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Approximating Common Fixed Point of Three C-α Nonexpansive Mappings

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Abstract

In this paper, we consider a new class of nonlinear mappings presented in [12] that generalizes two well-known classes of nonexpansive type mappings and extends some other classes of mappings. We introduce approximating common fixed point of three C-α nonexpansive mappings through weak and strong convergence of an iterative sequence in a uniformly convex Banach space. We also numerically illustrate the common fixed point approximations of the presented iteration for the three C-α nonexpansive mappings.

Keywords: *Local Fractional Derivative; Dirichlet boundary conditions; Spectral method; Separation of variables.*
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1. Introduction and Preliminaries

Throughout this paper, K be a nonempty convex subset of a Banach space X and $\varphi : K \to K$ be a mapping. We denote by $F(T)$ the set of fixed points of T. We denote by $F = \bigcap_{i=1}^{3} F(T_i)$ the set of a common fixed points of $T_i : K \to K, i = 1,2,3$.
A mapping T is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in X$. T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in X$ and $p \in F(T)$. In the past decades, many authors have been interested in some generalizations of nonexpansive mappings and established many iterative processes to approximate fixed points for generalized nonexpansive mappings(see [2], [3], [5], [10], [11], [12], [14], [18], [22], [23]). In 2008, Suzuki [14] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called *condition* (C) (herein referred as Suzuki generalized nonexpansive mapping), which properly includes the class of nonexpansive mappings. Let K be a nonempty closed and convex subset of a uniformly convex Banach space X. A mapping $T : K \to K$ is satisfy condition (C) if for all $x, y \in K$ $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.
Suzuki [14] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Lately, fixed-point approaches for Suzuki generalized nonexpansive mappings have been studied by a number of authors see e.g ([1], [4], [6], [15], [19], [20]).
In 2011, Aoyama and Kohsaka [3] introduced the class of $\alpha-$nonexpansive mappings in the setting of Banach spaces and obtained some fixed point results for such mappings. Let K be a nonempty closed and convex subset of a uniformly convex Banach space X. A mapping $T : K \to X$ is called a $\alpha-$*nonexpansive mapping* if there exists an $\alpha \in [0,1)$ such that for each $x, y \in K$

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

Note that Ariza-Ruiz et al. in [2] showed that the concept of $\alpha-$nonexpansive mapping is trivial for $\alpha < 0$. It is obvious that every nonexpansive mapping is $0-$nonexpansive and also every $\alpha-$nonexpansive mapping with a fixed point is quasi-nonexpansive (see [7]).
In [11], authors introduced the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings. A mapping $T : K \to K$ is called a *generalized* $\alpha-$*nonexpansive mapping* if there exists an $\alpha \in [0,1)$ and for each $x, y \in K$

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|.$$

More recently, a number of authors have been studied for numerical reckoning fixed points of generalized $\alpha-$nonexpansive mappings see e.g ([13], [16], [17]). In general, condition (C), $\alpha-$nonexpansive mapping and generalized $\alpha-$nonexpansive mapping are not continuous mappings (see examples [2], [4], [11], [14], [15], [16], [17]).

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Furthermore, in [12], authors presented the following new class of nonexpansive type mappings and obtained some fixed point results for this new class of mappings.
A mapping $T : K \to K$ is called C-α *nonexpansive mapping* if there exists an $\alpha \in [0,1)$ and for each $x, y \in K$,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies}$$
$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1-2\alpha)\|x - y\|^2.$$

A mapping satisfying the condition (C) is C-α nonexpansive mapping. An $\alpha-$nonexpansive mapping is a C-α nonexpansive mapping and also generalized $\alpha-$nonexpansive mapping is a C-α nonexpansive mapping, but from the examples given in [12] it can be seen that the reverse is not true.
The concept of approximating fixed points for generalized nonexpansive mappings plays an important role in the study of three-step iteration processes. Pant and Shukla [12] studied the Noor iteration scheme for C-α nonexpansive mapping. In 2000, Noor introduced the first three-step iteration scheme [8] and defined the following process: for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ defined by

$$\begin{cases} z_n &= (1-c_n)x_n + c_n Tx_n \\ y_n &= (1-b_n)x_n + b_n Tz_n \\ x_{n+1} &= (1-a_n)x_n +_n Ty_n, \forall n \in \mathbb{N} \end{cases}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\} \in (0,1)$.
Inspired and motivated by these facts, we introduce the following iterative scheme for three C-α nonexpansive mappings in uniformly convex Banach spaces. Let K be a nonempty convex subset of a Banach space X and $T_i : K \to K,\ i = 1,2,3$ be mappings. Then for arbitrary $x_1 \in K$, the scheme is defined as follows:

$$\left\{ \begin{array}{c} z_n = (1-c_n)x_n + c_n T_1 x_n \\ y_n = (1-b_n)z_n + b_n T_2 z_n \\ x_{n+1} = (1-a_n)y_n + a_n T_3 y_n, \forall n \in \mathbb{N}, \end{array} \right\} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0,1)$.
We then present the following three iteration schemes to approximate the fixed point for three mappings.
Let K be a nonempty convex subset of a Banach space X and $T_i : K \to K,\ i = 1,2,3$, be mappings. Then for arbitrary $x_1 \in K$, the scheme is defined as follows:

$$\left\{ \begin{array}{c} z_n = (1-c_n)x_n + c_n T_i x_n \\ y_n = (1-b_n)z_n + b_n T_i z_n \\ x_{n+1} = (1-a_n)y_n + a_n T_i y_n, \forall n \in \mathbb{N}, \end{array} \right\}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0,1)$.
In this paper let say the iterations: (1.2) for $i = 1$, (1.3) for $i = 2$, and (1.4) for $i = 3$, respectively. The aim of this paper is to introduce and study convergence problem of three-step iterative sequence (1.1) for three C-α nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize and extend some recent [12].
The following definitions will be needed in proving our main results.
A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta(\varepsilon) = inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\} > 0,$$

for all $0 < \varepsilon \leq 2$ $\Big($i.e., $\delta(\varepsilon)$ is a function $(0,2] \to (0,1)\Big)$.
Recall that a Banach space X is said to satisfy *Opial's condition* [9] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|.$$

Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to K is defined by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of $\{x_n\}$ relative to K is the set

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}.$$

It is known that, in uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one-point.

