$, (9)

v. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$. (10)

vi. If f is differentiable, then the derivative of the polynomial t is obtained as

$$T_\alpha(f)(t) = t^{1-\alpha}\frac{d}{dt}f(t). \tag{11}$$

Definition 2.2 [16] Let f be an $n-$times differentiable at x. Then, the conformable fractional derivative of f order α is defined by:

$$T_\alpha(f)(x) = \lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(x+\varepsilon x^{(\lceil\alpha\rceil-\alpha)}\right) - f^{(\lceil\alpha\rceil-1)}(x)}{\varepsilon}, \tag{12}$$

for all $x > 0, \alpha \in (n, n+1]$, $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Theorem 2.2 [16] Let f be an $n-$times differentiable at x. Then, it is obtained as

$$T_\alpha\big(f(x)\big) = x^{\lceil\alpha\rceil-\alpha}f^{\lceil\alpha\rceil}(x), \tag{13}$$

for all $x > 0, \alpha \in (n, n+1]$.

Definition 2.3 [26]

The Mittag-Leffler function E_a is given as follows:

$$E_a(z) = \sum_{n=0}^{\infty}\frac{z^a}{\Gamma(na+1)}. \tag{14}$$

Definition 2.4 [4] Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. Then, the conformable fractional Shehu transform (CFST) of order α of f is described by

$$ {}_{c}S_{\alpha}[f(t)] = V_{\alpha}(s;u) = \int_{0}^{\infty} exp\left(\frac{-st^{\alpha}}{u\alpha}\right) f(t) t^{\alpha-1} dt. \tag{15}$$

Definition 2.5 [4] Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. The conformable Shehu transform for the conformable fractional-order derivative of the function $f(t)$ is given by

$$V_{\alpha}[T_{\alpha} f(t)](v) = \frac{s}{u} V_{\alpha}(s;u) - f(0). \tag{16}$$

3. The Basic Idea of the Novel Numerical Techniques

Now, to clarify the main idea of Cq-SHATM and CSADM.

Case (i) Conformable q-Shehu Homotopy Analysis Transform Method

Now, to describe the fundamental idea of Cq-SHATM, examine the conformable time-fractional S-H equation with proportional delay:

$$D_t^{\alpha} w(x,t) + Aw(\rho_i x, \sigma_i t) + Hw(\rho_i x, \sigma_i t) = \varphi(x,t), t > 0, n-1 < \alpha \leq n, \tag{17}$$

where A is a linear operator, H is a nonlinear operator, $\varphi(x,t)$ is a source term, $\rho_i, \sigma_i \in (0,1)$ and D_t^{α} is a conformable fractional derivative of order α.

Implementing the CFST to Eq. (17) and utilizing the initial condition, it is generated as

$$\frac{s}{u} {}_{c}S_{\alpha}[w(x,t)] - w(x,0) + {}_{c}S_{\alpha}\left[Aw\left(\rho_i x, \sigma_i t\right)\right] + {}_{c}S_{\alpha}\left[Hw\left(\rho_i x, \sigma_i t\right)\right] = {}_{c}S_{\alpha}[\varphi(x,t)]. \tag{18}$$

By arranging the Eq. (18), Eq. (19) is generated as

$$ {}_{c}S_{\alpha}[w(x,t)] - \frac{u}{s} w(x,0) + \frac{u}{s} {}_{c}S_{\alpha}\left[Aw\left(\rho_i x, \sigma_i t\right)\right] + \frac{u}{s} {}_{c}S_{\alpha}\left[Hw\left(\rho_i x, \sigma_i t\right)\right] - \frac{u}{s} {}_{c}S_{\alpha}[\varphi(x,t)] = 0. \tag{19}$$

Utilizing the HAM, the nonlinear operator for real function $\psi(x,t;q)$ is defined as

$$N[\psi(x,t;q)\,] = {}_{c}S_{\alpha}[\psi(x,t;q)\,] - \frac{u}{s}\psi(x,t;q)\;(0^{+}) + \frac{u}{s}(\;{}_{c}S_{\alpha}[A\psi(\rho_i x,\sigma_i t;q)]$$

$$+\;{}_{c}S_{\alpha}[H\psi(\rho_i x,\sigma_i t;q)] - {}_{c}S_{\alpha}[\varphi(x,t)]), \tag{20}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

A homotopy is constructed as follows:

$$(1-nq)\;{}_{c}S_{\alpha}[\psi(x,t;q) - w_0(x,t)] = hqH^{*}(x,t)H[\psi(\rho_i x,\sigma_i t;q)], \tag{21}$$

where, $h \neq 0$ is an auxiliary parameter and ${}_{c}S_{\alpha}$ represents CFST. For $q=0$ and $q=\frac{1}{n}$, the results of Eq. (21) are as follows:

$$\psi(x,t;0) = w_0(x,t), \psi\left(x,t;\frac{1}{n}\right) = w(x,t), \tag{22}$$

Hence, by amplifying q from 0 to $\frac{1}{n}$, then the solution $\varphi(x,t;q)$ converges from $w_0(x,t)$ to the solution $w(x,t)$. Utilizing the Taylor theorem around q and then expanding $\psi(x,t;q)$, it is generated as

$$\psi(x,t;q) = w_0(x,t) + \sum_{i=1}^{\infty} w_m(x,t)q^m, \tag{23}$$

where,

$$w_m(x,t) = \frac{1}{m!}\frac{\partial^m \psi(x,t;q)}{\partial q^m}|_{q=0}. \tag{24}$$

