JOURNAL OF UNIVERSAL MATHEMATICS
VOL. 6 NO. 1 pp. 114-130 (2023)
ISSN-2618-5660
DOI: 10.33773/jum.1232944

(α, β)-INTERVAL VALUED INTUITIONISTIC FUZZY SETS DEFINED ON (α, β)-INTERVAL VALUED SET

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ABSTRACT. In this paper, (α, β)-interval valued set is studied. The order relation on (α, β)-interval valued set is defined. It is shown that (α, β)-interval valued set is complete lattice by giving the definitions of infumum and supremum on these sets. Then, negation function on these sets is introduced. With the help of (α, β)-interval valued set ,(α, β)-interval valued intuitionistic fuzzy sets are defined. The fundamental algebraic properties of these sets are examined. The level subsets of (α, β)-interval valued intuitionistic fuzzy sets are given. Some propositions and examples are studied.

1. INTRODUCTION

Fuzzy set theory was introduced by Zadeh in 1965 [15]. The concept of interval valued fuzzy set was introduced by Zadeh [16-18]. The basic properties of interval valued fuzzy sets were studied by many authors [7-10,13,14,16-18]. It is crucial to analyze the properties of interval fuzzy sets on different structures in these sense, the topological properties of interval valued fuzzy sets were studied by Mondal and Samantha [11].

Interval valued intuitionistic fuzzy sets which is the generalization of intuitionistic fuzzy sets and interval valued fuzzy sets were introduced by Atanassov and Gargov in 1989 [2]. Membership and non-membership functions on interval valued intuitionistic fuzzy sets are closed intervals whose the sum of supremums is equal to 1 or less than 1 of unit interval $\mathrm{I} = [0,1]$ [2]. Other properties of these sets were studied and the concept of intuitionistic fuzzy sets was introduced by Atanassov [1-5]. The topological properties of interval valued intuitionistic fuzzy sets were studied by Mondal and Samantha [12]. α-interval valued fuzzy sets were introduced by Çuvalcıoğlu, Bal and Çitil in 2022 [6].

2. PRELIMINARIES

In this paper, D(I) represents all closed intervals of unit interval $\mathrm{I} = [0,1]$. The elements of D(I) set are shown with capital letters such as M,N... In this place, M^{L} and M^{U} are called respectively lower end point and upper end point for interval $\mathrm{M} = [\mathrm{M}^{\mathrm{L}}, \mathrm{M}^{\mathrm{U}}]$.

Date: Received: 2023-01-12; Accepted: 2023-01-30.

2000 *Mathematics Subject Classification.* 18G50, 18G55, 16B50.

Key words and phrases. Interval Valued Intuitionistic Fuzzy Sets, (α,β)-Interval Valued Set, (α,β)-Interval Valued Intuitionistic Fuzzy Sets.

Definition 1: [6] $D(I_\alpha) = \{[M^L, M^U; \alpha] | \alpha \in I\}$ is called α-interval valued set. In order to make easy, it is shown that

$$\{[M^L, M^U; \alpha] | \alpha \in I\} = \{[M; \alpha] | M \in D(I) \text{ and } \alpha \in M\}$$

Order relation on $D(I_\alpha)$ is defined below.

Definition 2: [6] $\forall [M; \alpha], [N; \alpha] \in D(I_\alpha)$,

$$[M; \alpha] \leq [N; \alpha] :\Leftrightarrow M^L \leq N^L \text{ and } M^U \geq N^U$$

It is easily seen from definition,

$$[M; \alpha] < [N; \alpha]$$
$$\Leftrightarrow M^L < N^L, M^U \geq N^U \text{ or } M^L \leq N^L, M^U > N^U \text{ or } M^L < N^L, M^U > N^U$$

Proposition 1: [6] $(D(I_\alpha), \leq)$ is partial ordered set.

By the help of order relation on $D(I_\alpha)$, the definitions of supremum and infimum on this set are given below.

Definition 3: [6] $\forall [M; \alpha], [N; \alpha] \in D(I_\alpha)$,

i. $\inf\{[M; \alpha], [N; \alpha]\} = [\inf\{M^L, N^L\}, \sup\{M^U, N^U\}; \alpha]$

ii. $\sup\{[M; \alpha], [N; \alpha]\} = [\sup\{M^L, N^L\}, \inf\{M^U, N^U\}; \alpha]$

Lemma 1: [6] $(D(I_\alpha), \wedge, \vee)$ is complete lattice with units $[0,1; \alpha]$ and $[\alpha, \alpha; \alpha]$.

Proposition 2: [6] $\forall \alpha \in I$,

$$\bigcup_{\alpha \in I} D(I_\alpha) = D(I)$$

The following function is a negation function on $D(I_\alpha)$.

Proposition 3: [6] $\forall [M; \alpha] \in D(I_\alpha)$ and $\mathcal{N}: D(I_\alpha) \to D(I_\alpha)$,

$$\mathcal{N}([M; \alpha]) = [\alpha - M^L, 1 + \alpha - M^U; \alpha]$$

Definition 4: [6] Let X be universal set and $[A; \alpha]: X \to D(I_\alpha)$ be function.

$$[A; \alpha] = \{[\langle x, [A^L(x), A^U(x)] \rangle; \alpha] | x \in X\}$$

where; $A^L: X \to [0,1]$ and $A^U: X \to [0,1]$ are fuzzy sets.

In order to make easy, it is shown that;

$$\{[\langle x, [A^L(x), A^U(x)] \rangle; \alpha] | x \in X\} = \{[\langle x, A(x) \rangle; \alpha] | x \in X\}$$

$[A; \alpha]$ is called α −interval valued fuzzy set on X. The family of α −interval valued fuzzy sets on X is shown by α −IVFS(X).

Complement, inclusion, equation, intersection and union of α −interval valued fuzzy sets are given below.

Definition 5: [6] Let X be universal set and $[A; \alpha], [B; \alpha] \in \alpha$ −IVFS(X).

Λ is index set $\forall \lambda \in \Lambda$,

i. $[A^c; \alpha] = \{[< x, [\alpha - A^L(x), 1 + \alpha - A^U(x)] >; \alpha] | x \in X\}$

ii. $[A;\alpha] \sqsubseteq [B;\alpha] :\Leftrightarrow \forall x \in X, A^L(x) \leq B^L(x)$ and $A^U(x) \geq B^U(x)$

iii. $[A;\alpha] = [B;\alpha] :\Leftrightarrow \forall x \in X, A^L(x) = B^L(x)$ and $A^U(x) = B^U(x)$

iv. $[A \sqcap B;\alpha] = \{[< x, [\inf\{A^L(x), B^L(x)\}, \sup\{A^U(x), B^U(x)\}] >;\alpha] \mid x \in X\}$

v. $[A \sqcup B;\alpha] = \{[< x, [\sup\{A^L(x), B^L(x)\}, \inf\{A^U(x), B^U(x)\}] >;\alpha] \mid x \in X\}$

vi. $[\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha] = \{[< x, [\wedge_{\lambda\in\Lambda} {A_\lambda}^L(x), \vee_{\lambda\in\Lambda} {A_\lambda}^U(x)] >;\alpha] \mid x \in X\}$

vii. $[\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha] = \{[< x, [\vee_{\lambda\in\Lambda} {A_\lambda}^L(x), \wedge_{\lambda\in\Lambda} {A_\lambda}^U(x)] >;\alpha] \mid x \in X\}$

The algebraic properties of $\alpha-$interval valued fuzzy sets are expressed below.

Proposition 4: [6] Let X be universal set. $\forall [A;\alpha], [B;\alpha], [C;\alpha] \in \alpha-$IVFS(X) and Λ is index set $\forall \lambda \in \Lambda$,

i. $[A \sqcap B;\alpha] = [B \sqcap A;\alpha]$

ii. $[A \sqcup B;\alpha] = [B \sqcup A;\alpha]$

iii. $[A;\alpha] \sqcap ([B \sqcup C;\alpha]) = ([A \sqcap B;\alpha]) \sqcup ([A \sqcap C;\alpha])$

iv. $[A;\alpha] \sqcup ([B \sqcap C;\alpha]) = ([A \sqcup B;\alpha]) \sqcap ([A \sqcup C;\alpha])$

v. $[A;\alpha] \sqcap ([\sqcup_{\lambda\in\Lambda} B_\lambda;\alpha]) = [\sqcup_{\lambda\in\Lambda} (A \sqcap B_\lambda);\alpha]$

vi. $[A;\alpha] \sqcup ([\sqcap_{\lambda\in\Lambda} B_\lambda;\alpha]) = [\sqcap_{\lambda\in\Lambda} (A \sqcup B_\lambda);\alpha]$

Features about complement of $\alpha-$interval valued fuzzy sets are stated following proposition.

