



Parameter Estimation for Pareto Distribution and Type-II Fuzzy Logic

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Abstract

While parameter estimation is done by the classical methods, there are a number of assumptions need to be satisfied, in the linear regression analysis. Key assumptions of linear regression are; no auto correlation, no or little multicollinearity, homoscedasticity and the errors have normal distribution. In this work, the case that independent variable has Pareto distribution to be discussed and an algorithm using adaptive networks suggested to parameter estimation where the k which is one of the parameters of the fuzzy membership functions is fuzzy. Also the parameter of fuzzy membership function is fuzzy the estimation process is based on type-II fuzzy logic.

1. INTRODUCTION

The first serious step in fuzzy set theory has been taken in an article published in 1965 by Lotfi A. Zadeh which is introduced fuzzy set theory in detail. Over the last 30 years, studies on the theory of fuzzy sets have been conducted extensively [6, 17]. When there is an uncertainty about the membership functions fuzzy set is called as type-II fuzzy set. We can say that type-II fuzzy logic is a generalization of conventional fuzzy logic (type-I) in the sense that uncertainty is not only limited to the linguistic variables but also is present in the definition of the membership functions [2].

Studies on type-II fuzzy clusters briefly summarized as follows:

Karnik and Mendel (1999), defined the uncertainty of the rules in a type-II fuzzy inference system that the rules are uncertain. They applied a type-II fuzzy logic system to time varying channel equalization.

Türkşen (1999), proposed and discussed in fuzzy system development schema. For both the type-I and type-II fuzzy theory, they described the extraction of fuzzy sets and fuzzy rules with the application of an improved fuzzy clustering technique which is essentially an unsupervised learning of the fuzzy sets and rules from a given input-output data set.

Mendel and John (2002), defined a new representation theorem of type-II fuzzy sets and introduced formulas for the union, intersection, and complement for type-II fuzzy sets.

Mendel (2007), examined questions, such as "What is a type-II fuzzy set", "What is it different from a type-I fuzzy set", "the importance of definition of type-II fuzzy sets", "How and why are type-II fuzzy sets used in rule-based systems" and "How are the detailed computations for an interval type-II fuzzy logic system" in study titled an introduction to type-II fuzzy sets and systems.

Garg and Sharma (2012), developed a two-phase approach by taking the advantages of one of the evolutionary algorithms, namely the particle swarm optimization (PSO) for getting the global values of the distribution parameters.

Garg, Rani, Sharma and Vishwakarma (2014), presents a methodology for solving the multi-objective reliability optimization model in which parameters are considered as imprecise in terms of triangular interval data.

Garg (2016), studied the basic arithmetic operations for two generalized positive parabolic fuzzy numbers by using the concept of the distribution and complementary distribution functions.

Singh and Garg (2016), proposed a concept of type-II intuitionistic fuzzy set and hence under this environment, a family of distance measures based on Hamming, Euclidean and Hausdorff metrics are presented. Some of its desirable properties have also been investigated in details.

Almost all studies on parameter estimation in the literature realized under the condition of the data has normal distribution. In the real world problems normal distribution condition for independent variables may not achieved. Therefore, in this study, the case where the independent variable has the Pareto distribution is considered and the membership function which is obtained for Pareto distribution in this study is used for parameter estimation. Also, results are compare with the prediction values obtained from least square estimates.

Remainder of this paper is organized as follows. In the second part of the study will be definitions of type-II fuzzy logic method. In the third part, the membership function suitable for Pareto distribution is obtained. In the fourth part, an algorithm used fuzzy adaptive network based fuzzy inference system will be suggested to prediction of the unknown parameter of the regression model in the case of independent variable characterized by a Pareto membership function. A numerical application examining the work and validity of the suggested algorithm in the fifty part and in the last part a discussion and conclusion are provided. In section five a numerical application examining to show the validity of the suggested algorithm and in the last part the errors of the models obtained suggested adaptive network and the errors from least square method (LSM) are compared.

