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RESEARCH ARTICLE

HYPER-LEONARDO HYBRINOMIALS

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ABSTRACT

The aim of this paper is to define hyper-Leonardo hybrinomials as a generalization of the Leonardo Pisano hybrinomials and to examine some of their properties such as the recurrence relation, summation formula and generating function. Another aim is to introduce hyper-Leonardo hybrid numbers.

Keywords: Hyper-Leonardo numbers, Hyper-Leonardo Polynomials, Hybrinomials

1. INTRODUCTION

The integer sequences have an important role in the development of science and technology. Therefore, it is widely used especially in mathematics and many other sciences [1-5]. The most famous integer sequences are the Fibonacci and Lucas number sequences which are defined by the recurrence relations, respectively $(n \geq 2)$ [6]:

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_0 = 0,\ F_1 = 1 \tag{1}$$

$$L_n = L_{n-1} + L_{n-2} \text{ with } L_0 = 2,\ L_1 = 1. \tag{2}$$

The Leonardo sequence which has similar properties to the Fibonacci sequence is defined by Catarino and Borges [7], as follows:

$$Le_n = Le_{n-1} + Le_{n-2} + 1\ (n \geq 2), \tag{3}$$

with the initial conditions $Le_0 = Le_1 = 1$. Alp and Koçer [8] obtained some identities for the Leonardo numbers, and presented some relations among the Fibonacci, Lucas and Leonardo numbers. There are some papers on the generalization of the Leonardo numbers in the literature [9-12]. Kürüz et al. [13] preferred to call the Leonardo numbers as Leonardo Pisano numbers and defined Leonardo Pisano polynomials as

$$Le_n(x) = \begin{cases} 1, & n = 0,1 \\ x + 2, & n = 2 \\ 2xLe_{n-1}(x) - Le_{n-3}(x), & n \geq 3. \end{cases} \tag{4}$$

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Mersin and Bahşi [11] defined hyper-Leonardo numbers $Le_n^{(r)}$ as a generalization of the Leonardo numbers, by the formula

$$Le_n^{(r)} = \sum_{s=0}^{n} Le_s^{(r-1)} \text{ with } Le_n^{(0)} = Le_n,\ Le_0^{(r)} = Le_0 \text{ and } Le_1^{(r)} = r+1, \tag{5}$$

where r is a positive integer. Hyper-Leonardo numbers have the following recurrence relation for

$n \geq 1$ and $r \geq 1$ [11]:

$$Le_n^{(r)} = Le_{n-1}^{(r)} + Le_n^{(r-1)}. \tag{6}$$

Hyper-Leonardo polynomials are defined as [14]:

$$Le_n^{(r)}(x) = \sum_{s=0}^{n} Le_s^{(r-1)}(x) \text{ with } Le_n^{(0)}(x) = Le_n(x),\ Le_0^{(r)}(x) = 1,\ Le_1^{(r)}(x) = r+1. \tag{7}$$

Hyper-Leonardo polynomials have the following recurrence relation for $r \geq 1$ and $n \geq 1$ [14]:

$$Le_n^{(r)}(x) = Le_{n-1}^{(r)}(x) + Le_n^{(r-1)}(x)\ . \tag{8}$$

There is the relation between hyper-Leonardo polynomials and Leonardo Pisano polynomials for

$n \geq 1$ and $r \geq 1$ [14]:

$$Le_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} Le_s(x). \tag{9}$$

Hybrid number system $\mathbb{K}$, which is a generalization of complex, hyperbolic and dual numbers, is defined by Özdemir [15] as

$$\mathbb{K} = \{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = hi = \epsilon + i\}. \tag{10}$$

Hybrid numbers have been the subject of much research recently [16-30]. Szynal-Liana and Wloch [16] defined the n-th Fibonacci hybrid number as

$$HF_n = F_n + F_{n+1}i + F_{n+2}\epsilon + F_{n+3}h. \tag{11}$$

Kızılateş [24] introduced q-Fibonacci and q-Lucas hybrid numbers, and gave some of their algebraic properties. A new class of quaternions, octanions and sedenions called higher order Fibonacci hyper complex numbers whose components are higher order Fibonacci numbers are defined and some of their identities are examined by Kızılateş and Kone [25]. Polatlı [26] studied on divisibility identities of the Fibonacci and Lucas hybrid numbers. Fibonacci divisor hybrid numbers which are a generalization Fibonacci hybrid numbers are defined by using the Fibonacci divisor numbers [27]. Alp and Koçer [28] introduced hybrid-Leonardo numbers by using the Leonardo sequence as follows:

$$HLe_n = Le_n + Le_{n+1}i + Le_{n+2}\epsilon + Le_{n+3}h. \tag{12}$$

The generating function for the hybrid Leonardo numbers HLe_n is [28]

$$g(t)=\frac{HLe_0+t(-1+i-\epsilon-h)+t^2(1-i-\epsilon-3h)}{1-2t+t^3}. \qquad (13)$$

There are some relations between the hybrid-Leonardo numbers and Fibonacci hybrid numbers such as in [28]:

$$HLe_n=2HF_{n+1}-(1+i+\epsilon+h),\ (n\geq 0). \qquad (14)$$

Also, one can easily see the relation

$$2HF_n=HLe_n-HLe_{n-2},\ (n\geq 2) \qquad (15)$$

is valid. The Horadam hybrid quaternions and some special classes of number sequences such as Fibonacci, Lucas, Pell and Jacobsthal hybrid quaternions are introduced by Dağdeviren and Kürüz [29]. Mangueira et al. [30] defined Leonardo quaternions and Leonardo hybrid quaternions. The authors also presented the recurrence relation, characteristic equation, generating function, Binet formula for the Leonardo hybrid quaternions and its relations with the Fibonacci quaternions. Kürüz et al. [13] defined Leonardo Pisano hybrinomials by using the Leonardo Pisano polynomials as:

$$Le_n^{[H]}(x)=Le_n(x)+iLe_{n+1}(x)+\epsilon Le_{n+2}(x)+hLe_{n+3}(x). \qquad (16)$$

The Leonardo Pisano hybrinomials have the following recurrence relation and generating function, respectively [13]:

$$Le_n^{[H]}(x)=2xLe_{n-1}^{[H]}(x)-Le_{n-3}^{[H]}(x),\ (n\geq 3) \qquad (17)$$

and

$$g_{Le_n^{[H]}(x)}(\lambda)=\frac{Le_0^{[H]}(x)+\left(Le_1^{[H]}(x)-2xLe_0^{[H]}(x)\right)\lambda+\left(Le_2^{[H]}(x)-2xLe_1^{[H]}(x)\right)\lambda^2}{1-2x\lambda+\lambda^3}. \qquad (18)$$

By the motivation of the above papers, we define hyper-Leonardo hybrinomials and investigate their some algebraic and combinatoric properties. In addition, we introduce hyper-Leonardo hybrid numbers and examine some of their properties.

2. MAIN RESULTS

Definition 2.1. The n-th hyper-Leonardo hybrinomial is defined as

$$HLe_n^{(r)}(x)=\sum_{s=0}^{n}HLe_s^{(r-1)}(x) \qquad (19)$$

with the initial conditions $HLe_n^{(0)}(x)=HLe_n(x)$, $HLe_0^{(r)}(x)=HLe_0(x)$, where r is a positive integer and $HF_n(x)$ is the n-th Leonardo Pisano hybrinomial.

The first few hyper-Leonardo hybrinomials are

$$HLe_0^{(1)}(x) = 1 + i + \epsilon(x+2) + h(2x^2+4x-1),$$
$$HLe_1^{(1)}(x) = 2 + i(x+3) + \epsilon(2x^2+5x+1) + h(4x^3+10x^2+2x-2),$$
$$HLe_2^{(1)}(x) = (x+4) + i(2x^2+5x+2) + \epsilon(4x^3+10x^2+3x) + h(8x^4+20x^3+6x^2-x-4)$$

and

$$HLe_0^{(2)}(x) = 1 + i + \epsilon(x+2) + h(2x^2+4x-1),$$
$$HLe_1^{(2)}(x) = 3 + i(x+4) + \epsilon(2x^2+6x+3) + h(4x^3+12x^2+6x-3),$$
$$HLe_2^{(2)}(x) = (x+7) + i(2x^2+6x+6) + \epsilon(4x^3+12x^2+9x+3)$$
$$+h(8x^4+24x^3+18x^2+5x-7).$$

Definition 2.1 yields that the hyper-Leonardo hybrinomials have the following recurrence relation for $n \geq 1$ and $r \geq 1$:

$$HLe_n^{(r)}(x) = HLe_{n-1}^{(r)}(x) + HLe_n^{(r-1)}(x). \tag{20}$$

Note that, for $x = 1$, Definition 2.1 gives the following definition of hyper-Leonardo hybrid numbers. Therefore, the hyper-Leonardo hybrinomials are a generalization of the hyper-Leonardo hybrid numbers.