Proposition 5: [6] Let X be universal set. $\forall [A;\alpha], [B;\alpha] \in \alpha-$IVFS(X) and Λ is index set $\forall \lambda \in \Lambda$,

i. $[([A^c;\alpha])^c;\alpha] = [A;\alpha]$

ii. $([A \sqcap B;\alpha])^c = [A^c \sqcup B^c;\alpha]$

iii. $([A \sqcup B;\alpha])^c = [A^c \sqcap B^c;\alpha]$

iv. $([\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha])^c = [\sqcup_{\lambda\in\Lambda} A_\lambda^c;\alpha]$

v. $([\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha])^c = [\sqcap_{\lambda\in\Lambda} A_\lambda^c;\alpha]$

Proposition 6: [6] Let X be universal set. $\mathbf{0_X}: X \to [0,1;\alpha]$ and $\mathbf{1_X}: X \to [\alpha,\alpha;\alpha]$.

i. $(\mathbf{0_X})^c = \mathbf{1_X}$

ii. $(\mathbf{1_X})^c = \mathbf{0_X}$

Definition 6: [6] Let X be universal set and $[A;\alpha] \in \alpha-$IVFS(X).

$$[A;\alpha] \text{ has sup} - \text{property}$$
$$:\Leftrightarrow \forall x \in X, \exists [\lambda_1,\lambda_2;\alpha] \in D(I_\alpha) \ni [A(x);\alpha] = [\lambda_1,\lambda_2;\alpha]$$

Definition 7: [6] Let X be universal set and $[A;\alpha] \in \alpha-$IVFS(X).
$\forall [\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$,

$$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = \{x \in X \mid A^L(x) \geq \lambda_1 \text{and } A^U(x) \leq \lambda_2\}$$

The set $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ is called $[\lambda_1,\lambda_2;\alpha]$-level subset of $[A;\alpha]$. It is easily seen from definition, $[\lambda_1,\lambda_2;\alpha]$-level subsets of $[A;\alpha]$ are crisp sets.

Definition 8: [6] Let X be universal set and $[A;\alpha] \in \alpha-\mathrm{IVFS}(X)$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$,
$\forall[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$-level subsets of $[A;\alpha]$,

i. $A^L_{\lambda_1} = \{x \in X \mid A^L(x) \geq \lambda_1\}$

ii. $A^U_{\lambda_2} = \{x \in X \mid A^U(x) \leq \lambda_2\}$

iii. $B^L_{\lambda_1} = \{x \in X \mid B^L(x) \geq \lambda_1\}$

iv. $B^U_{\lambda_2} = \{x \in X \mid B^U(x) \leq \lambda_2\}$

The relations between level subsets of $\alpha-$interval valued fuzzy sets and crisp sets are given below.

Proposition 7: [6] Let X be universal set and $[A;\alpha],[B;\alpha] \in \alpha-\mathrm{IVFS}(X)$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$ and I is index set, $\forall i,j \in I$, $[\lambda_i,\lambda_j;\alpha] \in D(I_\alpha)$,

i. $x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \Leftrightarrow [A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha]$

ii. $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = A^L_{\lambda_1} \cap A^U_{\lambda_2}$

iii. $([A \sqcup B;\alpha])_{[\lambda_1,\lambda_2;\alpha]}$
$$= [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left(A^L_{\lambda_1} \cap B^U_{\lambda_2}\right) \cup \left(B^L_{\lambda_1} \cap A^U_{\lambda_2}\right)$$

iv. $([A \sqcap B;\alpha])_{[\lambda_1,\lambda_2;\alpha]} = [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$

v. $A^L_{\lambda_1} \supseteq A^L_{\lambda_2}$

vi. $A^U_{\lambda_1} \subseteq A^U_{\lambda_2}$

vii. $\bigcap_{i\in I} {A^L}_{\lambda_i} = {A^L}_{\wedge_{i\in I}\lambda_i}$

viii. $\bigcup_{j\in I} {A^U}_{\lambda_j} = {A^U}_{\vee_{j\in I}\lambda_j}$

Definition 9: [2] Let X be universal set.
M_A and $N_A: X \to D(I)$ such that $\forall x \in X, {M_A}^U(x) + {N_A}^U(x) \leq 1$,
$$A = \{< x, M_A(x), N_A(x) > | x \in X\}$$
is called interval valued intuitionistic fuzzy set. The family of interval valued intuitionistic fuzzy sets on X is shown by IVIFS(X).

Example 1: Let $X = \{a,b,c,d\}$.
$$A = \begin{Bmatrix} < a,[0.0,0.5],[0.2,0.4] >, < b,[0.1,0.3],[0.4,0.5] >, \\ < c,[0.2,0.7],[0.0,0.1] >, < d,[0.6,0.8],[0.1,0.2] > \end{Bmatrix}$$
is interval valued intuitionistic fuzzy set.

Definition 10: [2] Let X be universal set and A, B ∈ IVIFS(X).

i. $A \sqsubseteq_{\square,\inf} B :\Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x)$

ii. $A \sqsubseteq_{\square,\sup} B :\Leftrightarrow \forall x \in X, {M_A}^U(x) \leq {M_B}^U(x)$

iii. $A \sqsubseteq_{\lozenge,\inf} B :\Leftrightarrow \forall x \in X, {N_A}^L(x) \geq {N_B}^L(x)$

iv. $A \sqsubseteq_{\lozenge,\sup} B :\Leftrightarrow \forall x \in X, {N_A}^U(x) \geq {N_B}^U(x)$

v. $A \sqsubseteq_{\square} B :\Leftrightarrow A \sqsubseteq_{\square,\inf} B$ and $A \sqsubseteq_{\square,\sup} B$

vi. $A \sqsubseteq_{\Diamond} B :\Leftrightarrow A \sqsubseteq_{\Diamond,\inf} B$ and $A \sqsubseteq_{\Diamond,\sup} B$

vii. $A \sqsubseteq B :\Leftrightarrow A \sqsubseteq_{\Box} B$ and $A \sqsubseteq_{\Diamond} B$

viii. $A = B :\Leftrightarrow A \sqsubseteq B$ and $B \sqsubseteq A$

It is easily seen that from definition,

i. $A \sqsubseteq_{\Box} B \Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x)$ and ${M_A}^U(x) \leq {M_B}^U(x)$

ii. $A \sqsubseteq_{\Diamond} B \Leftrightarrow \forall x \in X, {N_A}^L(x) \geq {N_B}^L(x)$ and ${N_A}^U(x) \geq {N_B}^U(x)$

iii. $A \sqsubseteq B \Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x), {M_A}^U(x) \leq {M_B}^U(x)$

and ${N_A}^L(x) \geq {N_B}^L(x), {N_A}^U(x) \geq {N_B}^U(x)$

iv. $A = B \Leftrightarrow \forall x \in X, {M_A}^L(x) = {M_B}^L(x)$ and ${M_A}^U(x) = {M_B}^U(x)$

and ${N_A}^L(x) = {N_B}^L(x)$ and ${N_A}^U(x) = {N_B}^U(x)$

Definition 11: [5] Let X be universal set and A, B ∈ IVIFS(X).

i. $A^c = \{< x, N_A(x), M_A(x) > | x \in X\}$

ii. $$A \sqcap B = \begin{Bmatrix} < x, \left[\min\{{M_A}^L(x), {M_B}^L(x)\}, \min\{{M_A}^U(x), {M_B}^U(x)\}\right], \\ \left[\max\{{N_A}^L(x), {N_B}^L(x)\}, \max\{{N_A}^U(x), {N_B}^U(x)\}\right] > | x \in X \end{Bmatrix}$$

iii. $$A \sqcup B = \begin{Bmatrix} < x, \left[\max\{{M_A}^L(x), {M_B}^L(x)\}, \max\{{M_A}^U(x), {M_B}^U(x)\}\right], \\ \left[\min\{{N_A}^L(x), {N_B}^L(x)\}, \min\{{N_A}^U(x), {N_B}^U(x)\}\right] > | x \in X \end{Bmatrix}$$

Theorem 1: [5] Let X be universal set and A, B, C ∈ IVIFS(X).

i. $A \sqcap B = B \sqcap A$

ii. $A \sqcup B = B \sqcup A$

iii. $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C)$

iv. $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C)$

v. $(A \sqcap B) \sqcup C = (A \sqcup C) \sqcap (B \sqcup C)$

vi. $(A \sqcup B) \sqcap C = (A \sqcap C) \sqcup (B \sqcap C)$

Theorem 2: [5] Let X be universal set and A, B ∈ IVIFS(X).

i. $(A^c)^c = A$

ii. $(A^c \sqcap B^c)^c = A \sqcup B$

iii. $(A^c \sqcup B^c)^c = A \sqcap B$

Definition 12: [5] There are some special sets on vague set theories. These special sets on the theory of crisp set are null set and universal set. The special sets on interval valued intuitionistic fuzzy sets are given below.

i. $O^* = \{< x, [0,0], [1,1] > | x \in X\}$

ii. $U^* = \{< x, [0,0], [0,0] > | x \in X\}$

iii. $X^* = \{< x, [1,1], [0,0] > | x \in X\}$

It is easily seen that;

$$O^* \sqsubseteq U^* \sqsubseteq X^*$$

$\forall A \in IVIFS(X)$,

i. $A \sqcap O^* = O^*$

ii. $A \sqcup O^* = A$

3. (α, β)-INTERVAL VALUED INTUITIONISTIC FUZZY SETS

$D(I_\alpha)$ is all closed sub-intervals of $I = [0,1]$ including $\alpha \in [0,1]$.

Definition 13:

$$D(I_\alpha) \times D(I_\beta) = \{([M^L, M^U; \alpha], [N^L, N^U; \beta]) \mid M^U + N^U \leq 1 \text{ and } M \in D(I_\alpha), N \in D(I_\beta)\}$$

is called (α, β)-interval valued set.

To make clear, it is shown below,

$$\{([M^L, M^U; \alpha], [N^L, N^U; \beta]) \mid M^U + N^U \leq 1 \text{ and } M \in D(I_\alpha), N \in D(I_\beta)\}$$
$$= \{([M; \alpha], [N; \beta]) \mid M^U + N^U \leq 1 \text{ and } M \in D(I_\alpha), N \in D(I_\beta)\}$$

The order relation on $D(I_\alpha) \times D(I_\beta)$ is defined below.