Eq. (23) converges at $q=\frac{1}{n}$ for the convenient $w_0(x,t)$, n and h. Then, one gets one of the solutions of the original nonlinear equation of the form

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)\left(\frac{1}{n}\right)^m. \tag{25}$$

Differentiating the zeroth order deformation Eq. (21) $m-$times with respect to q and dividing by $m!$, respectively, then for $q=0$, one gets

$${}_{c}S_{\alpha}[w_m(x,t) - k_m w_{m-1}(x,t)] = hH^{*}(x,t)\mathcal{R}_m(\vec{w}_{m-1}), \tag{26}$$

where the vectors are given by

$$\vec{w}_m = \{w_0(x,t), w_1(x,t), \dots, w_m(x,t)\}. \quad (27)$$

When Eq. (26) is implemented to the inverse CFST, one has

$$w_m(x,t) = k_m w_{m-1}(x,t) + h\left({}_{c}S_\alpha\right)^{-1}[H^*(x,t)\mathcal{R}_m(\vec{w}_{m-1})], \quad (28)$$

where

$$\mathcal{R}_m(\vec{w}_{m-1}) = {}_{c}S_\alpha[w_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\frac{u}{s}w_0(x,t) + \frac{u}{s}\ {}_{c}S_\alpha[Aw_{m-1}(\rho_i x, \sigma_i t)$$

$$+H^*{}_{m-1}(x,t) - \varphi(x,t)], \quad (29)$$

and

$$k_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \quad (30)$$

Here, $H^*{}_m$ is homotopy polynomial and presented as

$$H^*{}_m = \frac{1}{m!}\frac{\partial^m \psi(x,t;q)}{\partial q^m}|_{q=0} \text{ and } \psi(x,t;q) = \psi_0 + q\psi_1 + q^2\psi_2 + \cdots. \quad (31)$$

Utilizing Eqs. (28) - (29), one gets

$$w_m(x,t) = (k_m + h)w_{m-1}(x,t) - \left(1 - \frac{k_m}{n}\right)\frac{u}{s}w_0(x,t) + h$$

$$\times \left({}_{c}S_\alpha\right)^{-1}\left[\left(\frac{u}{s}\ {}_{c}S_\alpha[Aw_{m-1}(\rho_i x, \sigma_i t) + H^*{}_{m-1}(x,t) - f(x,t)]\right)\right]. \quad (32)$$

Utilizing Cq-SHATM, then one has

$$w(x,t) = \sum_{c=0}^{\infty} w_c(x,t). \quad (33)$$

Case (ii) Conformable Shehu Adomian Decomposition Method

Now, to define the fundamental idea of CSADM, discuss the conformable time-fractional S-H equation with proportional delay:

$$D_t^\alpha w(x,t) + Aw(\rho_i x, \sigma_i t) + Hw(\rho_i x, \sigma_i t) = \varphi(x,t), t > 0, n-1 < \alpha \le n, \quad (34)$$

where A is a linear operator, H is a nonlinear operator, $\varphi(x,t)$ is a source term, $\rho_i, \sigma_i \in$

$(0,1)$ and D_t^{α} is a conformable fractional derivative of order α.

Implementing the CFST to Eq. (34) and utilizing the initial condition, it is generated as

$$\frac{s}{u}\ {}_cS_{\alpha}[w(x,t)] - w(x,0) + {}_cS_{\alpha}\left[Aw\left(\rho_i x, \sigma_i t\right)\right] + {}_cS_{\alpha}\left[Hw\left(\rho_i x, \sigma_i t\right)\right] = {}_cS_{\alpha}[\varphi(x,t)]. \quad (35)$$

By arranging the Eq. (35), Eq. (36) is generated as

$${}_cS_{\alpha}[w(x,t)] = \frac{u}{s} w(x,0) - \frac{u}{s}\ {}_cS_{\alpha}\left[Aw\left(\rho_i x, \sigma_i t\right) + Hw\left(\rho_i x, \sigma_i t\right) - \varphi(x,t)\right]. \quad (36)$$

To get Eq. (37), the inverse CFST is implemented.

$$w(x,t) = Q(x,t) - \left({}_cS_{\alpha}\right)^{-1}\left[\frac{u}{s}\ {}_cS_{\alpha}[Aw(\rho_i x, \sigma_i t) + Hw(\rho_i x, \sigma_i t)]\right], \quad (37)$$

where the non-homogenous term and the initial conditions result to $Q(x,t)$.