2. BASIC CONCEPTS OF TYPE-II FUZZY LOGIC

Let X be a classical set of objects, called the universe, whose generic elements are denoted by x . The membership in a crisp subset of $\mu(x)$ is often viewed as characteristic function μ_A from X to $\{0, 1\}$. When a set is a classical set, its membership function can take on only two values 0 and 1. If the valuation set is allowed to be the real interval $[0, 1]$, A is called a fuzzy set proposed by Zadeh [12, 19]. Fuzzy sets may be viewed as an extension and generalization of the basic concepts of classical sets [8]. If X is a collection of objects denoted generically by x , then a "fuzzy set" A in X is defined as a set of ordered pairs:

$$A = \{(x, \mu_A(x)|x \in X)\} \quad (2.1)$$

where $\mu_A(x)$ is called "membership function" for the fuzzy set A defined as $\mu_A(x): X \rightarrow [0,1]$. $\mu_A(x)$ indicates the degree of membership of x in X and its value lies between zero and one [2, 8, 12].

Type-II fuzzy systems are consist of fuzzy if-then rules that are includes type-II fuzzy sets. Basically, a type-II fuzzy set is a set in which we also have uncertainty about the membership function. We can say that type-II fuzzy logic is a generalization of traditional fuzzy logic (type- I). Uncertainty is not on the limited to the linguistic variable but also is present in the definition of the membership function [1, 2]. The concept of a type-II fuzzy set was introduced by Zadeh in 1975 as an extension of concept of an ordinary fuzzy set. A type-II fuzzy set is characterized by a fuzzy membership function, the membership degree of each element of this set is in $[0, 1]$. In this sense, differs from type-I fuzzy sets, because the degree of membership in the type-I fuzzy set is a crisp number in range of $[0, 1]$. Such sets can be used in situations where there is uncertainty about the membership degree and uncertainty in the shape of the membership function or in some of its parameters. The membership of an element in a set cannot determine as 0 or 1, type-I fuzzy set is used. Similarly, when the situation is so fuzzy that we have trouble determining the membership degree even as a crisp number in $[0, 1]$, fuzzy sets of type-II is used. In many real-world problems the exact form of the membership degree may not be identified. In type-II fuzzy sets the membership degrees are obtained via membership function like in type-I fuzzy sets.

Assuming that the fuzzy set has normal distribution, membership function is defined as;

$$\mu(x) = \exp\left\{-\left[\frac{x-m}{\sigma}\right]^2\right\} \quad (2.2)$$

where the parameter set of Normal Distribution are $\{m, \sigma\}$ and this parameters are crisp number. If the fuzzy set characterized by normal membership function with standard deviation σ and mean can take values in $[m_1, m_2]$, the membership function is defined as;

$$\mu(x) = \exp\left\{-\left[\frac{x-m}{\sigma}\right]^2\right\}; m \in [m_1, m_2], \tag{2.3}$$

and in this case $\mu(x)$ is a fuzzy set [2].

3. DETERMINATION OF THE OPTIMAL MEMBERSHIP FUNCTION FOR PARETO DISTRIBUTION

Membership functions may vary according to the structure of discussed problems. However, there are a number of common features of the membership functions; membership functions are continuous functions and converts the interval $[a, b]$ to interval $[0, 1]$ by aid of membership function $\mu(x)$. A membership function should provide the given conditions below for being an optimal membership function:

$$E\{\mu(x) \mid x \text{ is distributed according to the underlying probability density function}\} \geq 0,$$

$$0 \leq \mu(x) \leq 1,$$

$\int \mu^2(x) d(x)$ should be minimized.

Under these conditions optimal membership function is given in the form;

$$\mu(x) = \begin{cases} \lambda p(x) & \text{if } \lambda p(x) < 1 \\ 1 & \text{if } \lambda p(x) \geq 1 \end{cases} \tag{3.1}$$

Here,

$p(x)$: probability density function,

λ : constant [4].

In the given membership function the form of $p(x)$ is determined as the probability density function related to the interested distribution. However, the fixed element λ can be obtained by solving the following problem;

$$P: \min_{\mu} f(\mu) = \frac{1}{2} \int_{-\infty}^{+\infty} \mu^2(x) d(x),$$

$$G(\mu) = c - E\{\mu\} = c - \int_{-\infty}^{+\infty} \mu(x)p(x) d(x) \leq 0, \tag{3.2}$$

$$\mu \in \Omega = \{\mu(x) \mid 0 \leq \mu(x) \leq 1\}.$$

The problem P formed with the conditions described for optimal membership function and this problem is can be solved with the method of Lagrange multipliers for obtaining the fixed element λ . For this, the Lagrange function is written

$$L(\mu, \lambda) = \frac{1}{2} \int_{-\infty}^{+\infty} \mu^2(x) d(x) + \lambda \left\{ c - \int_{-\infty}^{+\infty} \mu(x)p(x) d(x) \right\}. \tag{3.3}$$