Definition 14: $\forall([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta]) \in D(I_\alpha) \times D(I_\beta)$,

$$([M; \alpha], [N; \beta]) \leq ([P; \alpha], [R; \beta]) :\Leftrightarrow [M; \alpha] \leq [P; \alpha] \text{ and } [N; \beta] \geq [R; \beta]$$

Here;

$$([M; \alpha], [N; \beta]) < ([P; \alpha], [R; \beta]) :\Leftrightarrow [M; \alpha] < [P; \alpha], [N; \beta] \geq [R; \beta] \text{ or}$$
$$[M; \alpha] \leq [P; \alpha], [N; \beta] > [R; \beta] \text{ or } [M; \alpha] < [P; \alpha], [N; \beta] > [R; \beta]$$

Proposition 8: $(D(I_\alpha) \times D(I_\beta), \leq)$ is partial ordered set.

Proof: $([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta]), ([S; \alpha], [T; \beta]) \in D(I_\alpha) \times D(I_\beta)$ are given arbitrary.

1. $M^L \leq M^L, M^U \geq M^U$ and $N^L \geq N^L, N^U \leq N^U$

$$\Rightarrow [M; \alpha] \leq [M; \alpha] \text{ and } [N; \beta] \geq [N; \beta] \Rightarrow ([M; \alpha], [N; \beta]) \leq ([M; \alpha], [N; \beta])$$

2. $([M; \alpha], [N; \beta]) \leq ([P; \alpha], [R; \beta])$ and $([M; \alpha], [N; \beta]) \geq ([P; \alpha], [R; \beta])$

$$\Rightarrow [M; \alpha] \leq [P; \alpha], [N; \beta] \geq [R; \beta] \text{ and } [M; \alpha] \geq [P; \alpha], [N; \beta] \leq [R; \beta]$$
$$\Rightarrow M^L \leq P^L, M^U \geq P^U, N^L \geq R^L, N^U \leq R^U \text{ and}$$
$$M^L \geq P^L, M^U \leq P^U, N^L \leq R^L, N^U \geq R^U$$
$$\Rightarrow M^L = P^L, M^U = P^U, N^L = R^L, N^U = R^U$$
$$\Rightarrow [M; \alpha] = [P; \alpha] \text{ and } [N; \beta] = [R; \beta] \Rightarrow ([M; \alpha], [N; \beta]) = ([P; \alpha], [R; \beta])$$

3. $([M; \alpha], [N; \beta]) \leq ([P; \alpha], [R; \beta])$ and $([P; \alpha], [R; \beta]) \leq ([S; \alpha], [T; \beta])$

$$\Rightarrow [M; \alpha] \leq [P; \alpha], [N; \beta] \geq [R; \beta] \text{ and } [P; \alpha] \leq [S; \alpha], [R; \beta] \geq [T; \beta]$$
$$\Rightarrow M^L \leq P^L, M^U \geq P^U, N^L \geq R^L, N^U \leq R^U \text{ and}$$
$$P^L \leq S^L, P^U \geq S^U, R^L \geq T^L, R^U \leq T^U$$
$$\Rightarrow M^L \leq S^L, M^U \geq S^U, N^L \geq T^L, N^U \leq T^U$$
$$\Rightarrow [M; \alpha] \leq [S; \alpha] \text{ and } [N; \beta] \geq [T; \beta] \Rightarrow ([M; \alpha], [N; \beta]) \leq ([S; \alpha], [T; \beta])$$

With the help of relation order on $D(I_\alpha) \times D(I_\beta)$, the definitions of supremum and infimum on this set are given below

Definition 15: $\forall([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta]) \in D(I_\alpha) \times D(I_\beta)$,

$$\inf\{([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta])\} = (\inf\{[M; \alpha], [P; \alpha]\}, \sup\{[N; \beta], [R; \beta]\})$$
$$\sup\{([M; \alpha], [N; \beta]), ([P; \alpha], [R; \beta])\} = (\sup\{[M; \alpha], [P; \alpha]\}, \inf\{[N; \beta], [R; \beta]\})$$

Lemma 2: $\left(D(I_\alpha)\times D(I_\beta),\wedge,\vee\right)$ is a complete lattice with units $([0,1-\beta;\alpha],[\beta,\beta;\beta])$ and $([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])$.
Proof: It is clear from known order relation on $\mathbb{R}$.

Remark 1: The intersection and union of the family of (α,β)-interval valued sets are again (α,β)-interval valued sets. If any function satisfies below conditions, then it is called negation function.

Definition 16: L is complete lattice with units 0 and 1. $\mathcal{N}: L\to L$ and $\forall a,b\in L$,

i. $\mathcal{N}(0)=1$ and $\mathcal{N}(1)=0$
ii. $\mathcal{N}(a)\le\mathcal{N}(b):\Leftrightarrow a\ge b$
iii. $\mathcal{N}(\mathcal{N}(a))=a$

We try to define a negation function on $D(I_\alpha)\times D(I_\beta)$ by the help of following relation, $\forall([M;\alpha],[N;\beta])\in D(I_\alpha)\times D(I_\beta)$,

$$\mathcal{N}\left(([M;\alpha],[N;\beta])\right)=([\alpha-M^L,\alpha-\beta+N^U;\alpha],[\beta-N^L,\beta-\alpha+M^U;\beta])$$

This relation on $\in D(I_\alpha)\times D(I_\beta)$ is a function. Indeed,
$([M;\alpha],[N;\beta])\in D(I_\alpha)\times D(I_\beta)$ is given arbitrary.

i. $M^L\le\alpha\Rightarrow 0\le\alpha-M^L\le\alpha$ and

$$N^U\ge\beta\ \Rightarrow\alpha-\beta+N^U\ge\alpha-\beta+\beta=\alpha$$

besides,

$$M^U+N^U\le 1\Rightarrow N^U\le 1-M^U\Rightarrow\alpha-\beta+N^U\le\alpha-\beta+1-M^U \text{ and } M^U\ge\alpha\Rightarrow\alpha-\beta+N^U\le\alpha-\beta+1-\alpha=1-\beta\le 1$$

From above consequences, we get that $[\alpha-M^L,\alpha-\beta+N^U;\alpha]$

ii. $N^L\le\beta\Rightarrow 0\le\beta-N^L\le\beta$ and

$$M^U\ge\alpha\Rightarrow\beta-\alpha+M^U\ge\beta-\alpha+\alpha=\beta$$

besides,

$$M^U+N^U\le 1\Rightarrow M^U\le 1-N^U\Rightarrow\beta-\alpha+M^U\le\beta-\alpha+1-N^U \text{ and } N^U\ \ge\beta\Rightarrow\beta-\alpha+M^U\le\beta-\alpha+1-\beta=1-\alpha\le 1$$

From above consequences, we get that $[\beta-N^L,\beta-\alpha+M^U;\beta]$

iii. $\alpha-\beta+N^U+\beta-\alpha+M^U=M^U+N^U\le 1$

From above results,

$$([\alpha-M^L,\alpha-\beta+N^U;\alpha],[\beta-N^L,\beta-\alpha+M^U;\beta])\in D(I_\alpha)\times D(I_\beta)$$

From previous discussions, we claim that $\mathcal{N}$ is negation function on $D(I_\alpha)\times D(I_\beta)$.

Proposition 9: $\forall([M;\alpha],[N;\beta])\in D(I_\alpha)\times D(I_\beta)$,
$\mathcal{N}: D(I_\alpha)\times D(I_\beta)\to D(I_\alpha)\times D(I_\beta)$,

$$\mathcal{N}\left(([M;\alpha],[N;\beta])\right)=([\alpha-M^L,\alpha-\beta+N^U;\alpha],[\beta-N^L,\beta-\alpha+M^U;\beta])$$

$\mathcal{N}$ satisfies conditions of Definition 16.

Proof: $([M;\alpha],[N;\beta]),([P;\alpha],[R;\beta])\in D(I_\alpha)\times D(I_\beta)$ are given arbitrary.