Let Eq. (38) be an infinite series solution form:

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t). \quad (38)$$

Eq. (37) is rewritten by utilizing Eq. (38).

$$\sum_{n=0}^{\infty} w_n(x,t) = Q(x,t) - \left({}_cS_{\alpha}\right)^{-1}\left[\frac{u}{s}\ {}_cS_{\alpha}\left[A\sum_{n=0}^{\infty} w_n(\rho_i x, \sigma_i t) + \sum_{n=0}^{\infty} B_n\right]\right], \quad (39)$$

where the B_n is the Adomian polynomials which represents the nonlinear term $Hw(\rho_i x, \sigma_i t)$.

It is found by comparing the two sides of the Eq. (39).

$$w_0(x,t) = Q(x,t), \quad (40)$$

$$w_1(x,t) = -\left({}_cS_{\alpha}\right)^{-1}\left[\frac{u}{s}\ {}_cS_{\alpha}[Aw_0(\rho_i x, \sigma_i t) + B_0]\right], \quad (41)$$

$$w_2(x,t) = -\left({}_cS_{\alpha}\right)^{-1}\left[\frac{u}{s}\ {}_cS_{\alpha}[Aw_1(\rho_i x, \sigma_i t) + B_1]\right], \quad (42)$$

$$w_3(x,t) = -\left({}_cS_{\alpha}\right)^{-1}\left[\frac{u}{s}\ {}_cS_{\alpha}[Aw_2(\rho_i x, \sigma_i t) + B_2]\right], \quad (43)$$

$\vdots$

The general formula for iteration is as follows:

$$w_{n+1}(x,t) = -\left({}_cS_\alpha\right)^{-1}\left[\frac{u}{s}\ {}_cS_\alpha[Aw_n(\rho_i x,\sigma_i t)+B_n]\right], n \geq 0. \quad (44)$$

As a result, an approximation of the solution can be expressed as

$$w(x,t) = \sum_{n=0}^{\infty} w_n(\rho_i x,\sigma_i t). \quad (45)$$

4. Applications

Examine the conformable time-fractional nonlinear S–H equation with proportional delay [24, 33]

$$D_t^\alpha w(x,t) + \frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu)w(x,t) + w^3\left(x,\frac{t}{2}\right) = 0, t > 0,$$

$$0 < \alpha \leq 1, \quad (46)$$

subject to initial condition

$$w(x,0) = \frac{1}{10}\sin\left[\frac{\pi x}{l}\right], \quad (47)$$

and boundary conditions

$$w(0,t) = 0, \qquad w_{xx}(l,t) = 0, \quad (48)$$

where $w(x,t)$ is temperature.

Case (i): Cq-SHATM solution for Eq. (46)

Implementing the CFST to Eq. (46) and utilizing the initial condition, it is generated as

$$\frac{s}{u}\ {}_cS_\alpha[w(x,t)] - w(x,0) + \ {}_cS_\alpha\left[\frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu)w(x,t)\right.$$

$$\left.+w^3\left(x,\frac{t}{2}\right)\right] = 0. \quad (49)$$

By arranging the Eq. (49), Eq. (50) is generated as

$$ {}_{c}S_{\alpha}[w(x,t)] - \frac{u}{s} w(x,0) + \frac{u}{s}\ {}_{c}S_{\alpha}\left[\frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu)w(x,t)\right. $$

$$ \left. + w^3\left(x, \frac{t}{2}\right)\right] = 0. \tag{50} $$

Utilizing the HAM, the nonlinear operator for real function $\psi(x,t;q)$ is defined as

$$ N[\psi(x,t;q)\,] = {}_{c}S_{\alpha}[\psi(x,t;q)\,] - \frac{u}{s}\psi(x,t;q)\ (0^{+}) + \frac{u}{s}\ {}_{c}S_{\alpha}\left[\frac{\partial^4 \psi(x,t;q)}{\partial x^4}\right. $$

$$ \left. + 2\frac{\partial^2 \psi(x,t;q)}{\partial x^2} + (1-\mu)\psi(x,t;q) + \psi^3\left(x, \frac{t}{2}; q\right)\right], \tag{51} $$

where $q\epsilon\left[0, \frac{1}{n}\right]$.

By applying the proposed algorithm, the $m-th$ order deformation equation is defined by

$$ {}_{c}S_{\alpha}[w_m(x,t) - k_m w_{m-1}(x,t)] = h\mathcal{R}_m(\vec{w}_{m-1}), \tag{52} $$

where,

$$ \mathcal{R}_m(\vec{w}_{m-1}) = {}_{c}S_{\alpha}[w_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\frac{u}{s} w_0(x,t) + \frac{\mathrm{u}}{s}\ {}_{c}S_{\alpha}\left[\frac{\partial^4 w_{m-1}(x,t)}{\partial x^4}\right. $$

$$ \left. + 2\frac{\partial^2 w_{m-1}(x,t)}{\partial x^2} + (1-\mu)w_{m-1}(x,t) + \sum_{r=0}^{m-1}\left(\sum_{j=0}^{r} V_r V_{r-j}\right)V_{m-1-r}\right]. \tag{53} $$