Where Lagrange multiplier $\lambda \geq 0$ and constant $c < 1$. When the membership function values given in Eq. (3.1) are insert into Eq. (3.3), we can obtain the two forms of Lagrange function. First one is obtain from $(\lambda p(x)) \leq 1$ and the other one is obtain from $(\lambda p(x)) > 1$. However, if $(\lambda p(x)) > 1$, then the solution of the Lagrange function is independent from λ . Thus, the stage of $(\lambda p(x)) \leq 1$ is hold on and;

$$\text{if } (\lambda p(x)) \leq 1 \Rightarrow \mu(x) = \lambda p(x).$$

In this stage, the Lagrange functions for the determining of λ is obtained as

$$L = \frac{1}{2} \int_{-\infty}^{+\infty} \lambda^2 p^2(x) d(x) - \lambda \int_{-\infty}^{+\infty} \lambda p^2(x) d(x) + \lambda c,$$

$$L = \frac{1}{2} \int_{-\infty}^{+\infty} \lambda^2 p^2(x) d(x) - \int_{-\infty}^{+\infty} \lambda^2 p^2(x) d(x) + \lambda c$$

$$L = -\frac{1}{2} \int_{-\infty}^{+\infty} \lambda^2 p^2(x) d(x) + \lambda c.$$

When this function's derivative is taken with respect to the constant element λ and is equalized to zero, the equation

$$\frac{\partial L}{\partial \lambda} = -\lambda \int_{-\infty}^{+\infty} P^2(x) d(x) + c \quad (3.4)$$

is obtained. With the solution of Eq. (3.4) the parameter λ is obtained as,

$$\lambda = \frac{c}{\int_{-\infty}^{+\infty} p^2(x) d(x)} \quad (3.5)$$

If X is a random variable from Pareto distribution its probability density function is

$$f(x; k, x_m) = k \frac{x_m^k}{x^{k+1}} \quad x \geq x_m. \quad (3.6)$$

Where x_m is the (necessarily positive) minimum possible value of X , and k is a positive parameter. The family of Pareto distributions is parameterized by two quantities, x_m and k . When this distribution is used to model the distribution of wealth, then the parameter k is called the Pareto index [5, 16, 20].

To obtain the optimal membership function suitable for Pareto distribution, firstly the λ parameter is determined. For this, the probability density function given by Eq. (3.6) is inserted into Eq. (3.5), then the fixed element λ is obtained as

$$\lambda = \frac{c}{\int_{x_m}^{+\infty} p^2(x) d(x)} = \frac{c}{\int_{x_m}^{+\infty} \left[k \frac{x_m^k}{x^{k+1}} \right]^2 d(x)} = \frac{c}{k^2 x_m^{2k} \int_{x_m}^{+\infty} \frac{1}{x^{2(k+1)}} d(x)}$$

$$\lambda = c \frac{2(k+1)x_m}{k^2} \quad (3.7)$$

The membership functions suitable for Pareto distribution is obtained using the λ parameter given by Eq. (3.7) as follows;

$$\mu(x) = \lambda p(x) = c \frac{2(k+1)x_m}{k^2} k \frac{x_m^k}{x^{k+1}} = c \frac{2(k+1)}{k} \left(\frac{x_m}{x} \right)^{k+1} \quad (3.8)$$

Consider the fuzzy set characterized by Pareto membership function with x_m is constant and Pareto index k can take values in $[k_1, k_2]$, the membership function is defined as

$$\mu(x) = c \frac{2(k+1)}{k} \left(\frac{x_m}{x} \right)^{k+1} \quad k \in [k_1, k_2] \quad (3.9)$$

and, in this case, obtain a different membership function curve corresponding to each value of k . Thus, each element of X is takes different membership degrees based values of k .

In this study, the unknown parameters of regression model will be obtained in the event of the one of the independent variable is fuzzy set this characterized by Pareto membership function and Pareto index of the membership function is a fuzzy number like as $k \in [k_1, k_2]$.

