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Chen-like Inequalities on Submanifolds of Cosymplectic 3-Space Forms

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Abstract

In this paper, some equalities and inequalities involving the Riemannian curvature invariants are obtained on 3-semi slant submanifolds of cosymplectic 3-space forms. Obtained relations for 3-semi slant submanifolds are examined on 3-slant, invariant, and totally real submanifolds.

Keywords: Curvature; Submanifold; Cosymplectic 3-Space Form.

Kosimplektik 3-Uzay Formlarının Altmanifoldları Üzerinde Chen-tipi Eşitsizlikler

Öz

Bu çalışmada kosimplektik 3-uzay formlarının 3-semi slant altmanifoldları üzerine Riemann eğrilik invaryantları içeren bazı eşitlik ve eşitsizlikler elde edilmiştir. 3-semi slant alt manifoldlar için elde edilen bağıntılar, 3-slant, invaryant ve total reel altmanifoldlar üzerinde incelenmiştir.

Anahtar Kelimeler: Eğrilik; Altmanifold; Kosimplektik 3-Uzay Form.

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1. Introduction

The concept of contact $3-$manifolds was originated by Y. Kuo [1] and C. Udrişte [2], independently. With the introduction of this concept, some classifications of contact $3-$manifolds were presented by many authors. For mathematical and physical applications of contact $3-$manifolds, we refer to [3-9], etc.

After the definition of Chen's slant submanifolds (cf. [10]), the problem of studying the geometry of slant submanifolds attracted a lot of attention. From this viewpoint, these submanifolds of almost contact metric $3-$manifolds were investigated by Malek and Balgeshir in [11, 12].

In the submanifold theory, the problem of finding basic relationships between curvature invariants is one of the most basic and interesting problems. In order to compare the curvature invariants of a Riemannian manifold and its submanifold, several inequalities were established by Chen [13-16], etc. Later, this problem has been studied by many authors in various submanifolds [17-24], etc.

In the first section of this study, some main formulas and notations for a Riemannian manifold and its submanifolds are expressed. In the second section, the definitions of contact $3-$manifolds and their submanifolds are given. An example of $3-$semi-slant submanifolds is presented. In the third section, some relations involving Ricci curvatures of cosymplectic $3-$space forms and their $3-$semi-slant, $3-$slant, invariant, and totally real submanifolds are examined. In the fourth section, some relations involving scalar curvatures and sectional curvatures of cosymplectic $3-$space forms and their $3-$semi-slant, $3-$slant, invariant and totally real submanifolds are obtained.

2. Preliminaries

Let $(\tilde{M},\tilde{g})$ be a $m-$dimensional Riemannian manifold. The sectional curvature of $\Pi=\mathrm{Span}\{Y,Z\}$ is formulated by

$$\tilde{K}(Y\wedge Z)=\frac{\tilde{g}(\tilde{R}(Y,Z)Z,Y)}{\tilde{g}(Y,Y)\tilde{g}(Z,Z)-\tilde{g}(Y,Z)^2},$$

where $\tilde{R}$ is the Riemannian curvature tensor field of $(\tilde{M},\tilde{g})$. Let $\{e_1,e_2,\ldots,e_m\}$ be an ortho-normal basis of $T_p\tilde{M}$ at $p\in\tilde{M}$. The Ricci curvature for $e_l, l\in\{1,2,\ldots,m\}$ is formulated by

$$\tilde{R}ic(e_l)=\sum_{j\neq l}^{m}\tilde{K}(e_l\wedge e_j) \tag{1}$$

and the scalar curvature at a point $p\in\tilde{M}$ is defined by

$$\tilde{\tau}(p)=\sum_{1\leq l<j\leq m}\tilde{K}(e_l\wedge e_j). \tag{2}$$

Let Π_n be an $n-$dimensional subsection of $T_p\tilde{M}$. If $n=m$, $\Pi_m=T_p\tilde{M}$. Let us choose an orthonormal basis $\{e_1,e_2,\ldots,e_n\}$ of Π_n. Then $n-$Ricci curvature of e_t, $t\in\{1,2,\ldots,n\}$, is formulated by

$$\tilde{R}ic_{\Pi_n}(e_t)=\sum_{j\neq t}^{n}\tilde{K}(e_t\wedge e_j) \tag{3}$$

and $n-$scalar curvature of Π_n is formulated by

$$\tilde{\tau}_{\Pi_n}(p)=\sum_{1\leq l<j\leq n}\tilde{K}(e_l\wedge e_j). \tag{4}$$

We note that if $n=m$, then $\tilde{R}ic_{\Pi_n}(e_t)=\tilde{R}ic_{T_p\tilde{M}}(e_t)$ and $\tilde{\tau}_{\Pi_n}(p)=\tilde{\tau}_{T_p\tilde{M}}(p)$.