Definition 2.2. The n-th hyper-Leonardo hybrid number $HLe_n^{(r)}$ is defined by

$$HLe_n^{(r)} = \sum_{s=0}^{n} HLe_s^{(r-1)} \tag{21}$$

with $HLe_n^{(0)} = HLe_n$ and $HLe_0^{(r)} = HLe_0$, where r is a positive integer and HLe_n is the n-th hybrid-Leonardo number.

Hyper-Leonardo hybrid numbers also have the following recurrence relation for $n \geq 1$ and $r \geq 1$:

$$HLe_n^{(r)} = HLe_{n-1}^{(r)} + HLe_n^{(r-1)}. \tag{22}$$

Table 1 contains the values of the hyper-Leonardo hybrid numbers.

Table 1. The first few values of the hyper-Leonardo hybrid numbers $HLe_n^{(r)}$.

	$r=0$	$r=1$	$r=2$	$r=3$
n = 0	$1+i+3\epsilon+5h$	$1+i+3\epsilon+5h$	$1+i+3\epsilon+5h$	$1+i+3\epsilon+5h$
n = 1	$1+3i+5\epsilon+9h$	$2+4i+8\epsilon+14h$	$3+5i+11\epsilon+19h$	$4+6i+14\epsilon+24h$
n = 2	$3+5i+9\epsilon+15h$	$5+9i+17\epsilon+29h$	$8+14i+28\epsilon+48h$	$12+20i+42\epsilon+72h$
n = 3	$5+9i+15\epsilon+25h$	$10+18i+32\epsilon+54h$	$18+32i+60\epsilon+102h$	$30+52i+102\epsilon+174h$
n = 4	$9+15i+25\epsilon+41h$	$19+33i+57\epsilon+95h$	$37+65i+117\epsilon+197h$	$67+117i+219\epsilon+371\text{h}$

Theorem 2.1. The generating function for the hyper-Leonardo hybrinomials is

$$G(r)=\sum_{n=0}^{\infty} HLe_n^{(r)}(x)t^n=\frac{HLe_0(x)+t\big(HLe_1(x)-2xHLe_0(x)\big)+t^2\big(HLe_2(x)-2xHLe_1(x)\big)}{(1-2xt^2+t^3)(1-t)^r}. \tag{23}$$

Proof. We use the induction method on r. Since

$$\begin{aligned} G(0) &= \sum_{n=0}^{\infty} HLe_n^{(0)}(x)t^n \\ &= \frac{HLe_0(x)+t(HLe_1(x)-2xHLe_0(x))+t^2(HLe_2(x)-2xHLe_1(x))}{(-2xt^2+t^3)(1-t)^0} \\ &= \sum_{n=0}^{\infty} HLe_n(x)t^n, \end{aligned}$$

the result is true for $r=0$. Suppose that the result is true for r. Then, we have

$$G(r)=\sum_{n=0}^{\infty} HLe_n^{(r)}(x)t^n=\frac{HLe_0(x)+t\big(HLe_1(x)-2xHLe_0(x)\big)+t^2\big(HLe_2(x)-2xHLe_1(x)\big)}{(1-2xt^2+t^3)(1-t)^r}.$$

Now, we must show that the result is true for $r+1$. Considering the Cauchy product, we have

$$\begin{aligned} G(r+1)=\sum_{n=0}^{\infty} HLe_n^{(r+1)}(x)t^n &= \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n} HLe_n^{(r)}(x)\right)t^n \\ &= \left(\sum_{i=0}^{\infty} HLe_i^{(r)}(x)t^i\right)\left(\sum_{j=0}^{\infty} t^j\right). \end{aligned}$$

Then, we have

$$G(r+1) = \frac{HLe_0(x) + t(HLe_1(x) - 2xHLe_0(x)) + t^2(HLe_2(x) - 2xHLe_1(x))}{(1 - 2xt^2 + t^3)(1-t)^{r+1}}.$$