Lemma 1.1. *[21]. Let $r > 0$ be a fixed real number. Then a Banach space X is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g\Big(\|x-y\|\Big)$$

for all $x, y \in B_r := \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0,1]$.

We now list some properties of mapping that satisfy C-α nonexpansive mapping. In what follows, we shall make use of the following lemmas.

Lemma 1.2. *Let K be a nonempty closed and convex subset of Banach space X. Let $T : K \to K$ be a C-α nonexpansive mapping for some $\alpha \in [0,1)$ such that $F(T) \neq \emptyset$. Then T is a quasi-nonexpansive.*

Proof. Let $x \in K$ and $p \in F(T)$. Then we have $\frac{1}{2}\|p - Tp\| = 0 \leq \|p - x\|$ implies that

$$\begin{aligned} \|Tx - p\|^2 &= \|Tx - Tp\|^2 \\ &\leq \alpha\|Tx - p\|^2 + \alpha\|x - Tp\|^2 + (1-2\alpha)\|x - p\|^2 \\ &\leq \alpha\|Tx - p\|^2 + \alpha\|x - p\|^2 + (1-2\alpha)\|x - p\|^2 \\ &\leq \alpha\|Tx - p\|^2 + (1-\alpha)\|x - p\|^2. \end{aligned}$$

So, we have $\|Tx - p\|^2 \leq \&\|x - p\|^2$. □

Lemma 1.3. *[12]. Suppose that K is a nonempty subset a Banach space X and $T : K \to K$ is a C-α nonexpansive mapping. Then $F(T)$ is closed. In addition, if K is convex and X is strictly convex, then $F(T)$ is convex.*

Proposition 1.4. *[12]. (`Demiclosedness principle`). Assume that K is a nonempty subset of a Banach space X which has the Opial property and $T : K \to K$ is a C-α nonexpansive mapping. If $\{x_n\}$ converges weakly to a point p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tp = p$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on X.*

2. Main results

In this section, we prove the three-step iterative scheme (1.1) to converge to a common fixed point for three C-α nonexpansive mappings in uniformly convex Banach space.

Lemma 2.1. *Let K be a nonempty bounded, closed, convex subset of a uniformly convex Banach space X. $T_i : K \to K,\ i = 1,2,3,$ be three C-α nonexpansive mappings for $\alpha \in [0,1)$ with $F \neq \emptyset$. For arbitrary chosen $x_0 \in K$, $\{x_n\}$ be a sequence generated by (1.1), then we have, for common fixed point p of $T_i,\ i = 1,2,3,$ $\lim_{n\to\infty} \|x_n - p\|$ exists.*

Proof. From Lemma 1.2, for any $p \in F, x \in K$ and $T_i : K \to K,\ i = 1,2,3,$ are C-α nonexpansive mappings , then we have for each $i = 1,2,3,$ $\frac{1}{2}\|p - T_ip\| = 0 \leq \|p - x\|$ implies that

$$\begin{aligned} \|T_ix - p\|^2 &= \|T_ix - T_ip\|^2 \\ &\leq \alpha\|T_ix - p\|^2 + \alpha\|x - T_ip\|^2 + (1-2\alpha)\|x - p\|^2 \qquad (2.1) \\ &\leq \alpha\|T_ix - p\|^2 + \alpha\|x - p\|^2 + (1-2\alpha)\|x - p\|^2 \\ &\leq \alpha\|T_ix - p\|^2 + (1-\alpha)\|x - p\|^2. \end{aligned}$$

So, for each $i = 1,2,3$, $\|T_ix - p\|^2 \leq \&\|x - p\|^2$. Thus for each $i = 1,2,3$, T_i C-α nonexpansive mappings are quasi-nonexpansive.
Now, using (1.1) and (2.1), we have,

$$\begin{aligned} \|z_n - p\| &= \|(1-c_n)x_n + c_nT_1x_n - p\| \qquad (2.2) \\ &= \|(1-c_n)(x_n - p) + c_n(T_1x_n - p)\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|T_1x_n - p\| \\ &\leq (1-c_n)\|x_n - p\| + c_n\|x_n - p\| = \|x_n - p\|. \end{aligned}$$

Using (1.1), (2.1) and (2.2), we get

$$\begin{aligned} \|y_n - p\| &= \|(1-b_n)z_n + b_nT_2z_n - p\| \qquad (2.3) \\ &= \|(1-b_n)(z_n - p) + b_n(T_2z_n - p)\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|T_2z_n - p\| \\ &\leq (1-b_n)\|z_n - p\| + b_n\|z_n - p\| = \|z_n - p\| \leq \|x_n - p\|. \end{aligned}$$

By using (1.1), (2.1), (2.2) and (2.3) , we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-a_n)y_n + a_nT_3y_n - p\| \\ &= \|(1-a_n)(y_n - p) + a_n(T_3y_n - p)\| \\ &\leq (1-a_n)\|y_n - p\| + a_n\|T_3y_n - p\| \\ &\leq (1-a_n)\|y_n - p\| + a_n\|y_n - p\| \\ &= \|y_n - p\| \leq \|x_n - p\|. \end{aligned}$$

Thus we have

$\|x_{n+1} - p\| \leq \|x_n - p\|$.

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for each p common fixed point of $T_i,\ i = 1,2,3$. It follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. $T_i : K \to K$, $i = 1,2,3$, be C-α nonexpansive mappings for $\alpha \in [0,1)$, common fixed point p of T_i, $i = 1,2,3$, and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $(0,1)$. Let $\{x_n\}$ be a sequence in K defined by (1.1), and parameters satisfy one of the following conditions:*

(1) If $\limsup_{n\to\infty} a_n < 1$ and $\liminf_{n\to\infty} a_n(1-a_n) > 0$,
(2) If $\limsup_{n\to\infty} b_n < 1$ and $\liminf_{n\to\infty} b_n(1-b_n) > 0$,
(3) If $\limsup_{n\to\infty} c_n < 1$ and $\liminf_{n\to\infty} c_n(1-c_n) > 0$.