Assume that (M,g) is a $k-$dimensional submanifold of $(\tilde{M},\tilde{g})$. The Gauss and Weingarten formulas are formulated by

$$\nabla_X Y=\nabla_X Y+\sigma(X,Y) \tag{5}$$

and

$$\nabla_X Y=-A_N X+\nabla_X^{\perp}N, \tag{6}$$

where $X,Y\in T_pM$, N is a unit normal vector, $\nabla_X Y, A_N X\in T_pM$ and $\sigma(X,Y),\nabla_X^{\perp}N\in T_p^{\perp}M$. Here, σ is the second fundamental form, A_N is the shape operator and $\nabla^{\perp}$ is the normal connection of M. It is well known that σ is associated to A_N by the following formula:

$$\tilde{g}(\sigma(X,Y),N) = g(A_N X,Y). \tag{7}$$

Denote the Riemannian curvature tensor of M by R. The Gauss equation is formulated by

$$g(R(X,Y)Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W) + \tilde{g}(\sigma(X,W),\sigma(Y,Z)) - \tilde{g}(\sigma(X,Z),\sigma(Y,W)) \tag{8}$$

for any $X,Y,Z,W \in T_pM$.

Let $\{e_1, e_2, \ldots, e_k\}$ be an orthonormal basis of T_pM. The main curvature vector field $\hbar$ is formulated by

$$\hbar = \frac{1}{k}\sum_{l=1}^{k}\sigma\left(e_l, e_l\right). \tag{9}$$

M is said to be totally geodesic if $\sigma = 0$, and it is said to be minimal if $\hbar = 0$. M is totally umbilical if and only if $\sigma\left(X,Y\right) = g\left(X,Y\right)\hbar$ is satisfied for all $X,Y \in T_pM$.

Let $\{e_{k+1}, e_{k+2}, \ldots, e_m\}$ be an orthonormal basis of $T_p^{\perp}M$ and e_s belongs to $\{e_{k+1}, e_{k+2}, \ldots, e_m\}$. Denote the intrinsic sectional curvature by $K(e_l \wedge e_j)$. In view of (8), if we put

$$\sigma_{lj}^{s} = \tilde{g}(\sigma\left(e_l, e_j\right), e_s) \qquad \text{and} \qquad \|\sigma\|^2 = \sum_{l,j=1}^{k}\tilde{g}(\sigma(e_l,e_j),\sigma(e_l,e_j)), \tag{10}$$

then we find

$$K(e_l \wedge e_j) = \tilde{K}(e_l \wedge e_j) + \sum_{s=k+1}^{m}\left(\sigma_{ll}^{s}\sigma_{jj}^{s} - (\sigma_{lj}^{s})^2\right). \tag{11}$$

From (11), it follows that

$$2\tau(p) = 2\tilde{\tau}\left(T_pM\right) + n^2\|\hbar\|^2 - \|\sigma\|^2, \tag{12}$$

where

$$\tilde{\tau}\left(T_pM\right) = \sum_{1\le l<j\le k}\tilde{K}_{lj}.$$

Moreover, there exists the following relation:

$$\|\sigma\|^2 = \frac{1}{2}k^2\|\hbar\|^2 + \frac{1}{2}\sum_{s=k+1}^{m}(\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{kk}^s)^2 + 2\sum_{s=k+1}^{m}\sum_{j=2}^{k}(\sigma_{1j}^s)^2$$
$$-2\sum_{s=k+1}^{m}\sum_{2\le l<j\le k}(\sigma_{ll}^s\sigma_{jj}^s - (\sigma_{lj}^s)^2). \tag{13}$$

For the basic concepts dealing with Riemannian manifolds, we refer to [16].

The relative null space at a point p in M is given by [14]

$$N_p = \left\{X \in T_pM \middle| \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\right\}. \tag{14}$$

We note that N_p is also said to be the kernel of σ at p [25].

The Chen invariant δ_M for a Riemannian submanifold M is formulated by [26]

$$\delta_M(p) = \tau(p) - \inf(K)(p), \tag{15}$$

where $\inf(K)(p) = \inf\{K(\Pi) : \Pi \text{ is a plane}\}$.