When Eq. (52) is implemented to the inverse CFST, one has

$$ w_m(x,t) = k_m w_{m-1}(x,t) + h\left({}_{c}S_{\alpha}\right)^{-1}[\mathcal{R}_m(\vec{w}_{m-1})]. \tag{54} $$

By the use of initial condition, then it is generated as

$$ w_0(x,t) = \frac{1}{10}\sin\left[\frac{\pi x}{l}\right]. \tag{55} $$

And by substituting $m = 1, m = 2$ in the Eq. (54) respectively, Eqs. (56)-(57) are generated as

$$w_1(x,t) = \frac{-ht^{\alpha}\sin\left[\frac{\pi x}{l}\right]}{10\alpha l^4}\left[\frac{l^4\cos^2\left[\frac{\pi x}{l}\right]}{100} + \left(\mu - \frac{101}{100}\right)l^4 + 2\pi^2 l^2 - \pi^4\right], \tag{56}$$

$$w_2(x,t) = (n+h)\left(\frac{-ht^{\alpha}\sin\left[\frac{\pi x}{l}\right]}{10\alpha l^4}\left[\frac{l^4\cos^2\left[\frac{\pi x}{l}\right]}{100} + \left(\mu - \frac{101}{100}\right)l^4 + 2\pi^2 l^2 - \pi^4\right]\right)$$

$$+\frac{3h^2t^{2\alpha}\sin\left[\frac{\pi x}{l}\right]}{2.10^5\alpha^2 l^8}\left[2^{-\alpha}\cos^4\left[\frac{\pi x}{l}\right]l^8 + 100l^4\left[\left[\left(\mu - \frac{51}{50}\right)l^4 + 2\pi^2 l^2 - \pi^4\right]2^{-\alpha}\right.\right.$$

$$\left.+\left(\frac{\mu}{3} - \frac{1}{3}\right)l^4 + 6\pi^2 l^2 - 27\pi^4\right]\cos^2\left[\frac{\pi x}{l}\right] + [(-100\mu + 101)l^8 - 200l^6\pi^2$$

$$+100l^4\pi^4]2^{-\alpha} + \left(\frac{10^4\mu^2}{3} - 6700\mu + \frac{10100}{3}\right)l^8 + \frac{40000}{3}\left(\mu - \frac{203}{200}\right)l^6\pi^2 - \frac{20000}{3}$$

$$\left.\times\left(\mu - \frac{621}{200}\right)l^4\pi^4 - \frac{40000}{3}l^2\pi^6 + \frac{10000}{3}\pi^8\right]. \tag{57}$$

Ultimately, the analytical solution of $w(x,t)$ is approximated by the truncated series

$$w(x,t) = \lim_{M\to\infty}\Theta_M(x,t), \tag{58}$$

where,

$$\Theta_M(x,t) = \sum_{m=1}^{M-1} w_m(x,t). \tag{59}$$

Thus, the Cq-SHATM solution of Eq. (46) is given by

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)\left(\frac{1}{n}\right)^m. \tag{60}$$

Case (ii): CSADM solution for Eq. (46)

Implementing the CFST to Eq. (46) and utilizing the initial condition, it is generated as

$$\frac{s}{u}\ {}_cS_{\alpha}[w(x,t)] - w(x,0) + {}_cS_{\alpha}\left[\frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu)w(x,t)\right.$$

$$\left.+w^3\left(x,\frac{t}{2}\right)\right] = 0. \tag{61}$$

By arranging the Eq. (61), Eq. (62) is generated as

$$ {}_cS_\alpha[w(x,t)] = \frac{u}{s} w(x,0) - \frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu)w(x,t)\right. $$

$$ \left. + w^3\left(x,\frac{t}{2}\right)\right]. \tag{62} $$

To get Eq. (63), the inverse CFST is implemented.

$$ w(x,t) = \frac{1}{10}\sin\left[\frac{\pi x}{l}\right] - \left({}_cS_\alpha\right)^{-1}\left[\frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^4 w(x,t)}{\partial x^4} + 2\frac{\partial^2 w(x,t)}{\partial x^2} + (1-\mu)w(x,t)\right.\right. $$

$$ \left.\left. + w^3\left(x,\frac{t}{2}\right)\right]\right]. \tag{63} $$

Assume the unknown $w(x,t)$ function is expressed in infinite series form:

$$ w(x,t) = \sum_{k=0}^{\infty} w_k(x,t). \tag{64} $$

Eq. (64) is substituted in Eq. (63), it is produced as

$$ \sum_{k=0}^{\infty} w_k(x,t) = \frac{1}{10}\sin\left[\frac{\pi x}{l}\right] - \left({}_cS_\alpha\right)^{-1}\left[\frac{u}{s}\ {}_cS_\alpha\left[\sum_{k=0}^{\infty}\frac{\partial^4 w_k(x,t)}{\partial x^4} + 2\sum_{k=0}^{\infty}\frac{\partial^2 w_k(x,t)}{\partial x^2}\right.\right. $$

$$ \left.\left. + (1-\mu)\sum_{k=0}^{\infty} w_k(x,t) + \sum_{k=0}^{\infty} A_k\right]\right], k \geq 0. \tag{65} $$

where A_k is Adomian polynomials.