1. $([M;\alpha],[N;\beta])=([P;\alpha],[R;\beta])\Rightarrow[M;\alpha]=[P;\alpha]$ and $[N;\beta]=[R;\beta]$

$$\Rightarrow M^L=P^L,M^U=P^U \text{ and } N^L=R^L,N^U=R^U$$
$$\Rightarrow\alpha-M^L=\alpha-P^L,\alpha-\beta+N^U=\alpha-\beta+R^U \text{ and}$$
$$\beta-N^L=\beta-R^L,\beta-\alpha+M^U=\beta-\alpha+P^U$$
$$([\alpha-M^L,\alpha-\beta+N^U;\alpha],[\beta-N^L,\beta-\alpha+M^U;\beta])=([\alpha-P^L,\alpha-\beta+R^U;\alpha],[\beta-R^L,\beta-\alpha+P^U;\beta])$$

$$\Rightarrow \mathcal{N}\big(([M;\alpha],[N;\beta])\big) = \mathcal{N}\big(([P;\alpha],[R;\beta])\big)$$

2. Now, it is shown that $\mathcal{N}$ satisfies the conditions of negation function,

i. $\mathcal{N}\big(([0,1-\beta;\alpha],[\beta,\beta;\beta])\big)$

$$= ([\alpha-0,\alpha-\beta+\beta;\alpha],[\beta-\beta,\beta-\alpha+1-\beta;\beta]) = ([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])$$
$$\mathcal{N}\big(([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])\big) = ([\alpha-\alpha,\alpha-\beta+1-\alpha;\alpha],[\beta-0,\beta-\alpha+\alpha;\beta])$$
$$= ([0,1-\beta;\alpha],[\beta,\beta;\beta])$$

ii. $\mathcal{N}\big(([M;\alpha],[N;\beta])\big) \leq \mathcal{N}\big(([P;\alpha],[R;\beta])\big)$

$$\Leftrightarrow \begin{pmatrix}[\alpha-M^L,\alpha-\beta+N^U;\alpha],\\ [\beta-N^L,\beta-\alpha+M^U;\beta]\end{pmatrix} \leq \begin{pmatrix}[\alpha-P^L,\alpha-\beta+R^U;\alpha],\\ [\beta-R^L,\beta-\alpha+P^U;\beta]\end{pmatrix}$$
$$\Leftrightarrow [\alpha-M^L,\alpha-\beta+N^U;\alpha] \leq [\alpha-P^L,\alpha-\beta+R^U;\alpha] \text{ and}$$
$$[\beta-N^L,\beta-\alpha+M^U;\beta] \geq [\beta-R^L,\beta-\alpha+P^U;\beta]$$
$$\Leftrightarrow \alpha-M^L \leq \alpha-P^L, \alpha-\beta+N^U \geq \alpha-\beta+R^U \text{ and}$$
$$\beta-N^L \geq \beta-R^L, \beta-\alpha+M^U \leq \beta-\alpha+P^U$$
$$\Leftrightarrow M^L \geq P^L, N^U \geq R^U, N^L \leq R^L, M^U \leq P^U$$
$$\Leftrightarrow [M;\alpha] \geq [P;\alpha] \text{ and } [N;\beta] \leq [R;\beta]$$
$$\Leftrightarrow ([M;\alpha],[N;\beta]) \geq ([P;\alpha],[R;\beta])$$

iii. $\mathcal{N}\Big(\mathcal{N}\big(([M;\alpha],[N;\beta])\big)\Big)$

$$= \mathcal{N}([\alpha-M^L,\alpha-\beta+N^U;\alpha],[\beta-N^L,\beta-\alpha+M^U;\beta])$$
$$= \begin{pmatrix}[\alpha-(\alpha-M^L),\alpha-\beta+\beta-\alpha+M^U;\alpha],\\ [\beta-(\beta-N^L),\beta-\alpha+\alpha-\beta+N^U;\beta]\end{pmatrix}$$
$$= ([M^L,M^U;\alpha],[N^L,N^U;\beta]) = ([M;\alpha],[N;\beta])$$

Definition 17: Let X be universal set.
For functions $[M_A;\alpha]: X \to D(I_\alpha)$ and $[N_A;\beta]: X \to D(I_\beta)$, $\forall x \in X, {M_A}^U(x) + {N_A}^U(x) \leq 1$,

$$[A;\alpha;\beta] = \{\langle x,[M_A(x);\alpha],[N_A(x);\beta]\rangle | x \in X\}$$

To make clear, it is denoted by;

$$\{\langle x,[M_A(x);\alpha],[N_A(x);\beta]\rangle | x \in X\} = \{\langle x,[A(x);\alpha;\beta]\rangle | x \in X\}$$

is called (α,β) −interval valued intuitionistic fuzzy set. The family of (α,β) −interval valued intuitionistic fuzzy sets on X is shown by (α, β)-IVIFS(X).

Some algebraic operations on (α, β)-IVIFS(X) are defined below.

Definition 18: Let X be universal set. [A; α; β], [B; α; β] ∈ (α, β)-IVIFS(X) and Λ is index set ∀λ ∈ Λ,

i. $$[A;\alpha;\beta]^c = \begin{Bmatrix} < x, \big[\alpha-{M_A}^L(x),\alpha-\beta+{N_A}^U(x);\alpha\big], \\ \big[\beta-{N_A}^L(x),\beta-\alpha+{M_A}^U(x);\beta\big] > | x \in X \end{Bmatrix}$$

ii. $[A;\alpha;\beta] \sqsubseteq [B;\alpha;\beta] :\Leftrightarrow \forall x \in X, {M_A}^L(x) \leq {M_B}^L(x), {M_A}^U(x) \geq {M_B}^U(x)$

$$\text{and } {N_A}^L(x) \geq {N_B}^L(x), {N_A}^U(x) \leq {N_B}^U(x)$$

iii. $[A;\alpha;\beta] \sqcap [B;\alpha;\beta]$

$$= \begin{Bmatrix} < x, \big[\inf\{{M_A}^L(x),{M_B}^L(x)\},\sup\{{M_A}^U(x),{M_B}^U(x)\};\alpha\big], \\ \big[\sup\{{N_A}^L(x),{N_B}^L(x)\},\inf\{{N_A}^U(x),{N_B}^U(x)\};\beta\big] > | x \in X \end{Bmatrix}$$

iv. $[A;\alpha;\beta] \sqcup [B;\alpha;\beta]$

$$= \begin{Bmatrix} < x, \big[\sup\{{M_A}^L(x),{M_B}^L(x)\},\inf\{{M_A}^U(x),{M_B}^U(x)\};\alpha\big], \\ \big[\inf\{{N_A}^L(x),{N_B}^L(x)\},\sup\{{N_A}^U(x),{N_B}^U(x)\};\beta\big] > | x \in X \end{Bmatrix}$$

v. $\sqcap_{\lambda\in\Lambda}[A;\alpha;\beta]_\lambda = \left\{\left\langle \begin{matrix} x, \left[\wedge_{\lambda\in\Lambda} M_{A_\lambda}^{L}(x), \vee_{\lambda\in\Lambda} M_{A_\lambda}^{U}(x);\alpha\right], \\ \left[\vee_{\lambda\in\Lambda} N_{A_\lambda}^{L}(x), \wedge_{\lambda\in\Lambda} N_{A_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x\in X\right\}$

vi. $\sqcup_{\lambda\in\Lambda}[A;\alpha;\beta]_\lambda = \left\{\left\langle \begin{matrix} x, \left[\vee_{\lambda\in\Lambda} M_{A_\lambda}^{L}(x), \wedge_{\lambda\in\Lambda} M_{A_\lambda}^{U}(x);\alpha\right], \\ \left[\wedge_{\lambda\in\Lambda} N_{A_\lambda}^{L}(x), \vee_{\lambda\in\Lambda} N_{A_\lambda}^{U}(x);\beta\right] \end{matrix} \right\rangle \middle| x\in X\right\}$

Example 2: Let $X = \{a, b, c, d\}$.

$$[A;\alpha;\beta] = \left\{\begin{matrix} < a,[0.1,0.3;0.3],[0.4,0.6;0.4] >, < b,[0.0,0.4;0.3],[0.3,0.6;0.4] >, \\ < c,[0.2,0.5;0.3],[0.1,0.4;0.4] >, < d,[0.15,0.45;0.3],[0.3,0.5;0.4] > \end{matrix}\right\}$$

$$[B;\alpha;\beta] = \left\{\begin{matrix} < a,[0.05,0.35;0.3],[0.25,0.65;0.4] >, < b,[0.15,0.45;0.3],[0.2,0.4;0.4] >, \\ < c,[0.1,0.3;0.3],[0.3,0.7;0.4] >, < d,[0.1,0.4;0.3],[0.15,0.55;0.4] > \end{matrix}\right\}$$

For $\alpha = 0.3$ and $\beta = 0.4$, A and B are (α,β)-interval valued intuitionistic fuzzy sets,

$$[A;\alpha;\beta]^c = \left\{\begin{matrix} < a,[0.2,0.5;0.3],[0.0,0.4;0.4] >, < b,[0.3,0.5;0.3],[0.1,0.5;0.4] >, \\ < c,[0.1,0.3;0.3],[0.3,0.6;0.4] >, < d,[0.15,0.4;0.3],[0.1,0.55;0.4] > \end{matrix}\right\}$$

$$[A;\alpha;\beta] \sqcap [B;\alpha;\beta] = \left\{\begin{matrix} < a,[0.05,0.35;0.3],[0.4,0.6;0.4] >, < b,[0.0,0.45;0.3],[0.3,0.4;0.4] >, \\ < c,[0.1,0.5;0.3],[0.3,0.4;0.4] >, < d,[0.1,0.45;0.3],[0.3,0.5;0.4] > \end{matrix}\right\}$$

$$[A;\alpha;\beta] \sqcup [B;\alpha;\beta] = \left\{\begin{matrix} < a,[0.1,0.3;0.3],[0.25,0.65;0.4] >, < b,[0.15,0.4;0.3],[0.2,0.6;0.4] >, \\ < c,[0.2,0.3;0.3],[0.1,0.7;0.4] >, < d,[0.1,0.4;0.3],[0.15,0.55;0.4] > \end{matrix}\right\}$$

Proposition 10: Let X be universal set.