Then $F \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_2z_n - z_n\| = 0$, $\lim_{n\to\infty} \|T_3y_n - y_n\| = 0$, $\lim_{n\to\infty} \|z_n - x_n\| = 0$, $\lim_{n\to\infty} \|z_n - y_n\| = 0$, $\lim_{n\to\infty} \|y_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_2x_n - x_n\| = 0$, $\lim_{n\to\infty} \|T_3x_n - x_n\| = 0$.

Proof. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exits for any $p \in F$. Then the sequence $\{x_n\}$ is bounded. $T_i : K \to K$, $i = 1,2,3$, are C-α nonexpansive mappings and $T_i : K \to K$, $i = 1,2,3$, has a common fixed point p. From (2.1) and Lemma 2.1, we see that $M_1 = \sup\{\|x_n\|, \|z_n\|, \|y_n\|, \|T_1x_n\|, \|T_2z_n\|, \|T_3y_n\| : n \in \mathbb{N}\} < \infty$. Also from (1.1), (2.1) and Lemma 1.1 , we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-c_n)x_n + c_nT_1x_n - p)\|^2 \\
&= \|(1-c_n)(x_n - p) + c_n(T_1x_n - p)\|^2 \\
&\leq (1-c_n)\|x_n - p\|^2 + c_n\|T_1x_n - p\|^2 - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big) \\
&\leq (1-c_n)\|x_n - p\|^2 + c_n\|x_n - p\|^2 - c_n(1-c_n)\Big(g(\|x_n - T_1x_n\|)\Big) \\
&= \|x_n - p\|^2 - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big).
\end{aligned}$$

Thus we have

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big). \tag{2.4}$$

Now by (1.1), (2.1), (2.4) and Lemma 1.1, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-b_n)z_n + b_nT_2z_n - p\|^2 \\
&= \|(1-b_n)(z_n - p) + b_n(T_2z_n - p)\|^2 \\
&\leq (1-b_n)\|z_n - p\|^2 + b_n\|z_n - p\|^2 - b_n(1-b_n)\Big(g\Big(\|z_n - T_2z_n\|\Big)\Big) \\
&\leq \|x_n - p\|^2 - b_n(1-b_n)\Big(g\Big(\|z_n - T_2z_n\|\Big)\Big) - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big).
\end{aligned}$$

So we have

$$\begin{aligned}
\|y_n - p\|^2 \leq\ & \|x_n - p\|^2 - b_n(1-b_n)\Big(g\Big(\|T_2z_n - T_2x_n\|\Big)\Big) \\
& - c_n(1-c_n)\Big(g\Big(\|T_1x_n - x_n\|\Big)\Big).
\end{aligned} \tag{2.5}$$

Moreover, by (1.1), (2.1), (2.5) and Lemma 1.1, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-a_n)y_n + a_nT_3y_n - p\|^2 \leq (1-a_n)\|y_n - p\|^2 + a_n\|T_3y_n - p\|^2 - a_n(1-a_n)\Big(g\Big(\|y_n - T_3y_n\|\Big)\Big) \\
&\leq (1-a_n)\|y_n - p\|^2 + a_n\|y_n - p\|^2 - a_n(1-a_n)\Big(g\Big(\|y_n - T_3y_n\|\Big)\Big) \\
&\leq \|y_n - p\|^2 - a_n(1-a_n)\Big(g\Big(\|y_n - T_3y_n\|\Big)\Big) - b_n(1-b_n)\Big(g\Big(\|z_n - T_2z_n\|\Big)\Big) \\
&\quad - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big) \\
&\leq \|x_n - p\|^2 - a_n(1-a_n)\Big(g\Big(\|y_n - T_3y_n\|\Big)\Big) - b_n(1-b_n)\Big(g\Big(\|z_n - T_2z_n\|\Big)\Big) - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq\ & \|x_n - p\|^2 - a_n(1-a_n)\Big(g\Big(\|y_n - T_3y_n\|\Big)\Big) - b_n(1-b_n)\Big(g\Big(\|z_n - T_2z_n\|\Big)\Big) \\
& - c_n(1-c_n)\Big(g\Big(\|x_n - T_1x_n\|\Big)\Big)
\end{aligned}$$

From the last inequality, we have

$$a_n(1-a_n)\Big(g\Big(\|y_n - T_3y_n\|\Big)\Big) \leq \Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big), \tag{2.6}$$

$$b_n(1-b_n)\Big(g\Big(\|z_n - T_2z_n\|\Big)\Big) \leq \Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big), \tag{2.7}$$

and

$$c_n(1-c_n)\Big(g\Big(\|x_n-T_1x_n\|\Big)\Big) \leq \Big(\|x_n-q\|^2-\|x_{n+1}-p\|^2\Big). \tag{2.8}$$

By condition $\limsup_{n\to\infty} a_n < 1$ and $\liminf_{n\to\infty} a_n(1-a_n) > 0$, then we have

$\lim_{n\to\infty} g\Big(\|y_n-T_3y_n\|\Big) = 0.$

From g is continuous strictly increasing with $g(0)=0$ then we have

$$\lim_{n\to\infty}\|y_n-T_3y_n\| = 0. \tag{2.9}$$

By using a similar method for inequalities (2.7) and (2.8) we have

$$\lim_{n\to\infty}\|z_n-T_2z_n\| = 0. \tag{2.10}$$

and

$$\lim_{n\to\infty}\|T_1x_n-x_n\| = 0. \tag{2.11}$$

Next, from (1.1) and (2.11), we have

$$\begin{aligned} \|z_n-x_n\| &\leq \|(1-c_n)x_n+c_nT_1x_n-x_n\| \\ &\leq (c_n)\|T_1x_n-x_n\|\to 0 \;\; as \;\; n\to\infty. \end{aligned} \tag{2.12}$$