Comparing both sides of Eq. (65) yields the result

$$ w_0(x,t) = \frac{1}{10}\sin\left[\frac{\pi x}{l}\right]. \tag{66} $$

And by substituting $k = 1, k = 2$ in the Eq. (65) respectively, Eqs. (67)-(68) are achieved as

$$ w_1(x,t) = \frac{t^\alpha \sin\left[\frac{\pi x}{l}\right]}{10\alpha l^4}\left[\frac{l^4\cos^2\left[\frac{\pi x}{l}\right]}{100} + \left(\mu - \frac{101}{100}\right) l^4 + 2\pi^2 l^2 - \pi^4\right], \tag{67} $$

$$w_2(x,t)=\frac{t^{2\alpha}\sin\left[\frac{\pi x}{l}\right]}{20\alpha^2 l^8}\Bigg[\frac{3}{10000}2^{-\alpha}\cos^4\left[\frac{\pi x}{l}\right]l^8+\frac{3}{100}l^4\Bigg[\Big[\Big(\mu-\frac{51}{50}\Big)l^4+2\pi^2l^2-\pi^4\Big]$$

$$\times 2^{-\alpha}+\Big(\frac{\mu}{3}-\frac{1}{3}\Big)l^4+6\pi^2l^2-27\pi^4\Bigg]\cos^2\left[\frac{\pi x}{l}\right]-\frac{3}{100}l^4\Big[\Big(\mu-\frac{101}{100}\Big)l^4+2\pi^2l^2$$

$$-\pi^4\Big]2^{-\alpha}+(-1+\mu)\Big(\mu-\frac{101}{100}\Big)l^8+4\Big(\mu-\frac{203}{200}\Big)l^6\pi^2-2\Big(\mu-\frac{621}{200}\Big)l^4\pi^4$$

$$-4l^2\pi^6+\pi^8\Big]\Bigg]. \tag{68}$$

Thus, it is obtained as

$$w(x,t)=\sum_{k=0}^{\infty}w_k(x,t)=w_0(x,t)+w_1(x,t)+w_2(x,t)=\frac{1}{10}\sin\left[\frac{\pi x}{l}\right]$$

$$+\frac{t^{\alpha}\sin\left[\frac{\pi x}{l}\right]}{10\alpha l^4}\left[\frac{l^4\cos^2\left[\frac{\pi x}{l}\right]}{100}+\Big(\mu-\frac{101}{100}\Big)l^4+2\pi^2l^2-\pi^4\right]+\frac{t^{2\alpha}\sin\left[\frac{\pi x}{l}\right]}{20\alpha^2l^8}\Big[\frac{3}{10000}2^{-\alpha}$$

$$\times\cos^4\left[\frac{\pi x}{l}\right]l^8+\frac{3}{100}l^4\Bigg[\Big[\Big(\mu-\frac{51}{50}\Big)l^4+2\pi^2l^2-\pi^4\Big]2^{-\alpha}+\Big(\frac{\mu}{3}-\frac{1}{3}\Big)l^4+6\pi^2l^2\Bigg]$$

$$-27\pi^4\Big]\cos^2\left[\frac{\pi x}{l}\right]-\frac{3}{100}l^4\Big[\Big(\mu-\frac{101}{100}\Big)l^4+2\pi^2l^2-\pi^4\Big]2^{-\alpha}+(-1+\mu)\Big(\mu-\frac{101}{100}\Big)l^8$$

$$+4\Big(\mu-\frac{203}{200}\Big)l^6\pi^2-2\Big(\mu-\frac{621}{200}\Big)l^4\pi^4-4l^2\pi^6+\pi^8\Big]\Big]. \tag{69}$$

Fig. 1 depicts Cq-SHATM graphs for varying α values.