$[A;\alpha;\beta], [B;\alpha;\beta], [C;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) and Λ is index set $\forall\lambda \in \Lambda$,

i. $[A;\alpha;\beta] \sqcap [B;\alpha;\beta] = [B;\alpha;\beta] \sqcap [A;\alpha;\beta]$

ii. $[A;\alpha;\beta] \sqcup [B;\alpha;\beta] = [B;\alpha;\beta] \sqcup [A;\alpha;\beta]$

iii. $[A;\alpha;\beta] \sqcap ([B;\alpha;\beta] \sqcup [C;\alpha;\beta])$
$$= ([A;\alpha;\beta] \sqcap [B;\alpha;\beta]) \sqcup ([A;\alpha;\beta] \sqcap [C;\alpha;\beta])$$

iv. $[A;\alpha;\beta] \sqcup ([B;\alpha;\beta] \sqcap [C;\alpha;\beta])$
$$= ([A;\alpha;\beta] \sqcup [B;\alpha;\beta]) \sqcap ([A;\alpha;\beta] \sqcup [C;\alpha;\beta])$$

v. $[A;\alpha;\beta] \sqcap (\sqcup_{\lambda\in\Lambda}[B;\alpha;\beta]_\lambda) = \sqcup_{\lambda\in\Lambda}([A;\alpha;\beta] \sqcap [B;\alpha;\beta]_\lambda)$

vi. $[A;\alpha;\beta] \sqcup (\sqcap_{\lambda\in\Lambda}[B;\alpha;\beta]_\lambda) = \sqcap_{\lambda\in\Lambda}([A;\alpha;\beta] \sqcup [B;\alpha;\beta]_\lambda)$

Proof: $[A;\alpha;\beta], [B;\alpha;\beta], [C;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) are given arbitrary.

i. $[A;\alpha;\beta] \sqcap [B;\alpha;\beta]$

$$= \left\{\begin{matrix} < x, \left[\inf\{M_A{}^L(x), M_B{}^L(x)\}, \sup\{M_A{}^U(x), M_B{}^U(x)\};\alpha\right], \\ \left[\sup\{N_A{}^L(x), N_B{}^L(x)\}, \inf\{N_A{}^U(x), N_B{}^U(x)\};\beta\right] > |x\in X \end{matrix}\right\}$$

$$= \left\{\begin{matrix} < x, \left[\inf\{M_B{}^L(x), M_A{}^L(x)\}, \sup\{M_B{}^U(x), M_A{}^U(x)\};\alpha\right], \\ \left[\sup\{N_B{}^L(x), N_A{}^L(x)\}, \inf\{N_B{}^U(x), N_A{}^U(x)\};\beta\right] > |x\in X \end{matrix}\right\} = [B;\alpha;\beta] \sqcap [A;\alpha;\beta]$$

ii. $[A;\alpha;\beta] \sqcup [B;\alpha;\beta]$

$$= \left\{\begin{matrix} < x, \left[\sup\{M_A{}^L(x), M_B{}^L(x)\}, \inf\{M_A{}^U(x), M_B{}^U(x)\};\alpha\right], \\ \left[\inf\{N_A{}^L(x), N_B{}^L(x)\}, \sup\{N_A{}^U(x), N_B{}^U(x)\};\beta\right] > |x\in X \end{matrix}\right\}$$

$$= \left\{\begin{matrix} < x, \left[\sup\{M_B{}^L(x), M_A{}^L(x)\}, \inf\{M_B{}^U(x), M_A{}^U(x)\};\alpha\right], \\ \left[\inf\{N_B{}^L(x), N_A{}^L(x)\}, \sup\{N_B{}^U(x), N_A{}^U(x)\};\beta\right] > |x\in X \end{matrix}\right\} = [B;\alpha;\beta] \sqcup [A;\alpha;\beta]$$

iii. $[A;\alpha;\beta] \sqcap ([B;\alpha;\beta] \sqcup [C;\alpha;\beta])$

$$
\begin{aligned}
&= [A;\alpha;\beta] \sqcap \left\{\begin{array}{l} <x, \left[\begin{array}{c} \sup\{M_B{}^L(x), M_C{}^L(x)\}, \\ \inf\{M_B{}^U(x), M_C{}^U(x)\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \inf\{N_B{}^L(x), N_C{}^L(x)\}, \\ \sup\{N_B{}^U(x), N_C{}^U(x)\}; \beta \end{array}\right] > |x \in X \end{array}\right\} \\
&= \left\{\begin{array}{l} <x, \left[\begin{array}{c} \inf\{M_A{}^L(x), \sup\{M_B{}^L(x), M_C{}^L(x)\}\}, \\ \sup\{M_A{}^U(x), \inf\{M_B{}^U(x), M_C{}^U(x)\}\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \sup\{N_A{}^L(x), \inf\{N_B{}^L(x), N_C{}^L(x)\}\}, \\ \inf\{N_A{}^U(x), \sup\{N_B{}^U(x), N_C{}^U(x)\}\}; \beta \end{array}\right] > |x \in X \end{array}\right\} \\
&= \left\{\begin{array}{l} <x, \left[\begin{array}{c} \sup\{\inf\{M_A{}^L(x), M_B{}^L(x)\}, \inf\{M_A{}^L(x), M_C{}^L(x)\}\}, \\ \inf\{\sup\{M_A{}^U(x), M_B{}^U(x)\}, \sup\{M_A{}^U(x), M_C{}^U(x)\}\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \inf\{\sup\{N_A{}^L(x), N_B{}^L(x)\}, \sup\{N_A{}^L(x), N_C{}^L(x)\}\}, \\ \sup\{\inf\{N_A{}^U(x), N_B{}^U(x)\}, \inf\{N_A{}^U(x), N_C{}^U(x)\}\}; \beta \end{array}\right] > |x \in X \end{array}\right\} \\
&= \left\{\begin{array}{l} <x, [\inf\{M_A{}^L(x), M_B{}^L(x)\}, \sup\{M_A{}^U(x), M_B{}^U(x)\}; \alpha], \\ [\sup\{N_A{}^L(x), N_B{}^L(x)\}, \inf\{N_A{}^U(x), N_B{}^U(x)\}; \beta] > |x \in X \end{array}\right\} \\
&\sqcup \left\{\begin{array}{l} <x, [\inf\{M_A{}^L(x), M_C{}^L(x)\}, \alpha, \sup\{M_A{}^U(x), M_C{}^U(x)\}; \alpha], \\ [\sup\{N_A{}^L(x), N_C{}^L(x)\}, \beta, \inf\{N_A{}^U(x), N_C{}^U(x)\}; \beta] > |x \in X \end{array}\right\} \\
&= ([A;\alpha;\beta] \sqcap [B;\alpha;\beta]) \sqcup ([A;\alpha;\beta] \sqcap [C;\alpha;\beta])
\end{aligned}
$$

iv. $[A;\alpha;\beta] \sqcup ([B;\alpha;\beta] \sqcap [C;\alpha;\beta])$

$$
\begin{aligned}
&= [A;\alpha;\beta] \sqcup \left\{\begin{array}{l} <x, \left[\begin{array}{c} \inf\{M_B{}^L(x), M_C{}^L(x)\}, \\ \sup\{M_B{}^U(x), M_C{}^U(x)\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \sup\{N_B{}^L(x), N_C{}^L(x)\}, \\ \inf\{N_B{}^U(x), N_C{}^U(x)\}; \beta \end{array}\right] > |x \in X \end{array}\right\} \\
&= \left\{\begin{array}{l} <x, \left[\begin{array}{c} \sup\{M_A{}^L(x), \inf\{M_B{}^L(x), M_C{}^L(x)\}\}, \\ \inf\{M_A{}^U(x), \sup\{M_B{}^U(x), M_C{}^U(x)\}\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \inf\{N_A{}^L(x), \sup\{N_B{}^L(x), N_C{}^L(x)\}\}, \\ \sup\{N_A{}^U(x), \inf\{N_B{}^U(x), N_C{}^U(x)\}\}; \beta \end{array}\right] > |x \in X \end{array}\right\} \\
&= \left\{\begin{array}{l} <x, \left[\begin{array}{c} \inf\{\sup\{M_A{}^L(x), M_B{}^L(x)\}, \sup\{M_A{}^L(x), M_C{}^L(x)\}\}, \\ \sup\{\inf\{M_A{}^U(x), M_B{}^U(x)\}, \inf\{M_A{}^U(x), M_C{}^U(x)\}\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \sup\{\inf\{N_A{}^L(x), N_B{}^L(x)\}, \inf\{N_A{}^L(x), N_C{}^L(x)\}\}, \\ \inf\{\sup\{N_A{}^U(x), N_B{}^U(x)\}, \sup\{N_A{}^U(x), N_C{}^U(x)\}\}; \beta \end{array}\right] > |x \in X \end{array}\right\} \\
&= \left\{\begin{array}{l} <x, [\sup\{M_A{}^L(x), M_B{}^L(x)\}, \inf\{M_A{}^U(x), M_B{}^U(x)\}; \alpha], \\ [\inf\{N_A{}^L(x), N_B{}^L(x)\}, \sup\{N_A{}^U(x), N_B{}^U(x)\}; \beta] > |x \in X \end{array}\right\} \\
&\sqcap \left\{\begin{array}{l} <x, [\sup\{M_A{}^L(x), M_C{}^L(x)\}, \inf\{M_A{}^U(x), M_C{}^U(x)\}; \alpha], \\ [\inf\{N_A{}^L(x), N_C{}^L(x)\}, \sup\{N_A{}^U(x), N_C{}^U(x)\}; \beta] > |x \in X \end{array}\right\} \\
&= ([A;\alpha;\beta] \sqcup [B;\alpha;\beta]) \sqcap ([A;\alpha;\beta] \sqcup [C;\alpha;\beta])
\end{aligned}
$$