4. AN ALGORITHM FOR PARAMETER ESTIMATION BASED FUZZY INFERENCE SYSTEM

Fuzzy adaptive network is a structure that allows the use fuzzy inference system to prediction the unknown parameters of fuzzy regression analysis. An adaptive network is a multilayer feed forward neural network. A good approach can be obtained for the regression function by using to learning algorithms and update approaches which are developed for fuzzy neural networks. Used for obtaining a good approach to regression functions and formed via neural and connections, such an adaptive network consist of five layers [9]. Each node in the first layer is produced the membership function based linguistic input, so the output of this layer is membership function. The nodes in second layer are produced w^L weight based input signals

and number of nodes in this layer is equal to number of nodes combination which are located in sub group of first layer. Third layer includes normalization function of output from second layer. Each node in fourth layer corresponds to the result of if-then rule. Finally, the fifth layer is weighed sum of all output from fourth layer [3]. The process of determining parameters of regression model begins with determining class numbers of independent variables and a priori parameters. The priori parameters are characterized the distribution. In this work since the independent variables come from Pareto distribution, we are interested in parameters related to Pareto membership function. In the case of independent variables come from Pareto distribution, the algorithm to obtain the unknown parameter of regression model is determined as follows.

STEP 1: Class numbers related to the data set associated with the independent variables are determined intuitively.

STEP 2: Priori parameters are determined.

STEP 3: Weights w^L are counted, which are then used to form matrix B , to be used in forming the posteriori parameter set. \bar{w}^L weights are outputs from the third layer of the adaptive network and calculated using the membership function of the distribution family of the independent variable. When independent variable numbers are indicated with p and if the fuzzy class number associated with each variable is indicated by $l_i; i = 1, \dots, p$ the fuzzy rule number is indicated by $L = \prod_{i=1}^p l_i$. The neural functions in the first layer of adaptive network are defined by membership function belonging to distribution of independent variable, as follows;

$$f_{1,h} = \mu_{F_h}(x_i).$$

Different membership function can be defined for F_h . Since the independent variable from Pareto distribution with priori parameter set $\{x_{m_h}, k_h\}$ the membership function is defined as

$$\mu(x) = c \frac{2(k+1)}{k} \left(\frac{x_m}{x}\right)^{k+1}.$$

Here k is a fuzzy parameter and takes values in the range of $k \in [k_1, k_2]$. Membership degrees for the independent variables are determinate from the defined membership function. w^L weights are obtained from the multiplication of these membership degrees and defined as

$$w^L = \mu_{F_L}(x_i) \cdot \mu_{F_L}(x_j).$$

\bar{w}^L weights are normalization of the w^L and determined by $\bar{w}^L = \frac{w^L}{\sum_{L=1}^m w^L}$.

STEP 4: When the one of the priori parameter k is a fuzzy number, the posterior parameter set c_i^L which is the unknown coefficients of regression model obtained as a fuzzy number shape of $c_i^L = (a_i^L, b_i^L)$ ($i = 1, \dots, p$). Under this condition, the equality $Z = (B^T B)^{-1} B^T Y$ is used for determining the a posteriori parameter set.

Here B and Y defined as

$$B = \begin{bmatrix} \bar{w}_1^1 & \dots & \bar{w}_1^m, & \bar{w}_1^1 x_{11} & \dots & \bar{w}_1^m x_{11}, & \dots & , & \bar{w}_1^1 x_{p1} & \dots & \bar{w}_1^m x_{p1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \bar{w}_k^l x_{jk} & \vdots & \ddots & \vdots & \\ \bar{w}_n^1 & \dots & \bar{w}_n^m, & \bar{w}_n^1 x_{1n} & \dots & \bar{w}_n^m x_{1n}, & \dots & , & \bar{w}_n^1 x_{pn} & \dots & \bar{w}_n^m x_{pn}, \end{bmatrix}$$

$$Y = [y_1, y_2, \dots, y_n]^T$$

STEP 5: By using the posteriori parameter set $c_i^L = (a_i^L, b_i^L)$ obtained in Step 4, the regression model indicated by

$$Y^L = c_0^L + c_1^L x_1 + c_2^L x_2 + \dots + c_p^L x_p$$

Setting out from the models and weights specified in Step 3, the prediction values are obtained using $\hat{Y} = \sum_{L=1}^m \bar{w}^L Y^L$.