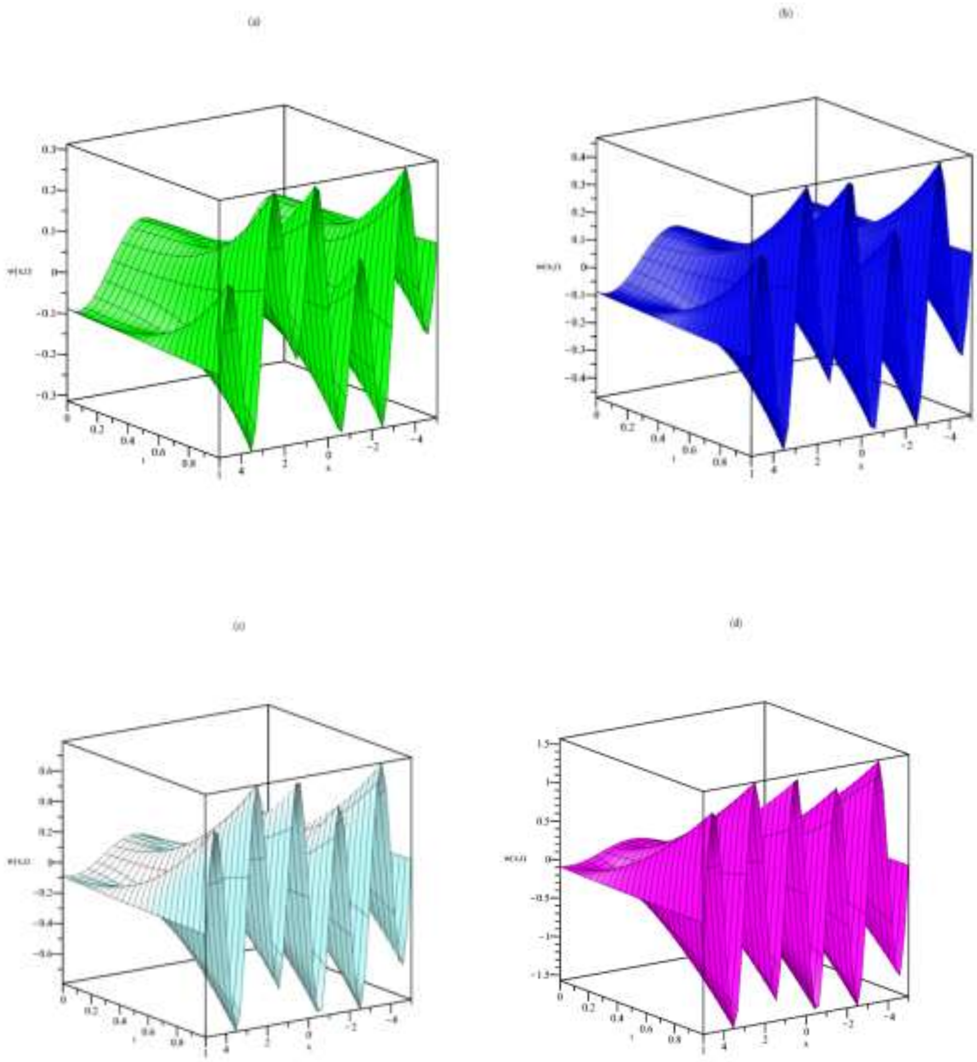

Fig. 1. (a) For $\alpha = 1$, Cq-HATM solution **(b)** For $\alpha = 0.85$, Cq-HATM solution **(c)** For $\alpha = 0.70$, Cq-HATM solution **(d)** For $\alpha = 0.55$, Cq-HATM solution at $l = 3, \mu = 0.3, h = -1, n = 1$.

3D graphs for CSADM solution by different α values are depicted in Fig. 2.