Also, from (1.1) and (2.10), we have

$$\begin{aligned} \|y_n-z_n\| &\leq \|(1-b_n)z_n+b_nT_2z_n-z_n\| \\ &\leq (b_n)\|T_2z_n-z_n\|\to 0 \;\; as \;\; n\to\infty. \end{aligned} \tag{2.13}$$

By (2.10) and (2.12) we have

$$\|T_2z_n-x_n\| \leq \|T_2z_n-z_n\|+\|z_n-x_n\|\to 0 \;\; as \;\; n\to\infty. \tag{2.14}$$

Moreover from (2.12) and (2.13)

$$\|y_n-x_n\| \leq \|y_n-z_n\|+\|z_n-x_n\|\to 0 \;\; as \;\; n\to\infty. \tag{2.15}$$

By (2.9) and (2.15) we have

$$\|T_3y_n-x_n\| \leq \|T_3y_n-y_n\|+\|y_n-x_n\|\to 0 \;\; as \;\; n\to\infty. \tag{2.16}$$

Next

$$\begin{aligned} \|T_2x_n-z_n\|^2 &\leq \Big(\|T_2x_n-T_2z_n\|+\|T_2z_n-z_n\|\Big)^2 \\ &= \|T_2x_n-T_2z_n\|^2+\|T_2z_n-z_n\|^2+2\Big(\|T_2x_n-T_2z_n\|\|T_2z_n-z_n\|\Big) \\ &\leq \alpha\|T_2x_n-z_n\|^2+\alpha\|T_2z_n-x_n\|^2+(1-2\alpha)\|x_n-z_n\|^2+4M_1\|T_2z_n-z_n\|+\|T_2z_n-z_n\|^2. \end{aligned}$$

Then from (2.10), (2.12) and (2.14), we obtain

$$\|T_2x_n-z_n\|^2 \leq \frac{\alpha}{1-\alpha}\|T_2z_n-x_n\|^2+\frac{(1-2\alpha)}{(1-\alpha)}\|x_n-z_n\|^2+\frac{4M_1}{(1-\alpha)}\|T_2z_n-z_n\|+\frac{1}{(1-\alpha)}\|T_2z_n-z_n\|^2 \tag{2.17}$$

Thus from (2.12) and (2.17) we obtain

$$\|T_2x_n-x_n\| \leq \|T_2x_n-z_n\|+\|z_n-x_n\|\to 0 \;\; as \;\; n\to\infty. \tag{2.18}$$

Next

$$\begin{aligned} \|T_3x_n-y_n\|^2 &\leq \Big(\|T_3x_n-T_3y_n\|+\|T_3y_n-y_n\|\Big)^2 \\ &= \|T_3x_n-T_2y_n\|^2+\|T_3y_n-y_n\|^2+2\Big(\|T_3x_n-T_3y_n\|\|T_3y_n-y_n\|\Big) \\ &\leq \alpha\|T_3x_n-y_n\|^2+\alpha\|T_3y_n-x_n\|^2+(1-2\alpha)\|x_n-y_n\|^2+4M_1\|T_3y_n-y_n\|+\|T_3y_n-y_n\|^2. \end{aligned}$$

Then from (2.9),(2.15) and (2.16) we obtain

$$\|T_3x_n - y_n\|^2 \leq \frac{\alpha}{1-\alpha}\|T_3y_n - x_n\|^2 + \frac{(1-2\alpha)}{(1-\alpha)}\|x_n - y_n\|^2 + \frac{4M_1}{(1-\alpha)}\|T_3y_n - y_n\| + \frac{1}{(1-\alpha)}\|T_3y_n - y_n\|^2 \to 0 \ \ as \ \ n \to \infty.$$

Thus from (2.15) and (2.19) we obtain

$$\|T_3x_n - x_n\| \leq \|T_3x_n - y_n\| + \|y_n - x_n\| \to 0 \ \ as \ \ n \to \infty. \tag{2.19}$$

Thus from (2.11), (2.18) and (2.19) we obtain

$\lim_{n\to\infty} \|T_3x_n - x_n\| = 0,\ \lim_{n\to\infty} \|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0.$

Conversely, assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|T_3x_n - x_n\| = 0,\ \lim_{n\to\infty} \|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$. For each $i = 1,2,3$, there are bounded subsequences $\{T_ix_{n_k}\}$ of $\{T_ix_n\}$ such that $\lim_{k\to\infty} \|T_ix_{n_k} - x_{n_k}\| = 0$. Suppose $p \in A(K,\{x_{n_k}\})$. Let $M_2 = \sup\{\|x_{n_k}\|, \|T_ix_{n_k}\|, \|T_ip\|, \|p\| : k \in \mathbb{N},\ i = 1,2,3\} < \infty$. For $\alpha \in [0,1)$ and $i = 1$, by Lemma 1.2, we obtain

$$\begin{aligned}
\|x_{n_k} - T_1p\|^2 &\leq \Big(\|x_{n_k} - T_1x_{n_k}\| + \|T_1x_{n_k} - T_1p\|\Big)^2 \\
&= \|x_{n_k} - T_1x_{n_k}\|^2 + \|T_1x_{n_k} - T_1p\|^2 + 2\Big(\|x_{n_k} - T_1x_{n_k}\|\|T_1x_{n_k} - T_1p\|\Big) \\
&\leq \alpha\|T_1x_{n_k} - p\|^2 + \alpha\|x_{n_k} - T_1p\|^2 + (1-2\alpha)\|x_{n_k} - p\|^2 + 2\Big(\|T_1x_{n_k} - T_1p\|\|x_{n_k} - T_1x_{n_k}\|\Big) + \|x_{n_k} - T_1x_{n_k}\|^2.
\end{aligned}$$

$$\begin{aligned}
(1-\alpha)\|x_{n_k} - T_1p\|^2 &\leq (1+\alpha)\|T_1x_{n_k} - p\|^2 + 2\Big(\alpha\|x_{n_k} - p\|\|x_{n_k} - T_1x_{n_k}\|\Big) + \\
&+ 2\Big(\|x_{n_k} - T_1x_{n_k}\|\|x_{n_k} - T_1p\|\Big)\|x_{n_k} - T_1x_{n_k}\| + (1-\alpha)\|x_{n_k} - p\|^2.
\end{aligned}$$