STEP 6: Error related to the model is measured as $\varepsilon = \frac{\sum_{k=1}^n (Y_k - \hat{Y}_k)^2}{n}$. If $\varepsilon < \emptyset$, then the posteriori parameter has been obtained as parameters of regression models to be formed, the process is determined. If $\varepsilon \geq \emptyset$, then Step 6 begins. Here \emptyset is a law stable value determined by decision maker.

STEP 7: Priori parameters specified in Step 2, are updated.

STEP 8: Predictions for each priori parameter obtained by change and error criterion related to these predictions are counted. The lowest of error criterion is defined. Priori parameters giving the lowest error specified, and prediction obtained via the models related to these parameters is taken as output.

5. NUMERICAL EXAMPLE

Table 1. Predictions and Error Values for Data Set

Observed Number	X_1	X_2	Y	$\hat{Y}_{(Network)_i}$	$e_{(Network)_i}$	$\hat{Y}_{(LSM)_i}$	$e_{(LSM)_i}$
1	7,08	0,00978	17,1510	(30.3686;3.0461)	-13.2176	37.6245	-20.4735
2	7,27	0,00986	24,2520	(31.6734;3.2014)	-7.4214	37.6096	-13.3576
3	5,83	0,00751	40,9530	(35.4948;3.6214)	5.4582	40.5707	0.3823
4	4,84	0,00604	31,7880	(37.0639;3.9012)	-5.2759	42.3774	-10.5894
5	7,92	0,00609	40,6780	(37.2324;3.7522)	3.4456	42.3246	-1.6466
6	6,33	0,00748	41,0380	(37.4618;3.8210)	3.5762	40.7614	0.2766
7	5,37	0,00716	45,4170	(37.9336;3.7241)	7.4834	41.1616	4.2554
8	4,77	0,00618	27,9600	(38.6279;3.9240)	-10.6679	42.3289	-14.3689
9	5,54	0,00589	38,7620	(38.3834;3.8595)	0.3786	42.6363	-3.8743
10	8,03	0,00764	36,6670	(36.3231;3.6012)	0.3439	40.4830	-3.8160
11	5,94	0,00792	43,4360	(38.1646;3.8624)	5.2714	40.3231	3.1129
12	6,48	0,00749	43,0870	(38.5354;3.9425)	4.5516	40.8490	2.2380
13	5,83	0,00717	44,2140	(38.7200;3.9025)	5.4940	41.2239	2.9901
14	6,82	0,00623	41,5770	(39.4323;3.9525)	2.1447	42.3446	-0.7676
15	7,54	0,00770	44,6000	(38.2569;3.7895)	6.3431	40.5768	4.0232
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665	8,09	0,01639	30,9170	(30.7690;3.0865)	0.1480	32.2422	-1.3252
666	6,43	0,01417	42,9090	(33.7263;3.3565)	9.1827	34.9594	7.9496
667	9,40	0,01155	45,6820	(37.1599;3.6954)	8.5221	38.1000	7.5820
668	5,39	0,01127	24,3580	(37.5565;3.7556)	-13.1985	38.4581	-14.1001
669	7,66	0,00992	35,7490	(39.4029;3.9568)	-3.6539	40.1182	-4.3692
670	9,99	0,00701	47,4810	(43.0997;4.5624)	4.3813	43.5641	3.9169
671	9,78	0,00586	45,5350	(44.7210;4.4986)	0.8140	45.0009	0.5341
672	9,46	0,00560	53,3880	(45.1159;4.5565)	8.2721	45.3344	8.0536
ERROR				$\varepsilon_{Network} = 40.2540$		$\varepsilon_{LSM} = 41.1848$	

Data set used in the application is consists of two independent variables and one dependent variable. This set includes 672 observations and located in the Table 1. In addition, the predictions and errors are located in this table.

According to the Kolmogorov-Smirnov goodness-of-fit test, independent variable X_2 has Pareto distribution with $(k = 2.75, x_m = 0.005)$ parameters. The calculated values and the table values of the Kolmogorov-Smirnov test are shown in Table 2.