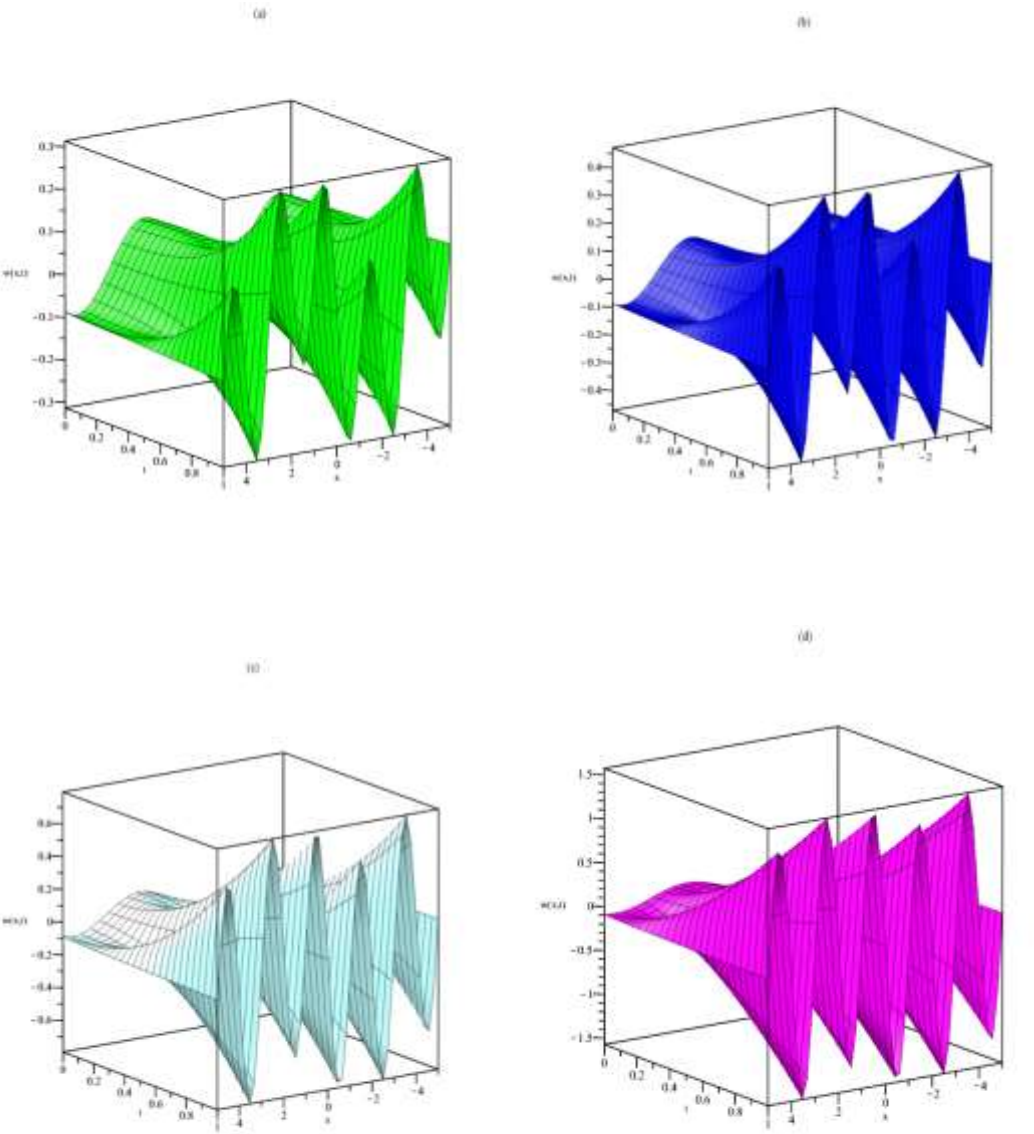

Fig. 2. (a) For $\alpha = 1$, CSADM solution **(b)** For $\alpha = 0.85$, CSADM solution **(c)** For $\alpha = 0.70$, CSADM solution **(d)** For $\alpha = 0.55$, CSADM solution at $l = 3$, $\mu = 0.3$.

The 2D graphs for Cq-SHATM and CSADM solutions by varying α values are depicted in Fig. 3.

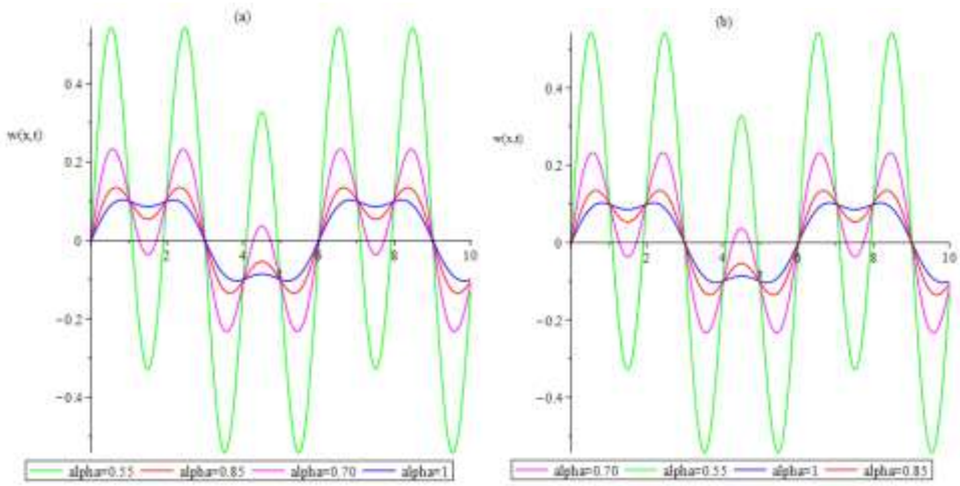

Fig. 3. (a) The 2D graph of the Cq-SHATM solution **(b)** The 2D graph of the CSADM solution for Eq. (46) $l = 3, \mu = 0.3, h = -1, n = 1, t = 0.5$ with distinct α.

Table 1. Numerical solution for CTFSHE with proportional delay by Cq-SHATM in Eq. (46) at $l = 3, \mu = 0.3, h = -1, n = 1$ with distinct values of x and t for diverse α.

x	t	$\alpha = 0.55$	$\alpha = 0.7$	$\alpha = 0.85$	$\alpha = 1$
0.1	0.1	0.02183028447	0.01291273226	0.01123977954	0.01080371354
	0.2	0.04424028868	0.01905705819	0.01322693156	0.01157758999
	0.3	0.07538926723	0.02951424222	0.01717936025	0.01323869603
	0.4	0.11413415110	0.04452665732	0.02364124959	0.01625653178
	0.5	0.15975804650	0.06425651933	0.03307019241	0.02110059736
0.2	0.1	0.04257638075	0.02554657740	0.02233151817	0.02148433319
	0.2	0.08528146383	0.03728604551	0.02614832257	0.02298328891
	0.3	0.14459942930	0.05722474844	0.03370223434	0.02617081677
	0.4	0.21836156800	0.08582641819	0.04602940632	0.03194008309
	0.5	0.30520635540	0.12340073200	0.06400074178	0.04118425422
0.3	0.1	0.06124815856	0.03763908101	0.03313122954	0.03192058385
	0.2	0.12022345020	0.05392822667	0.03847762419	0.03405148484
	0.3	0.20203971040	0.08149160008	0.04896492223	0.03850888590
	0.4	0.30372595600	0.12097428210	0.06602189143	0.04652241783
	0.5	0.42341350400	0.17280532760	0.09084848998	0.05932171152
0.4	0.1	0.07703466085	0.04895918477	0.04350275723	0.04199476250
	0.2	0.14673554940	0.06835696686	0.04996455267	0.04462863072
	0.3	0.24323940020	0.10098682410	0.06246431482	0.05000189431
	0.4	0.36308038950	0.14762002080	0.08268659535	0.05956057903
	0.5	0.50407192140	0.20876628860	0.11204438030	0.07475071070
0.5	0.1	0.08937217283	0.05931938852	0.05332101722	0.05159436599