Thus we have

$$\|x_{n_k} - T_1p\|^2 \leq \frac{(1+\alpha)}{(1-\alpha)}\|T_1x_{n_k} - p\|^2 + \frac{2}{(1-\alpha)}\Big(\alpha\|x_{n_k} - p\| + \|x_{n_k} - T_1p\|\Big)\|x_{n_k} - T_1x_{n_k}\| + \|x_{n_k} - p\|^2. \tag{2.20}$$

Therefore

$$\|x_{n_k} - T_1p\|^2 \leq \frac{(1+\alpha)}{(1-\alpha)}\|T_1x_{n_k} - p\|^2 + \frac{4M_2(1+\alpha)}{(1-\alpha)}\|x_{n_k} - T_1x_{n_k}\| + \|x_{n_k} - p\|^2.$$

Take the both side limsup, then we have

$$\limsup_{k\to\infty}\|x_{n_k} - T_1p\|^2 \leq \frac{(1+\alpha)}{(1-\alpha)}\limsup_{k\to\infty}\|T_1x_{n_k} - p\|^2 + \frac{4M_2(1+\alpha)}{(1-\alpha)}\limsup_{k\to\infty}\|x_{n_k} - T_1x_{n_k}\| + \limsup_{k\to\infty}\|x_{n_k} - p\|^2.$$

Thus we have for $T_1 : K \to K,\ i = 1$

$$r(T_1p,\{x_{n_k}\}) = \limsup_{k\to\infty}\|x_{n_k} - T_1p\| \leq \limsup_{k\to\infty}\|x_{n_k} - p\| = r(p,\{x_{n_k}\}).$$

This implies that for $i = 2,3$, we also obtain

$$r(T_2p,\{x_{n_k}\}) = \limsup_{k\to\infty}\|x_{n_k} - T_2p\| \leq \limsup_{k\to\infty}\|x_{n_k} - p\| = r(p,\{x_{n_k}\})$$

and

$$r(T_3p,\{x_{n_k}\}) = \limsup_{k\to\infty}\|x_{n_k} - T_3p\| \leq \limsup_{k\to\infty}\|x_{n_k} - p\| = r(p,\{x_{n_k}\}).$$

These mean that for each $i = 1,2,3$, $T_ip \in A(K,\{x_{n_k}\})$. Since X is uniformly Banach space, $A(K,\{x_n\})$ is singleton, hence for each $i = 1,2,3$, $T_ip = p$. This completes the proof. □

In the next result, we prove the weak convergence of the iterative scheme (1.1) for three C-α nonexpansive mappings with $\alpha \in [0,1)$ in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.3. *Let X be a uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let $T_i : K \to K,\ i = 1,2,3,$ be three C-α nonexpansive mappings for $\alpha \in [0,1)$. Assume that $p \in F$ is a common fixed point of $T_i,\ i = 1,2,3$. Let $\{x_n\}$ be a sequence in K defined by (1.1) where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,1)$ and satisfy the conditions of Theorem 2.1. Then $\{x_n\}$ converges weakly to a common fixed point of p $T_i, i = 1,2,3$.*

Proof. Since $F \neq \emptyset$, it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Now, we show that $\{x_n\}$ has a unique weak subsequential limit in F. We assume that ω_1 and ω_2 are weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. From Theorem 2.1, we have $\lim_{n\to\infty} \|T_3x_n - x_n\| = 0,\ \lim_{n\to\infty} \|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$. Moreover by Proposition 1.1, $I - T_i$ for $i = 1,2,3$ are demiclosed at zero. This implies that $(I - T_i)\omega_1 = 0,\ i = 1,2,3$, that is $T_i\omega_1 = \omega_1,\ i = 1,2,3$. Similarly $T_i\omega_2 = \omega_2,\ i = 1,2,3$. Now, we show the uniqueness. If $\omega_1 \neq \omega_2$, then by the Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \omega_1\| &= \lim_{j\to\infty} \|x_{n_j} - \omega_1\| < \lim_{j\to\infty} \|x_{n_j} - \omega_2\| = \lim_{n\to\infty} \|x_n - \omega_2\| \\
&= \lim_{k\to\infty} \|x_{n_k} - \omega_2\| < \lim_{k\to\infty} \|x_{n_k} - \omega_1\| = \lim_{n\to\infty} \|x_n - \omega_1\|
\end{aligned}$$

This is a contradiction. So, $\omega_1 = \omega_2$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $T_i, i = 1,2,3$. This completes the proof. □

Finally, we prove our strong convergence theorem as follows.

Theorem 2.4. *Let X be a real uniformly convex Banach space, K be a nonempty compact convex subset of X and for $\alpha \in [0,1)$, $T_i : K \to K,\ i = 1,2,3$, be three C-α nonexpansive mappings with $F \neq \emptyset$. Let $\{x_n\}$ be a sequence in K defined by (1.1) where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0,1)$ for all $n \in \mathbb{N}$, and satisfy the conditions of Theorem 2.1. Then $\{x_n\}$ converges strongly to a common fixed point of $T_i,\ i = 1,2,3$.*

Proof. By Theorem 2.1, we have $\lim_{n\to\infty} \|T_3x_n - x_n\| = 0,\ \lim_{n\to\infty} \|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p$ as $k \to \infty$. Let $M_3 = \sup\{\|x_{n_k}\|, \|T_ix_{n_k}\|, \|T_ip\|, \|p\| : k \in \mathbb{N}, i = 1,2,3\} < \infty$. Then from (2.20), choose $i = 1$, by Lemma 1.2, we obtain for $\alpha \in [0,1)$

$$\|x_{n_k} - T_1p\|^2 \le \frac{(1+\alpha)}{(1-\alpha)} \|T_1x_{n_k} - p\|^2 + \frac{4M_3(1+\alpha)}{(1-\alpha)} \|x_{n_k} - T_1x_{n_k}\| + \|x_{n_k} - p\|^2$$