Corollary 2.1. The generating function for the hyper-Leonardo hybrid numbers is

$$g(r) = \sum_{n=0}^{\infty} HLe_n^{(r)} t^n = \frac{HLe_0 + t(-1 + i - \epsilon - h) + t^2(1 - i - \epsilon - 3h)}{(1 - 2t + t^3)(1-t)^r}. \tag{24}$$

Theorem 2.2. If $n \geq 1$ and $r \geq 1$, then there is the relation between the hyper-Leonardo hybrinomials and Leonardo Pisano hybrinomials:

$$HLe_n^{(r)}(x) = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HLe_s(x). \tag{25}$$

Proof. The symmetric infinite matrix with entries a_n^r has the following recurrence relation [31]:

$$a_n^0 = a_n,\ a_0^n = a^n\ (n \geq 0), \tag{26}$$

$$a_n^r = a_n^{r-1} + a_{n-1}^r\ \ (n \geq 1, r \geq 1), \tag{27}$$

where (a_n) and (a^n) are two real initial sequences. Also the entries a_n^r have the following symmetric relation [32]:

$$a_n^r = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} a_0^i + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} a_s^0. \tag{28}$$

For the case $a_n^r = HLe_n^{(r)}(x)$, equation (28) is of the form:

$$HLe_n^{(r)}(x) = \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} HLe_0^{(i)}(x) + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} HLe_s^{(0)}(x). \tag{29}$$

By using the initial conditions in Definition 2.1, we get

$$\begin{aligned} HLe_n^{(r)}(x) &= \sum_{i=1}^{r} \binom{n+r-i-1}{n-1} HLe_0(x) + \sum_{s=1}^{n} \binom{n+r-s-1}{r-1} HLe_s(x) \\ &= \sum_{i=0}^{r-1} \binom{n+r-i-2}{n-1} HLe_0(x) + \sum_{s=0}^{n-1} \binom{n+r-s-2}{r-1} HLe_{s+1}(x) \\ &= HLe_0(x) \sum_{k=0}^{r-1} \binom{n+k-1}{n-1} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} HLe_{n-b}(x), \end{aligned}$$

where $k = r - i - 1$ and $b = n - s - 1$.

By means of [33], we have

$$\sum_{t=a}^{c} \binom{t}{a} = \binom{c+1}{a+1}. \tag{30}$$

Thus,

$$\begin{aligned} HLe_n^{(r)}(x) &= HLe_0(x)\binom{n+r-1}{n} + \sum_{b=0}^{n-1} \binom{r+b-1}{r-1} HLe_{n-b}(x) \\ &= \sum_{b=0}^{n} \binom{r+b-1}{r-1} HLe_{n-b}(x) \\ &= \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HLe_s(x). \end{aligned}$$

Corollary 2.2. If $n \geq 1$ and $r \geq 1$, then there is the relation between the hyper-Leonardo hybrid numbers and hybrid-Leonardo numbers:

$$HLe_n^{(r)} = \sum_{s=0}^{n} \binom{n+r-s-1}{r-1} HLe_s. \tag{31}$$

Theorem 2.3. If $n \geq 2$ and $r \geq 1$, then the summation formula is valid for the hyper-Leonardo hybrinomials:

$$\sum_{s=0}^{r} HLe_n^{(s)}(x) = HLe_{n+1}^{(r)}(x) + (1-2x)HLe_n(x) + HLe_{n-2}(x). \tag{32}$$

Proof. By using Theorem 2.2 and equation (30), we get

$$\begin{aligned} \sum_{s=1}^{r} HLe_n^{(s)}(x) &= \sum_{s=1}^{r} \left(\sum_{t=0}^{n} \binom{n+s-t-1}{s-1} HLe_t(x) \right) \\ &= \sum_{t=0}^{n} \left(HLe_t(x) \sum_{s=1}^{r} \binom{n+s-t-1}{s-1} \right) \\ &= \sum_{t=0}^{n} \binom{n+r-t}{r-1} HLe_t(x) \\ &= \sum_{t=0}^{n+1} \binom{n+r-t}{r-1} HLe_t(x) - HLe_{n+1}(x). \end{aligned}$$