v. $[A;\alpha;\beta] \sqcap (\sqcup_{\lambda\in\Lambda} [B;\alpha;\beta]_\lambda)$

$$= [A;\alpha;\beta] \sqcap \left\{ \langle \begin{matrix} x, \left[\bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[\bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\}$$

$$= \left\{ \langle \begin{matrix} x, \left[M_A{}^{L}(x) \wedge \bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \vee \bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[N_A{}^{L}(x) \vee \bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \wedge \bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\}$$

$$= \left\{ \langle \begin{matrix} x, \left[\bigvee_{\lambda\in\Lambda} M_A{}^{L}(x) \wedge M_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} M_A{}^{U}(x) \vee M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[\bigwedge_{\lambda\in\Lambda} N_A{}^{L}(x) \vee N_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} N_A{}^{U}(x) \wedge N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\}$$

$$= \sqcup_{\lambda\in\Lambda} \left\{ \langle \begin{matrix} x, \left[M_A{}^{L}(x) \wedge M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \vee M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[N_A{}^{L}(x) \vee N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \wedge N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\} = \sqcup_{\lambda\in\Lambda} \left([A;\alpha;\beta] \sqcap [B;\alpha;\beta]_\lambda \right)$$

vi. $[A;\alpha;\beta] \sqcup \left(\sqcap_{\lambda\in\Lambda} [B;\alpha;\beta]_\lambda \right)$

$$= [A;\alpha;\beta] \sqcup \left\{ \langle \begin{matrix} x, \left[\bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[\bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\}$$

$$= \left\{ \langle \begin{matrix} x, \left[M_A{}^{L}(x) \vee \bigwedge_{\lambda\in\Lambda} M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \wedge \bigvee_{\lambda\in\Lambda} M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[N_A{}^{L}(x) \wedge \bigvee_{\lambda\in\Lambda} N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \vee \bigwedge_{\lambda\in\Lambda} N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\}$$

$$= \left\{ \langle \begin{matrix} x, \left[\bigwedge_{\lambda\in\Lambda} M_A{}^{L}(x) \vee M_{B_\lambda}^{L}(x), \bigvee_{\lambda\in\Lambda} M_A{}^{U}(x) \wedge M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[\bigvee_{\lambda\in\Lambda} N_A{}^{L}(x) \wedge N_{B_\lambda}^{L}(x), \bigwedge_{\lambda\in\Lambda} N_A{}^{U}(x) \vee N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\}$$

$$= \sqcap_{\lambda\in\Lambda} \left\{ \langle \begin{matrix} x, \left[M_A{}^{L}(x) \vee M_{B_\lambda}^{L}(x), M_A{}^{U}(x) \wedge M_{B_\lambda}^{U}(x); \alpha \right], \\ \left[N_A{}^{L}(x) \wedge N_{B_\lambda}^{L}(x), N_A{}^{U}(x) \vee N_{B_\lambda}^{U}(x); \beta \right] \end{matrix} \rangle \,|\, x \in X \right\} = \sqcap_{\lambda\in\Lambda} \left([A;\alpha;\beta] \sqcup [B;\alpha;\beta]_\lambda \right)$$

Proposition 11: Let X be universal set. $[A;\alpha;\beta], [B;\alpha;\beta] \in (\alpha,\beta)\text{-IVIFS}(X)$ and Λ is index set $\forall \lambda \in \Lambda$,

i. $(([A;\alpha;\beta])^c)^c = [A;\alpha;\beta]$

ii. $([A;\alpha;\beta] \sqcap [B;\alpha;\beta])^c = ([A;\alpha;\beta])^c \sqcup ([B;\alpha;\beta])^c$

iii. $([A;\alpha;\beta] \sqcup [B;\alpha;\beta])^c = ([A;\alpha;\beta])^c \sqcap ([B;\alpha;\beta])^c$

iv. $(\sqcap_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c = \sqcup_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$

v. $(\sqcup_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c = \sqcap_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$

Proof: $[A;\alpha;\beta], [B;\alpha;\beta] \in (\alpha,\beta)\text{-IVIFS}(X)$ are given arbitrary.

i. $([A;\alpha;\beta])^c = \left\{ \begin{matrix} < x, \left[\alpha - M_A{}^{L}(x), \alpha - \beta + N_A{}^{U}(x); \alpha \right], \\ \left[\beta - N_A{}^{L}(x), \beta - \alpha + M_A{}^{U}(x); \beta \right] > |x \in X \end{matrix} \right\}$

$$\Rightarrow (([A;\alpha;\beta])^c)^c$$
$$= \left\{ \begin{array}{c} < x, \left[\alpha - \left(\alpha - {M_A}^L(x)\right), \alpha - \beta + \beta - \alpha + {M_A}^U(x); \alpha\right], \\ \left[\beta - \left(\beta - {N_A}^L(x)\right), \beta - \alpha + \alpha - \beta + {N_A}^U(x); \beta\right] > |x \in X \end{array} \right\}$$
$$= \left\{< x, \left[{M_A}^L(x), {M_A}^U(x); \alpha\right], \left[{N_A}^L(x), {N_A}^U(x); \beta\right] > |x \in X\right\} = [A;\alpha;\beta]$$

ii. $([A;\alpha;\beta] \sqcap [B;\alpha;\beta])^c$

$$= \left\{ \begin{array}{c} < x, \left[\begin{array}{c} \alpha - \left(\inf\left\{{M_A}^L(x), {M_B}^L(x)\right\}\right), \\ \alpha - \beta + \left(\inf\left\{{N_A}^U(x), {N_B}^U(x)\right\}\right); \alpha \end{array}\right], \\ \left[\begin{array}{c} \beta - \left(\sup\left\{{N_A}^L(x), {N_B}^L(x)\right\}\right), \\ \beta - \alpha + \left(\sup\left\{{M_A}^U(x), {M_B}^U(x)\right\}\right); \beta \end{array}\right] > |x \in X \end{array} \right\}$$
$$= \left\{ \begin{array}{c} < x, \left[\begin{array}{c} \sup\left\{\alpha - {M_A}^L(x), \alpha - {M_B}^L(x)\right\}, \\ \inf\left\{\alpha - \beta + {N_A}^U(x), \alpha - \beta + {N_B}^U(x)\right\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \inf\left\{\beta - {N_A}^L(x), \beta - {N_B}^L(x)\right\}, \\ \sup\left\{\beta - \alpha + {M_A}^U(x), \beta - \alpha + {M_B}^U(x)\right\}; \beta \end{array}\right] > |x \in X \end{array} \right\}$$
$$= \left\{ \begin{array}{c} < x, \left[\alpha - {M_A}^L(x), \alpha - \beta + {N_A}^U(x); \alpha\right], \\ \left[\beta - {N_A}^L(x), \beta - \alpha + {M_A}^U(x); \beta\right] > |x \in X \end{array} \right\} \sqcup \left\{ \begin{array}{c} < x, \left[\alpha - {M_B}^L(x), \alpha - \beta + {N_B}^U(x); \alpha\right], \\ \left[\beta - {N_B}^L(x), \beta - \alpha + {M_B}^U(x); \beta\right] > |x \in X \end{array} \right\}$$
$$= ([A;\alpha;\beta])^c \sqcup ([B;\alpha;\beta])^c$$

iii. $([A;\alpha;\beta] \sqcup [B;\alpha;\beta])^c$

$$= \left\{ \begin{array}{c} < x, \left[\begin{array}{c} \alpha - \left(\sup\left\{{M_A}^L(x), {M_B}^L(x)\right\}\right), \\ \alpha - \beta + \left(\sup\left\{{N_A}^U(x), {N_B}^U(x)\right\}\right); \alpha \end{array}\right], \\ \left[\begin{array}{c} \beta - \left(\inf\left\{{N_A}^L(x), {N_B}^L(x)\right\}\right), \\ \beta - \alpha + \left(\inf\left\{{M_A}^U(x), {M_B}^U(x)\right\}\right); \beta \end{array}\right] > |x \in X \end{array} \right\}$$
$$= \left\{ \begin{array}{c} < x, \left[\begin{array}{c} \inf\left\{\alpha - {M_A}^L(x), \alpha - {M_B}^L(x)\right\}, \\ \sup\left\{\alpha - \beta + {N_A}^U(x), \alpha - \beta + {N_B}^U(x)\right\}; \alpha \end{array}\right], \\ \left[\begin{array}{c} \sup\left\{\beta - {N_A}^L(x), \beta - {N_B}^L(x)\right\}, \\ \inf\left\{\beta - \alpha + {M_A}^U(x), \beta - \alpha + {M_B}^U(x)\right\}; \beta \end{array}\right] > |x \in X \end{array} \right\}$$
$$= \left\{ \begin{array}{c} < x, \left[\alpha - {M_A}^L(x), \alpha - \beta + {N_A}^U(x); \alpha\right], \\ \left[\beta - {N_A}^L(x), \beta - \alpha + {M_A}^U(x); \beta\right] > |x \in X \end{array} \right\} \sqcap \left\{ \begin{array}{c} < x, \left[\alpha - {M_B}^L(x), \alpha - \beta + {N_B}^U(x); \alpha\right], \\ \left[\beta - {N_B}^L(x), \beta - \alpha + {M_B}^U(x); \beta\right] > |x \in X \end{array} \right\}$$
$$= ([A;\alpha;\beta])^c \sqcap ([B;\alpha;\beta])^c$$