	0.2	0.16327207310	0.08012797347	0.06040908768	0.05457808832
	0.3	0.26527172420	0.11481262220	0.07383620710	0.06044936981
	0.4	0.39177182290	0.16420575430	0.09538171431	0.07072940311
	0.5	0.54049036960	0.22885219870	0.12653512450	0.08693938079

Table 2. Numerical solution for CTFSHE with proportional delay by CSADM in Eq. (46) at $l = 3, \mathrm{\mu} = 0.3, \mathrm{h} = -1, \mathrm{n} = 1$ with distinct values of x and t for diverse α.

$\boldsymbol{x}$	$\boldsymbol{t}$	$\boldsymbol{\alpha = 0.55}$	$\boldsymbol{\alpha = 0.7}$	$\boldsymbol{\alpha = 0.85}$	$\boldsymbol{\alpha = 1}$
0.1	0.1	0.02183028449	0.01291273227	0.01123977954	0.01080371354
	0.2	0.04257638078	0.02554657741	0.02233151817	0.02148433319
	0.3	0.07538926730	0.02951424223	0.01717936024	0.01323869604
	0.4	0.11413415110	0.04452665737	0.02364124959	0.01625653178
	0.5	0.15975804660	0.06425651940	0.03307019240	0.02110059736
0.2	0.1	0.04257638078	0.02554657741	0.02233151817	0.02148433319
	0.2	0.08528146380	0.03728604552	0.02614832257	0.02298328891
	0.3	0.14459942930	0.05722474846	0.03370223434	0.02617081677
	0.4	0.21836156780	0.08582641824	0.04602940631	0.03194008310
	0.5	0.30520635540	0.12340073210	0.06400074176	0.04118425421
0.3	0.1	0.06124815859	0.03763908100	0.03313122954	0.03192058385
	0.2	0.12022345020	0.05392822668	0.03847762418	0.03405148484
	0.3	0.20203971040	0.08149160004	0.04896492222	0.03850888590
	0.4	0.30372595600	0.12097428210	0.06602189138	0.04652241784
	0.5	0.42341350380	0.17280532740	0.09084848993	0.05932171152
0.4	0.1	0.07703466090	0.04895918477	0.04350275723	0.04199476250
	0.2	0.14673554940	0.06835696685	0.04996455267	0.04462863072
	0.3	0.24323940030	0.10098682410	0.06246431480	0.05000189431
	0.4	0.36308039000	0.14762002090	0.08268659528	0.05956057904
	0.5	0.50407192200	0.20876628870	0.11204438030	0.07475071071
0.5	0.1	0.08937217288	0.05931938852	0.05332101722	0.05159436600
	0.2	0.16327207330	0.08012797344	0.06040908767	0.05457808832
	0.3	0.26527172420	0.11481262210	0.07383620710	0.06044936982
	0.4	0.39177182320	0.16420575420	0.09538171429	0.07072940311
	0.5	0.54049037020	0.22885219880	0.12653512450	0.08693938080

5. Results and Discussion

For CTFSHE with proportional delay, Fig. 1 depicts the 3D graphs of the temperature $w(x,t)$ derived by Cq-SHATM for various values of $\alpha = 0.55$, $\alpha = 0.70$, $\alpha = 0.85$, and $\alpha = 1$. The 3D graphs of the temperature $w(x,t)$ acquired by CSADM for various values of $\alpha = 0.55$, $\alpha = 0.70$, $\alpha = 0.85$, and $\alpha = 1$ are shown in Fig. 2. Fig. 3 illustrates the behavior of Cq-SHATM and CSADM solutions of CTFSHE with proportional delay for various values α using 2D graphs. Table 1 displays the temperature $w(x,t)$ obtained via Cq-SHATM for different values of $\alpha = 0.55$, $\alpha = 0.70$, $\alpha = 0.85$, and $\alpha = 1$ for Eq. (46). Also, for Eq. (46), Table 2 exhibits the graphs of the temperature $w(x,t)$ generated using CSADM for various values of $\alpha = 0.55$, $\alpha = 0.70$, $\alpha = 0.85$, and $\alpha = 1$.

6. Conclusion

CTFSHE with proportional delay are analyzed via Cq-SHATM and CSADM in this work. In addition, the 2D and 3D graphs illustrating the solutions to these equations for various values of have been generated using the MAPLE software. The general structure of the surface graphs created by the Maple software for Eq. (46) is noted to vary. It can be observed from Tables 1-2 that close results were obtained from both techniques. It is possible to conclude that the recently developed methods for solving nonlinear conformable time-fractional partial differential equations with proportional delay are both advantageous and effective.

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