Then, by considering equation (17), we have

$$\begin{aligned}\sum_{s=0}^{r} HLe_n^{(s)}(x) &= HLe_{n+1}^{(r)}(x) - HLe_{n+1}(x) + HLe_n(x)\\ &= HLe_{n+1}^{(r)}(x) - (2xHLe_n(x) - HLe_{n-2}(x)) + HLe_n(x)\\ &= HLe_{n+1}^{(r)}(x) + (1-2x)HLe_n(x) + HLe_{n-2}(x).\end{aligned}$$

Corollary 2.3. If $n \geq 1$ and $r \geq 1$, then there is the summation formula for the hyper-Leonardo hybrid numbers:

$$\sum_{s=0}^{r} HLe_n^{(s)} = HLe_{n+1}^{(r)} - 2HF_n. \tag{33}$$

Theorem 2.4. For $n \geq 3$ and $r \geq 1$, there is the recurrence relation for the hyper-Leonardo hybrinomials:

$$HLe_n^{(r)}(x) = 2xHLe_{n-1}^{(r)}(x) - HLe_{n-3}^{(r)}(x) + \binom{n+r-1}{r-1}HLe_0(x)$$
$$-\binom{n+r-2}{r-1}(2x-1+i(x-2)+\epsilon+h) - \binom{n+r-3}{r-1}\big(x-2+i+\epsilon+h(x+2)\big). \tag{34}$$

Proof. Considering Theorem 2.2 and equation (17), we have

$$\begin{aligned}HLe_n^{(r)}(x) &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HLe_s(x)\\ &= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\big(2xHLe_{s-1}(x) - HLe_{s-3}(x)\big)\\ &= 2x\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HLe_{s-1}(x) - \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}HLe_{s-3}(x)\\ &= 2x\sum_{s=-1}^{n-1}\binom{n+r-(s+1)-1}{r-1}HLe_s(x) - \sum_{s=-3}^{n-3}\binom{n+r-(s+3)-1}{r-1}HLe_s(x)\\ &= 2x\left(\sum_{s=-1}^{n-1}\binom{(n-1)+r-s-1}{r-1}HLe_s(x) + \binom{n+r-1}{r-1}HLe_{-1}(x)\right)\\ &\qquad -\left(\sum_{s=-3}^{n-3}\binom{(n-3)+r-s-1}{r-1}HLe_s(x) + \binom{n+r-3}{r-1}HLe_{-1}(x)\right.\\ &\qquad\qquad \left. + \binom{n+r-2}{r-1}HLe_{-2}(x) + \binom{n+r-1}{r-1}HLe_{-3}(x)\right).\end{aligned}$$

Thus,

$$HLe_n^{(r)}(x) = 2x\left(HLe_n^{(r)}(x) + \binom{n+r-1}{r-1}(x-2+i+\epsilon+(x+2)h)\right)$$

$$-\left(HLe_{n-3}^{(r)}(x) + \binom{n+r-3}{r-1}(x-2+i+\epsilon+(x+2)h)\right.$$

$$+\binom{n+r-2}{r-1}(2x-1+i(x-2)+\epsilon+h)$$

$$\left.+\binom{n+r-1}{r-1}(2x^2-4x-1+i(2x-1)+\epsilon(x-2)+h)\right)$$

$$= 2xHLe_{n-1}^{(r)}(x) - HLe_{n-3}^{(r)}(x) + \binom{n+r-1}{r-1}\left(1+i+\epsilon(x+2)+h(2x^2+4x-1)\right)$$

$$-\binom{n+r-2}{r-1}(2x-1+i(x-2)+\epsilon+h) - \binom{n+r-3}{r-1}\left(x-2+i+\epsilon+h(x+2)\right).$$

Corollary 2.4. For $n \geq 3$ and $r \geq 1$, the recurrence relation is valid for the hyper-Leonardo hybrid numbers:

$$HLe_n^{(r)} = 2HLe_{n-1}^{(r)} - HLe_{n-3}^{(r)} + \binom{n+r-1}{r-1}HLe_0 - \binom{n+r-2}{r-1}(1-i+\epsilon+h)$$

$$-\binom{n+r-3}{r-1}(-1+i+\epsilon+3h). \tag{35}$$

Theorem 2.5. If $n \geq 1$ and $r \geq 1$ then, there is the relation between the hyper-Leonardo polynomials and hyper-Leonardo hybrinomials:

$$Le_n^{(r)}(x) + iLe_{n+1}^{(r)}(x) + \epsilon Le_{n+2}^{(r)}(x) + hLe_{n+3}^{(r)}(x) - HLe_n^{(r)}(x) = \binom{n+r}{r-1}(i+\epsilon+h(x+2))$$

$$+\binom{n+r+1}{r-1}(\epsilon+h) + \binom{n+r+2}{r-1}h. \tag{36}$$

Proof. By using equations (9), (16) and (25), we have

$$Le_n^{(r)}(x) + iLe_{n+1}^{(r)}(x) + \epsilon Le_{n+2}^{(r)}(x) + hLe_{n+3}^{(r)}(x) - HLe_n^{(r)}(x)$$

$$= \sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s(x) + i\sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}Le_s(x)$$

$$+\epsilon\sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}Le_s(x) + h\sum_{s=0}^{n+3}\binom{(n+3)+r-s-1}{r-1}Le_s(x)$$

$$-\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}\left(Le_s(x)+iLe_{s+1}(x)+\epsilon Le_{s+2}(x)+hLe_{s+3}(x)\right).$$

Then,

$$Le_n^{(r)}(x)+iLe_{n+1}^{(r)}(x)+\epsilon Le_{n+2}^{(r)}(x)+hLe_{n+3}^{(r)}(x)-HLe_n^{(r)}(x)$$

$$=\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s(x)+i\sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}Le_s(x)$$

$$+\epsilon\sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}Le_s(x)+h\sum_{s=0}^{n+3}\binom{(n+3)+r-s-1}{r-1}Le_s(x)$$

$$-\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_s(x)-i\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_{s+1}(x)$$

$$-\epsilon\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_{s+2}(x)-h\sum_{s=0}^{n}\binom{n+r-s-1}{r-1}Le_{s+3}(x)$$

$$=i\left(\sum_{s=0}^{n+1}\binom{(n+1)+r-s-1}{r-1}Le_s(x)-\sum_{s=1}^{n+1}\binom{(n+1)+r-s-1}{r-1}Le_s(x)\right)$$

$$+\epsilon\left(\sum_{s=0}^{n+2}\binom{(n+2)+r-s-1}{r-1}Le_s(x)-\sum_{s=2}^{n+2}\binom{(n+2)+r-s-1}{r-1}Le_s(x)\right)$$

$$+h\left(\sum_{s=0}^{n+3}\binom{(n+3)+r-s-1}{r-1}Le_s(x)-\sum_{s=3}^{n+3}\binom{(n+3)+r-s-1}{r-1}Le_s(x)\right)$$

$$=i\binom{n+r}{r-1}Le_0(x)+\epsilon\left(\binom{n+r+1}{r-1}Le_0(x)+\binom{n+r}{r-1}Le_1(x)\right)$$

$$+h\left(\binom{n+r+2}{r-1}Le_0(x)+\binom{n+r+1}{r-1}Le_1(x)+\binom{n+r}{r-1}Le_2(x)\right)$$

$$=\binom{n+r}{r-1}(i+\epsilon+h(x+2))+\binom{n+r+1}{r-1}(\epsilon+h)+\binom{n+r+2}{r-1}h.$$

Corollary 2.5. If $n\geq 1$ and $r\geq 1$ then, there is the relation between the hyper-Leonardo numbers and hyper-Leonardo hybrid numbers:

$$Le_n^{(r)}+iLe_{n+1}^{(r)}+\epsilon Le_{n+2}^{(r)}+hLe_{n+3}^{(r)}-HLe_n^{(r)}=\binom{n+r}{r-1}(i+\epsilon+3h)+\binom{n+r+1}{r-1}(\epsilon+h)$$

$$+\binom{n+r+2}{r-1}h. \qquad (37)$$

CONCLUSION

Hybrid number system which is considered a generalization of complex, hyperbolic and dual numbers is a subject that has attracted the attention of the much researchers, recently. In this paper, we defined hyper-Leonardo hybrinomials as a generalization of the Leonardo Pisano hybrinomials and investigated some of their properties such as the generating function, recurrence relations and summation formulas. Additionally, we obtained hyper-Leonardo hybrid numbers from the hyper-Leonardo hybrinomials for $x = 1$, and we also presented some of their similar properties.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

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