iv. $\sqcap_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda = \left\{ \left\langle \begin{array}{c} x, \left[\wedge_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \vee_{\lambda\in\Lambda} M_{A_\lambda}^U(x); \alpha\right], \\ \left[\vee_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \wedge_{\lambda\in\Lambda} N_{A_\lambda}^U(x); \beta\right] \end{array} \right\rangle |x \in X \right\}$

$$\Rightarrow (\sqcap_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c$$
$$= \left\{ \left\langle \begin{array}{c} x, \left[\alpha - \bigwedge_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \alpha - \beta + \bigwedge_{\lambda\in\Lambda} N_{A_\lambda}^U(x); \alpha\right], \\ \left[\beta - \bigvee_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \beta - \alpha + \bigvee_{\lambda\in\Lambda} M_{A_\lambda}^U(x); \beta\right] \end{array} \right\rangle |x \in X \right\}$$
$$= \left\{ \left\langle \begin{array}{c} x, \left[\bigvee_{\lambda\in\Lambda} \alpha - M_{A_\lambda}^L(x), \bigwedge_{\lambda\in\Lambda} \alpha - \beta + N_{A_\lambda}^U(x); \alpha\right], \\ \left[\bigwedge_{\lambda\in\Lambda} \beta - N_{A_\lambda}^L(x), \bigvee_{\lambda\in\Lambda} \beta - \alpha + M_{A_\lambda}^U(x); \beta\right] \end{array} \right\rangle |x \in X \right\} = \sqcup_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$$

v. $\sqcup_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda = \left\{ \left\langle \begin{array}{c} x, \left[\vee_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \wedge_{\lambda\in\Lambda} M_{A_\lambda}^U(x); \alpha\right], \\ \left[\wedge_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \vee_{\lambda\in\Lambda} N_{A_\lambda}^U(x); \beta\right] \end{array} \right\rangle |x \in X \right\}$

$$\Rightarrow (\sqcup_{\lambda\in\Lambda} [A;\alpha;\beta]_\lambda)^c$$

$$= \left\{ \left\langle \begin{array}{c} x, \left[\alpha - \bigvee_{\lambda\in\Lambda} M_{A_\lambda}^L(x), \alpha-\beta+\bigvee_{\lambda\in\Lambda} N_{A_\lambda}^U(x);\alpha\right], \\ \left[\beta - \bigwedge_{\lambda\in\Lambda} N_{A_\lambda}^L(x), \beta-\alpha+\bigwedge_{\lambda\in\Lambda} M_{A_\lambda}^U(x);\beta\right] \end{array} \right\rangle | x\in X \right\}$$

$$= \left\{ \left\langle \begin{array}{c} x, \left[\bigwedge_{\lambda\in\Lambda} \alpha - M_{A_\lambda}^L(x), \bigvee_{\lambda\in\Lambda} \alpha-\beta+N_{A_\lambda}^U(x);\alpha\right], \\ \left[\bigvee_{\lambda\in\Lambda} \beta - N_{A_\lambda}^L(x), \bigwedge_{\lambda\in\Lambda} \beta-\alpha+M_{A_\lambda}^U(x);\beta\right] \end{array} \right\rangle | x\in X \right\} = \sqcap_{\lambda\in\Lambda} ([A;\alpha;\beta]_\lambda)^c$$

Proposition 12: Let X be universal set.

Functions $\mathbf{0_X}: X \to ([0,1-\beta;\alpha],[\beta,\beta;\beta])$ and $\mathbf{1_X}: X \to ([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])$

i. $(\mathbf{0_X})^c = \mathbf{1_X}$

ii. $(\mathbf{1_X})^c = \mathbf{0_X}$

Proof:

i. $(\mathbf{0_X})^c = \big(([0,1-\beta;\alpha],[\beta,\beta;\beta])\big)^c$

$$= ([\alpha-0,\alpha-\beta+\beta;\alpha],[\beta-\beta,\beta-\alpha+1-\beta;\beta])$$
$$= ([\alpha,\alpha;\alpha],[0,1-\alpha;\beta]) = \mathbf{1_X}$$

i. $(\mathbf{1_X})^c = \big(([\alpha,\alpha;\alpha],[0,1-\alpha;\beta])\big)^c$

$$= ([\alpha-\alpha,\alpha-\beta+1-\alpha;\alpha],[\beta-0,\beta-\alpha+\alpha;\beta])$$
$$= ([0,1-\beta;\alpha],[\beta,\beta;\beta]) = \mathbf{0_X}$$

Definition 19: Let X be universal set and $[A;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X).
$[A;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) has sup-property: $\Leftrightarrow$ $\forall x \in X$,
$\exists([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha)\times D(I_\beta) \ni [A;\alpha;\beta] = ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$

Definition 20: Let X be universal set and $[A;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X).
$\forall([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha)\times D(I_\beta)$,

$$[A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$$
$$= \{x\in X | [M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \text{ and } [N_A(x);\beta] \leq [\theta_1,\theta_2;\beta]\}$$

$[A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is called $([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$-level subset of $[A;\alpha;\beta]$. It is easily seen that from definition,
$([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$-level subsets of $[A;\alpha;\beta]$ are crisp sets . Besides,

$$[M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \Rightarrow {M_A}^L(x) \geq \lambda_1 \text{ and } {M_A}^U(x) \leq \lambda_2$$
$$[N_A(x);\beta] \leq [\theta_1,\theta_2;\beta] \Rightarrow {N_A}^L(x) \leq \theta_1 \text{ and } {N_A}^U(x) \geq \theta_2$$

Proposition 13: Let X be universal set. $\forall[A;\alpha;\beta],[B;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X), $\forall([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha)\times D(I_\beta)$,

i. $x \in [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$\Leftrightarrow ([M_A(x);\alpha],[N_A(x);\beta]) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$

ii. $[A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])} = [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]}$

iii. $([A;\alpha;\beta] \sqcup [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$= \left([M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [M_B(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left({M_A}^L{}_{\lambda_1} \cap {M_B}^U{}_{\lambda_2}\right)\right) \cup \left({M_B}^L{}_{\lambda_1} \cap {M_A}^U{}_{\lambda_2}\right) \cap \left([N_A(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup [N_B(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup \left({N_A}^L{}_{\theta_1} \cap {N_B}^U{}_{\theta_2}\right) \cup \left({N_B}^L{}_{\theta_1} \cap {N_A}^U{}_{\theta_2}\right)\right)$$

iv. $([A;\alpha;\beta] \sqcap [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$= [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])} \cap [B;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$$

Proof: $[A;\alpha;\beta], [B;\alpha;\beta] \in (\alpha,\beta)$-IVIFS(X) and $([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta]) \in D(I_\alpha) \times D(I_\beta)$ are given arbitrary.

i. $x \in [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$

$$\Leftrightarrow [M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \text{ and } [N_A(x);\beta] \leq [\theta_1,\theta_2;\beta]$$
$$\Leftrightarrow ([M_A(x);\alpha],[N_A(x);\beta]) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$

ii. $x \in [A;\alpha;\beta]_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is given arbitrary.

$$([M_A(x);\alpha],[N_A(x);\beta]) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$
$$\Leftrightarrow [M_A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \text{and } [N_A(x);\beta] \leq [\theta_1,\theta_2;\beta]$$
$$\Leftrightarrow x \in [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ and } x \in [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]}$$
$$\Leftrightarrow x \in [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]}$$

iii. $x \in ([A;\alpha;\beta] \sqcup [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is given arbitrary.