Letting $k \to \infty$, we get $T_1p = p$. By using a similar method, $p = T_2p$ and then we have $p = T_3p$. Thus we have $\{x_{n_k}\}$ converges to common fixed point p of $T_i, i = 1,2,3$. Since by Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F$, so $\{x_n\}$ converges strongly to a common fixed point of $T_i, i = 1,2,3$. □

3. Examples

Now we give the examples of $T_i : K \to K,\ i = 1,2,3$, be three C-α nonexpansive mappings with $\alpha \in [0,1)$ which are not generalized $\alpha-$nonexpansive mappings.

Example 3.1. *Let $K = [0,5] \subset \mathbb{R}$ endowed with usual norm in $\mathbb{R}$. Define a mapping $T_1 : K \to K$ by*

$$T_1x = \begin{cases} \frac{x}{4}, & x \neq 5 \\ \frac{13}{4}, & x = 5 \end{cases}$$

To verify that for $\alpha = \frac{3}{4}$, T_1 is a C-$\frac{3}{4}$ nonexpansive mapping, we consider the following cases:
Case I:*If $x, y \neq 5$, then*

$$\begin{aligned}
\alpha|T_1x - y|^2 + \alpha|T_1y - x|^2 + (1-2\alpha)|x-y|^2 &= \frac{3}{4}|T_1x - y|^2 + \frac{3}{4}|T_1y - x|^2 - \frac{1}{2}|x-y|^2 \\
&= \frac{3}{4}(\frac{1}{4}x - y)^2 + \frac{3}{4}(\frac{1}{4}y - x)^2 - \frac{1}{2}(x-y)^2 \\
&= \frac{3}{4}(\frac{1}{16}x^2 - \frac{1}{2}xy + y^2) + \frac{3}{4}(\frac{1}{16}y^2 - \frac{1}{2}xy + x^2) - \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 \\
&= \frac{3}{64}x^2 - \frac{3}{8}xy + \frac{3}{4}y^2 + \frac{3}{64}y^2 - \frac{3}{8}xy + \frac{3}{4}x^2 - \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 \\
&= \left(\frac{1}{4}x - \frac{1}{4}y\right)^2 + \frac{15}{64}x^2 + \frac{15}{64}y^2 + \frac{3}{8}xy \ge \left|\frac{1}{4}x - \frac{1}{4}y\right|^2 = |T_1x - T_1y|^2.
\end{aligned}$$

Case II:*If $x = 5, y \neq 5$, then*

$$\begin{aligned}
\alpha|T_1x - y|^2 + \alpha|T_1y - x|^2 + (1-2\alpha)|x-y|^2 &= \frac{3}{4}|T_1x - y|^2 + \frac{3}{4}|T_1y - x|^2 - \frac{1}{2}|x-y|^2 \\
&= \frac{3}{4}(\frac{13}{4} - y)^2 + \frac{3}{4}(\frac{1}{4}y - 5)^2 - \frac{1}{2}(5-y)^2 \\
&= \frac{3}{4}(\frac{169}{16} - \frac{13}{2}y + y^2) + \frac{3}{4}(\frac{1}{16}y^2 - \frac{5}{2}y + 25) - \frac{25}{2} + 5y - \frac{1}{2}y^2 \\
&= (\frac{13}{4} - \frac{1}{4}y)^2 + \frac{15}{64}y^2 - \frac{1}{8}y + \frac{231}{64} \ge \left|\frac{13}{4} - \frac{1}{4}y\right|^2 = |T_1x - T_1y|^2.
\end{aligned}$$

Since for $y \in [0,5)$, $\frac{15}{64}y^2 - \frac{1}{8}y + \frac{231}{64} \ge 0$, then T_1 is a C-$\frac{3}{4}$ nonexpansive mapping.
Contrarily at $x = 3, y = 5$; we get

$$\frac{1}{2}|x - T_1x| = \frac{1}{2}\left|3 - \frac{3}{4}\right| = \frac{9}{8} \le 2 = |x-y|.$$

Then, we have

$$\alpha|T_1x-y|+\alpha|T_1y-x|+(1-2\alpha)|x-y| = \alpha\left|\frac{3}{4}-5\right|+\alpha\left|\frac{13}{4}-3\right|+(1-2\alpha)|3-5| = 2+\frac{1}{2}\alpha < \left|\frac{3}{4}-\frac{13}{4}\right| = \frac{10}{4} = 2+\frac{1}{2} = |T_1x-T_1y|.$$

Hence T_1 is not a generalized $\frac{3}{4}-$nonexpansive mapping.

Example 3.2. *Let $K=[0,5]\subset\mathbb{R}$ endowed with usual norm in $\mathbb{R}$. Define a mapping $T_2: K\to K$ by*

$$T_2x=\begin{cases}\frac{x}{3}, & x\neq 5\\ \frac{11}{3}, & x=5\end{cases}$$

To verify that for $\alpha=\frac{3}{4}$, T_2 is a C-$\frac{3}{4}$ nonexpansive mapping, we consider the following cases:
Case I:*If $x,y\neq 5$, then*