Table 2. Goodness-of-fit to Pareto Distribution

n	k	K-S (calculated value)	K-S (table)
672	2.75	0.0386	<0.0629

Applying least squares, the estimated regression model according to the data set given in Table 1 is obtained as;

$$Y = 48.4102 + 0.000036X_1 - 1029.4453X_2$$

and the algorithm proposed in section four was counted with a program written in MATLAB. From the program, the regression models based fuzzy inference systems are as follows

$$\hat{Y}_1 = (736824.9045; 86,699) + (549.6362; 0.0642)X_1 + (9026.4144; 0.2733)X_2$$

$$\hat{Y}_2 = (-807771.2970 ; 95,0552) + (538.5523; 0.0637)X_1 - (11172.1242; 0.2665)X_2$$

6. CONCLUSIONS

The independent variable X_2 comes from a Pareto distribution, and regression models are formed using membership functions that are appropriate to the Pareto distribution. Since the parameter k located in Pareto membership function is fuzzy parameter and takes values in the range $k \in [k_1, k_2]$ the unknown parameters of regression model are obtained as fuzzy numbers. The prediction values obtain from adaptive network and the prediction values obtained from least square estimates are compared. According to the indicated error criterion, the errors related to the predictions that are obtained from the network are less than errors obtained from the least square estimates.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors

REFERENCES

- [1] Aisbett, J., Rickart, J. T., Morgenthaler, D., Multivariate modeling and type-II fuzzysets, Fuzzy Sets and Systems 163: 78-95 (2011).
- [2] Castillo, O., Melin, P. Type-II fuzzylogic: theory and applications, Sipringer, (2008).
- [3] Cheng, C.B., Lee, E.S., Switching regression analysis by fuzzy adaptive network, European Journal of Operational Research 128: 647-668, (2001).
- [4] Civanlar , M.R., Trussell, H.J., Constructing membership functions using statistical data fuzzy sets and systems, 18: 1-13(1986).
- [5] Cramer, H.,Mathematical methods of statistics, Princeton University Press, (1963).

- [6] Garg, H., Sharma, S. P., A two-phase approach for reliability and maintainability analysis of an industrial system, *International Journal of Reliability, Quality and Safety Engineering*, 19: 1250013 (19 pages) (2012).
- [7] Garg, H., Rani, M., Sharma, S.P., Vishwakarma, Y., Intuitionistic fuzzy optimization technique for solving multi-objective reliability optimization problems in interval environment, *Expert Systems with Applications* 41: 3157–3167 (2014).
- [8] Garg, H., Arithmetic Operations on Generalized Parabolic Fuzzy Numbers and Its Application, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* DOI 10.1007/s40010-016-0278-9, Springer, (2016).
- [9] Hisao, I., Manabu, N., Fuzzy regression using asymmetric fuzzy coefficients and fuzzied neural networks, *Fuzzy Sets and Systems* 119: 273-290 (2001).
- [10] Karnik N.K., Mendel, J.M., Type-II Fuzzy logic systems, *IEEE Transaction on Fuzzy Systems* 7: 643-658(1999).
- [11] Karnik N.K., Mendel, J.M., Operations on type-II fuzzy sets, *Fuzzy Sets and Systems* 122: 327-348 (2002).
- [12] Lai, Y.Y and Hwang C.L., *Fuzzy mathematical programming methods and application*, Springer-Verlag (1992).
- [13] Mendel, J.M., John, R.I.B., Type-II fuzzy sets made simple, *IEEE Transaction on Fuzzy Systems* 10: 117-127 (2002).
- [14] Mendel, J. M., Type-2 fuzzy sets and systems: An overview, *IEEE Computational Intelligence Magazine* February 21-29 (2007).
- [15] Mendel, J. M., Advances in type-II fuzzy sets and systems, *Information Sciences* 177: 84-110 (2007).
- [16] Meyer, P.L., *Introductory probability and statistical applications*, Second Edition, Addison-Wesley Publishing Company, USA, (1970).
- [17] Singh, S., Garg, H., Distance measures between type-II intuitionistic fuzzy sets and their application to multicriteria decision-making process, DOI 10.1007/s10489-016-0869-9, Springer Science+Business Media New York, Springer, (2016).
- [18] Türkşen, I.B., Type-I and Type-II fuzzy systems modelling, *Fuzzy Sets and Systems* 106: 11-34 (1999).
- [19] Zadeh, L.A., Fuzzy sets, *Information and Control* 8: 338-353 (1965).
- [20] Zisheng O., Chi, X., Generalized Pareto distribution fit to medical insurance claim data, *Appl. Math. J. Chinese Univ. Ser. B.* 21: 21-29 (2006).