$$\left([M_{([A;\alpha;\beta]\sqcup[B;\alpha;\beta])}(x);\alpha],[N_{([A;\alpha;\beta]\sqcup[B;\alpha;\beta])}(x);\beta]\right) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$
$$\Leftrightarrow \begin{pmatrix} \begin{bmatrix} \sup\{M_A{}^L(x), M_B{}^L(x)\}, \\ \inf\{M_A{}^U(x), M_B{}^U(x)\};\alpha \end{bmatrix}, \\ \begin{bmatrix} \inf\{N_A{}^L(x), N_B{}^L(x)\}, \\ \sup\{N_A{}^U(x), N_B{}^U(x)\};\beta \end{bmatrix} \end{pmatrix} \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$
$$\Leftrightarrow \begin{bmatrix} \sup\{M_A{}^L(x), M_B{}^L(x)\}, \\ \inf\{M_A{}^U(x), M_B{}^U(x)\};\alpha \end{bmatrix} \geq [\lambda_1,\lambda_2;\alpha]$$
$$\text{and } \begin{bmatrix} \inf\{N_A{}^L(x), N_B{}^L(x)\}, \\ \sup\{N_A{}^U(x), N_B{}^U(x)\};\beta \end{bmatrix} \leq [\theta_1,\theta_2;\beta]$$
$$\Leftrightarrow \sup\{M_A{}^L(x), M_B{}^L(x)\} \geq \lambda_1, \inf\{M_A{}^U(x), M_B{}^U(x)\} \leq \lambda_2 \text{ and}$$
$$\inf\{N_A{}^L(x), N_B{}^L(x)\} \leq \theta_1, \sup\{N_A{}^U(x), N_B{}^U(x)\} \geq \theta_2$$
$$\Leftrightarrow \{M_A{}^L(x) \geq \lambda_1 \text{ or } M_B{}^L(x) \geq \lambda_1\} \text{ and } \{M_A{}^U(x) \leq \lambda_2 \text{ or } M_B{}^U(x) \leq \lambda_2\}$$
$$\text{and } \{N_A{}^L(x) \leq \theta_1 \text{ or } N_B{}^L(x) \leq \theta_1\} \text{ and } \{N_A{}^U(x) \geq \theta_2 \text{ or } N_B{}^U(x) \geq \theta_2\}$$
$$\Leftrightarrow \{M_A{}^L(x) \geq \lambda_1 \text{ and } M_A{}^U(x) \leq \lambda_2\} \text{ or } \{M_B{}^L(x) \geq \lambda_1 \text{ and } M_B{}^U(x) \leq \lambda_2\}$$
$$\text{or } \{M_A{}^L(x) \geq \lambda_1 \text{ and } M_B{}^U(x) \leq \lambda_2\} \text{ or } \{M_B{}^L(x) \geq \lambda_1 \text{ and } M_A{}^U(x) \leq \lambda_2\} \text{ and}$$
$$\{N_A{}^L(x) \leq \theta_1 \text{ and } N_A{}^U(x) \geq \theta_2\} \text{ or } \{N_B{}^L(x) \leq \theta_1 \text{ and } N_B{}^U(x) \geq \theta_2\}$$
$$\text{or } \{N_A{}^L(x) \leq \theta_1 \text{ and } N_B{}^U(x) \geq \theta_2\} \text{ or } \{N_B{}^L(x) \leq \theta_1 \text{ and } N_A{}^U(x) \geq \theta_2\}$$
$$\Leftrightarrow x \in [M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ or } x \in [M_B(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ or } \left\{x \in \left({M_A{}^L}_{\lambda_1} \cap {M_B{}^U}_{\lambda_2}\right)\right\}$$
$$\text{or } \left\{x \in \left({M_B{}^L}_{\lambda_1} \cap {M_A{}^U}_{\lambda_2}\right)\right\} \text{ and } x \in [N_A(x);\beta]_{[\theta_1,\theta_2;\beta]} \text{ or } x \in [N_B(x);\beta]_{[\theta_1,\theta_2;\beta]}$$
$$\text{or } \left\{x \in \left({N_A{}^L}_{\theta_1} \cap {N_B{}^U}_{\theta_2}\right)\right\} \text{ or } \left\{x \in \left({N_B{}^L}_{\theta_1} \cap {N_A{}^U}_{\theta_2}\right)\right\}$$
$$\Leftrightarrow x \in \left([M_A(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [M_B(x);\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left({M_A{}^L}_{\lambda_1} \cap {M_B{}^U}_{\lambda_2}\right)\right) \cup \left({M_B{}^L}_{\lambda_1} \cap {M_A{}^U}_{\lambda_2}\right) \quad \cap$$
$$\left([N_A(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup [N_B(x);\beta]_{[\theta_1,\theta_2;\beta]} \cup \left({N_A{}^L}_{\theta_1} \cap {N_B{}^U}_{\theta_2}\right) \cup \left({N_B{}^L}_{\theta_1} \cap {N_A{}^U}_{\theta_2}\right)\right)$$

iv. $x \in ([A;\alpha;\beta] \sqcap [B;\alpha;\beta])_{([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])}$ is given arbitrary.

$$\left(M_{([A;\alpha;\beta]\sqcap[B;\alpha;\beta])}(x), N_{([A;\alpha;\beta]\sqcap[B;\alpha;\beta])}(x)\right) \geq ([\lambda_1,\lambda_2;\alpha],[\theta_1,\theta_2;\beta])$$

$$\Leftrightarrow \begin{pmatrix} \begin{bmatrix} \inf\{M_A{}^L(x), M_B{}^L(x)\}, \\ \sup\{M_A{}^U(x), M_B{}^U(x)\}; \alpha \end{bmatrix}, \\ \begin{bmatrix} \sup\{N_A{}^L(x), N_B{}^L(x)\}, \\ \inf\{N_A{}^U(x), N_B{}^U(x)\}; \beta \end{bmatrix} \end{pmatrix} \geq ([\lambda_1, \lambda_2; \alpha], [\theta_1, \theta_2; \beta])$$

$$\Leftrightarrow \begin{bmatrix} \inf\{M_A{}^L(x), M_B{}^L(x)\}, \\ \sup\{M_A{}^U(x), M_B{}^U(x)\}; \alpha \end{bmatrix} \geq [\lambda_1, \lambda_2; \alpha]$$

$$\text{and } \begin{bmatrix} \sup\{N_A{}^L(x), N_B{}^L(x)\}, \\ \inf\{N_A{}^U(x), N_B{}^U(x)\}; \beta \end{bmatrix} \leq [\theta_1, \theta_2; \beta]$$

$$\Leftrightarrow \inf\{M_A{}^L(x), M_B{}^L(x)\} \geq \lambda_1 \text{ and } \sup\{M_A{}^U(x), M_B{}^U(x)\} \leq \lambda_2$$

$$\text{and } \sup\{N_A{}^L(x), N_B{}^L(x)\} \leq \theta_1 \text{ and } \inf\{N_A{}^U(x), N_B{}^U(x)\} \geq \theta_2$$

$$\Leftrightarrow \{M_A{}^L(x) \geq \lambda_1 \text{ and } M_B{}^L(x) \geq \lambda_1\} \text{ and } \{M_A{}^U(x) \leq \lambda_2 \text{ and } M_B{}^U(x) \leq \lambda_2\}$$

$$\text{and } \{N_A{}^L(x) \leq \theta_1 \text{ and } N_B{}^L(x) \leq \theta_1\} \text{ and } \{N_A{}^U(x) \geq \theta_2 \text{ and } N_B{}^U(x) \geq \theta_2\}$$

$$\Leftrightarrow \{M_A{}^L(x) \geq \lambda_1 \text{ and } M_A{}^U(x) \leq \lambda_2\} \text{ and } \{M_B{}^L(x) \geq \lambda_1 \text{ and } M_B{}^U(x) \leq \lambda_2\}$$

$$\text{and } \{N_A{}^L(x) \leq \theta_1 \text{ and } N_A{}^U(x) \geq \theta_2\} \text{ and } \{N_B{}^L(x) \leq \theta_1 \text{ and } N_B{}^U(x) \geq \theta_2\}$$

$$\Leftrightarrow [M_A(x); \alpha] \geq [\lambda_1, \lambda_2; \alpha] \text{ and } [M_B(x); \alpha] \geq [\lambda_1, \lambda_2; \alpha]$$

$$\text{and } [N_A(x); \beta] \leq [\theta_1, \theta_2; \beta] \text{ and } [N_B(x); \beta] \leq [\theta_1, \theta_2; \beta]$$

$$\Leftrightarrow \{x \in [M_A(x); \alpha]_{[\lambda_1, \lambda_2; \alpha]} \text{ and } x \in [N_A(x); \beta]_{[\theta_1, \theta_2; \beta]}\}$$

$$\text{and } \{x \in [M_B(x); \alpha]_{[\lambda_1, \lambda_2; \alpha]} \text{ and } x \in [N_B(x); \beta]_{[\theta_1, \theta_2; \beta]}\}$$

$$\Leftrightarrow x \in [A; \alpha; \beta]_{([\lambda_1, \lambda_2; \alpha], [\theta_1, \theta_2; \beta])} \cap [B; \alpha; \beta]_{([\lambda_1, \lambda_2; \alpha], [\theta_1, \theta_2; \beta])}$$

Example 3: Let $X = \{a, b, c, d\}$.

$$A = \begin{Bmatrix} < a, [0.1, 0.6; 0.4], [0.1, 0.4; 0.3] >, < b, [0.2, 0.5; 0.4], [0.2, 0.4; 0.3] >, \\ < c, [0.3, 0.6; 0.4], [0.1, 0.3; 0.3] >, < d, [0.4, 0.6; 0.4], [0.1, 0.3; 0.3] > \end{Bmatrix},$$

For $\alpha = 0.4$ and $\beta = 0.3$, $[A; \alpha; \beta]$ is (α, β)-interval valued intuitionistic fuzzy set.

i. $([0.0, 0.5; 0.4], [0.2, 0.4; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,

$$A_{([0.0, 0.5; 0.4], [0.2, 0.4; 0.3])} = \{b\}$$

ii. $([0.3, 0.6; 0.4], [0.1, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,

$$A_{([0.3, 0.6; 0.4], [0.1, 0.3; 0.3])} = \{c, d\}$$

iii. $([0.2, 0.7; 0.4], [0.2, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,

$$A_{([0.2, 0.7; 0.4], [0.2, 0.3; 0.3])} = \{b, c, d\}$$

iv. $([0.0, 0.7; 0.4], [0.3, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$,

$$A_{([0.0, 0.7; 0.4], [0.3, 0.3; 0.3])} = \{a, b, c, d\} = X$$

v. $([0.1, 0.4; 0.4], [0.0, 0.3; 0.3]) \in D(I_\alpha) \times D(I_\beta)$, $A_{([0.1, 0.4; 0.4], [0.0, 0.3; 0.3])} = \emptyset$

4. CONCLUSION

In this study, the definition of (α, β)-interval set is given. It is shown that (α, β)-interval set is lattice by giving of definitions of order relation, infimum and supremum on this set. Afterwards, the definition of negation function on this set is given by the help of negation function on crisp sets and fuzzy sets.

In terms of above definitions and information, the definition of (α, β)-interval valued intuitionistic fuzzy set is introduced. The definitions of intersection, union and complement on this set are introduced and the fundamental algebraic properties of this set are studied. In addition, the level subset of (α, β)-interval valued intuitionistic fuzzy set is given.

5. ACKNOWLEDGMENTS

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest.

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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