$$\begin{aligned}\alpha|T_2x-y|^2+\alpha|T_2y-x|^2+(1-2\alpha)|x-y|^2 &= \frac{3}{4}|T_2x-y|^2+\frac{3}{4}|T_2y-x|^2-\frac{1}{2}|x-y|^2\\ &= \frac{3}{4}(\frac{1}{3}x-y)^2+\frac{3}{4}(\frac{1}{3}y-x)^2-\frac{1}{2}(x-y)^2\\ &= \frac{3}{4}(\frac{1}{9}x^2-\frac{2}{3}xy+y^2)+\frac{3}{4}(\frac{1}{9}y^2-\frac{2}{3}xy+x^2)-\frac{1}{2}x^2+xy-\frac{1}{2}y^2\\ &= \frac{3}{36}x^2-\frac{1}{2}xy+\frac{3y^2}{4}+\frac{3y^2}{36}-\frac{1}{2}xy+\frac{3x^2}{4}-\frac{1}{2}x^2+xy-\frac{1}{2}y^2\\ &= x^2(\frac{1}{12}+\frac{3}{4}-\frac{1}{2})+y^2(\frac{1}{12}+\frac{3}{4}-\frac{1}{2})=\frac{1}{3}x^2+\frac{1}{3}y^2\\ &= (\frac{1}{3}x-\frac{1}{3}y)^2+\frac{2}{9}(x^2+y^2+xy)\geq\left|\frac{1}{3}x-\frac{1}{3}y\right|^2=|T_2x-T_2y|^2.\end{aligned}$$

Case II:*If $x=5, y\neq 5$, then*

$$\begin{aligned}\alpha|T_2x-y|^2+\alpha|T_2y-x|^2+(1-2\alpha)|x-y|^2 &= \frac{3}{4}|T_2x-y|^2+\frac{3}{4}|T_2y-x|^2-\frac{1}{2}|x-y|^2\\ &= \frac{3}{4}(\frac{11}{3}-y)^2+\frac{3}{4}(\frac{1}{3}y-5)^2-\frac{1}{2}(5-y)^2\\ &= \frac{3}{4}(\frac{121}{9}-\frac{22}{3}y+y^2)+\frac{3}{4}(\frac{1}{9}y^2-\frac{10}{3}y+25)-\frac{25}{2}+5y-\frac{1}{2}y^2\\ &= (\frac{11}{3}-\frac{1}{3}y)^2+\frac{2}{9}y^2-\frac{5}{9}y+\frac{26}{9}\geq\left|\frac{11}{3}-\frac{1}{3}y\right|^2=|T_2x-T_2y|^2.\end{aligned}$$

Since $\frac{2}{9}y^2-\frac{5}{9}y+\frac{26}{9}\geq 0$, T_2 is a C-$\frac{3}{4}$ nonexpansive mapping.
Contrarily at $x=3, y=5$; we get

$$\frac{1}{2}|x-T_2x|=\frac{1}{2}\left|3-\frac{3}{3}\right|=1\leq 2=|x-y|$$

Then, we have

$$\alpha|T_2x-y|+\alpha|T_2y-x|+(1-2\alpha)|x-y| = \alpha\left|\frac{3}{3}-5\right|+\alpha\left|\frac{11}{3}-3\right|+(1-2\alpha)|3-5| = 2+\frac{2}{3}\alpha < \left|\frac{3}{3}-\frac{11}{3}\right| = \frac{8}{3} = 2+\frac{2}{3} = |T_2x-T_2y|.$$

Hence T_2 is not a generalized $\frac{3}{4}-$nonexpansive mapping.

Example 3.3. *Let $K=[0,5]\subset\mathbb{R}$ endowed with usual norm in $\mathbb{R}$. Define a mapping $T_3: K\to K$ by*

$$T_3x=\begin{cases}\frac{x}{2}, & x\neq 5\\ \frac{7}{2}, & x=5\end{cases}$$

To verify that for $\alpha=\frac{3}{4}$, T_3 is a C-$\frac{3}{4}$ nonexpansive mapping, we consider the following cases:
Case I:*If $x,y\neq 5$, then*

$$\begin{aligned}\alpha|T_3x-y|^2+\alpha|T_3y-x|^2+(1-2\alpha)|x-y|^2 &= \frac{3}{4}|T_3x-y|^2+\frac{3}{4}|T_3y-x|^2-\frac{1}{2}|x-y|^2\\ &= \frac{3}{4}(\frac{1}{2}x-y)^2+\frac{3}{4}(\frac{1}{2}y-x)^2-\frac{1}{2}(x-y)^2\\ &= \frac{3}{4}(\frac{1}{4}x^2-xy+y^2)+\frac{3}{4}(\frac{1}{4}y^2-xy+x^2)-\frac{1}{2}x^2+xy-\frac{1}{2}y^2\\ &= \frac{3}{16}x^2-\frac{3}{4}xy+\frac{3}{4}y^2+\frac{3}{16}y^2-\frac{3}{4}xy+\frac{3}{4}x^2-\frac{1}{2}x^2+xy-\frac{1}{2}y^2\end{aligned}$$

$$= x^2(\frac{3}{16}+\frac{3}{4}-\frac{1}{2})+y^2(\frac{3}{16}+\frac{3}{4}-\frac{1}{2})-\frac{1}{2}xy = \frac{7}{16}x^2+\frac{7}{16}y^2-\frac{1}{2}xy$$
$$= (\frac{1}{2}x-\frac{1}{2}y)^2+\frac{3}{16}x^2+\frac{3}{16}y^2 \geq \left|\frac{1}{2}x-\frac{1}{2}y\right|^2 = |T_3x-T_3y|^2$$

Case II: *If $x=5, y\neq 5$, then*

$$\begin{aligned}
\alpha|T_3x-y|^2+\alpha|T_3y-x|^2+(1-2\alpha)|x-y|^2 &= \frac{3}{4}|T_3x-y|^2+\frac{3}{4}|T_3y-x|^2-\frac{1}{2}|x-y|^2\\
&= \frac{3}{4}(\frac{7}{2}-y)^2+\frac{3}{4}(\frac{1}{2}y-5)^2-\frac{1}{2}(5-y)^2\\
&= \frac{3}{4}(\frac{49}{4}-7y+y^2)+\frac{3}{4}(\frac{y^2}{4}-5y+25)-\frac{25}{2}+5y-\frac{1}{2}y^2\\
&= y^2(\frac{3}{4}+\frac{3}{16}-\frac{1}{2})+y(5-\frac{21}{4}-\frac{15}{4})+\frac{147}{16}+\frac{75}{4}-\frac{25}{2}\\
&= \frac{7}{16}y^2-4y+\frac{247}{16}=(\frac{7}{2}-\frac{1}{2}y)^2+\frac{3}{16}y^2-\frac{1}{2}y+\frac{51}{16} \geq \left|\frac{7}{2}-\frac{1}{2}y\right|^2 = |T_3x-T_3y|^2
\end{aligned}$$

Since $\frac{3}{16}y^2-\frac{1}{2}y+\frac{51}{16}\geq 0$, then T_3 is a C-$\frac{3}{4}$ nonexpansive mapping.
Contrarily at $x=5, y=3.4$; we get

$$\frac{1}{2}|x-T_3x| = \frac{1}{2}\left|5-\frac{7}{2}\right| = \frac{3}{4} = 0.75 \leq 1.6 = |x-y|$$

Then, we have

$$\begin{aligned}
\alpha|T_3x-y|+\alpha|T_3y-x|+(1-2\alpha)|x-y| &= \alpha\left|\frac{7}{2}-3.4\right|+\alpha\left|\frac{3.4}{2}-5\right|+(1-2\alpha)|5-3.4| = 1.6+(0.2)\alpha\\
&< \left|\frac{7}{2}-\frac{3.4}{2}\right| = 1.8 = |T_3x-T_3y|.
\end{aligned}$$

Hence T_3 is not a generalized $\frac{3}{4}$−nonexpansive mapping.

Let $a_n=b_n=c_n=0.75$ for all $n\in\mathbb{N}$. We compute that the sequence $\{x_n\}$ generated by iterative schemes (1.1)-(1.4) converge to a fixed point 0 of $T_i,\ i=1,2,3$, which is shown by the Table 1. Also we compute that the sequences $\{x_n\}$ generated by iterative schemes (1.1)-(1.4) converge to a common fixed point 0 of $T_i,\ i=1,2,3$, which is shown by Figure 1.

Table 1: Sequences generated by (1.1)-iteration, (1.2)-iteration, (1.3)-iteration and (1.4)-iteration for $T_i,\ i=1,2,3$, mappings defined in Example 3.1, Example 3.2 and Example 3.3.

	(1.1)-iteration	(1.2)-iteration	(1.3)-iteration	(1.4)-iteration
x_1	5.0000000000	5.0000000000	5.0000000000	5.0000000000
x_2	1.1523437500	0.7058105469	1.0000000000	1.5136718750
x_3	0.1575469971	0.0591047406	0.1250000000	0.3695487976
x_4	0.0215396285	0.0049494448	0.0156250000	0.0902218744
x_5	0.0029448711	0.0004144677	0.0019531250	0.0220268248
x_6	0.0004026191	0.0000347076	0.0002441406	0.0053776428
x_7	0.0000550456	0.0000029064	0.0000305176	0.0013129011
x_8	0.0000075258	0.0000002434	0.0000038147	0.0003205325
x_9	0.0000010289	0.0000000204	0.0000004768	0.0000782550
x_{10}	0.0000001407	0.0000000017	0.0000000596	0.0000191052
x_{11}	0.0000000192	0.0000000001	0.0000000075	0.0000046644
x_{12}	0.0000000026	0.0000000000	0.0000000009	0.0000011388
x_{13}	0.0000000004	0.0000000000	0.0000000001	0.0000002780
x_{14}	0.0000000000	0.0000000000	0.0000000000	0.0000000679
x_{15}	0.0000000000	0.0000000000	0.0000000000	0.0000000166
x_{16}	0.0000000000	0.0000000000	0.0000000000	0.0000000040
x_{17}	0.0000000000	0.0000000000	0.0000000000	0.0000000010
x_{18}	0.0000000000	0.0000000000	0.0000000000	0.0000000002
x_{19}	0.0000000000	0.0000000000	0.0000000000	0.0000000000

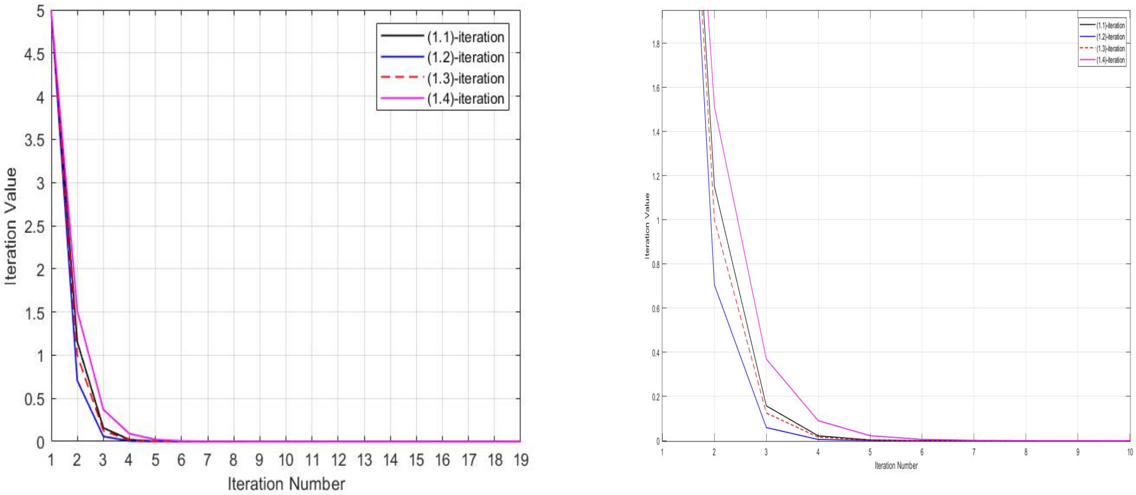

Figure 3.1: Convergences of (1.1)-iteration, (1.2)-iteration, (1.3)-iteration and (1.4)-iteration to the common fixed point 0 of $T_i,\ i = 1,2,3$, mappings defined in Example 3.1, Example 3.2, Example 3.3.

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