3. Submanifolds of Contact 3-Space Forms

Definition 1. [1] A differentiable manifold $\tilde{M}$ admitting an almost contact $3-$structure $(\xi_l, \eta_l, \varphi_l)_{l\in\{1,2,3\}}$ is said to be an almost contact $3-$structure manifold. An almost contact $3-$structure manifold is denoted by $(\tilde{M}, \xi_l, \eta_l, \varphi_l)_{l\in\{1,2,3\}}$.

For $(\tilde{M}, \xi_l, \eta_l, \varphi_l)_{l\in\{1,2,3\}}$, the following relations hold:

$$\varphi_l\,\xi_j = -\varphi_j\xi_l = \xi_n, \quad \eta_l\varphi_j = -\eta_j\varphi_l = \eta_n, \quad \eta_l\xi_j = 0 \tag{16}$$

and

$$\varphi_l \circ \varphi_j - \eta_j \otimes \xi_l = -\varphi_j \circ \varphi_l + \eta_l \otimes \xi_j = \varphi_n, \tag{17}$$

where (l, j, n) is a cyclic permutation of $(1,2,3)$. If $(\tilde{M}, \xi_l, \eta_l, \varphi_l)_{l\in\{1,2,3\}}$ includes a Riemannian metric $\tilde{g}$ given by

$$\tilde{g}(\varphi_l Y,\varphi_l Z)=\tilde{g}(Y,Z)-\eta_l(Y)\eta_l(Z) \tag{18}$$

for any $Y,Z\in T_p\tilde{M}$, then $(\tilde{M},\tilde{g},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$ is said to be an almost contact metric $3-$ structure manifold. From the Eq. (18), we have

$$\tilde{g}(\varphi_l Y,Z)=-\tilde{g}(Y,\varphi_l Z). \tag{19}$$

$(\tilde{M},\tilde{g},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$ is called a cosymplectic $3-$ manifold if

$$\tilde{\nabla}\varphi_l=0 \tag{20}$$

is satisfied. It is said to be a Sasakian $3-$ manifold if

$$(\tilde{\nabla}_Y\varphi_l)Z=\tilde{g}(Y,Z)\xi_l-\eta_l(Z)Y \tag{21}$$

is provided.

In a similar manner to the concept of holomorphic sectional curvature on Hermitian or contact metric manifolds, we can state the concept of φ_l-holomorphic sectional curvature on $(\tilde{M},\tilde{g},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$ in such a way:

Definition 2. [11] A plane Π is said to be a φ_l-section if there exists a unit vector $X\in T_p\tilde{M}$ orthogonal to ξ_l, where $\{X,\varphi_l X\}$ is an orthonormal basis on Π for some $l\in\{1,2,3\}$. The φ_l-holomorphic sectional curvature of a φ_l-section is defined by

$$\tilde{K}(X\wedge\varphi_l X)=\tilde{g}(\tilde{R}(X,\varphi_l X)\varphi_l X,X).$$

A cosymplectic $3-$manifold $(\tilde{M},\tilde{g},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$ becomes a cosymplectic $3-$space form if it is of constant φ_l-holomorphic sectional curvature c. A cosymplectic $3-$space form is shown by $\tilde{M}(c)$.

If $\tilde{M}(c)$ is a cosymplectic $3-$space form, then the Riemannian curvature is satisfied the following relation [1]:

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = \frac{c}{4}\{ & g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ & + \sum_{n=1}^{3}\big[g(X,\varphi_n W)g(Y,\varphi_n Z) - g(X,\varphi_n Z)g(Y,\varphi_n W) \\ & - 2g(X,\varphi_n Y)g(Z,\varphi_n W) - g(X,W)\eta_n(Y)\eta_n(Z) \\ & + g(X,Z)\eta_n(Y)\eta_n(W) - g(Y,Z)\eta_n(X)\eta_n(W) \\ & + g(Y,W)\eta_n(X)\eta_n(Z)\big],\end{aligned}$$

(22)

for any $X,Y,Z,W \in \tilde{M}$.

Assume that (M,g) is a $k-$dimensional submanifold of $(\tilde{M},\tilde{g},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$. For any vector field X in T_pM, we can write $\varphi_l X$ as follows:

$$\varphi_l X = P_l X + F_l X, \tag{23}$$

where $P_l X \in T_pM$ and $F_l X \in T_p^{\perp}M$ for $l \in \{1,2,3\}$.

We can express the following:

$$\|P_l\|^2 = \sum_{j,n=1}^{k} g(P_l e_j, e_n)^2 \tag{24}$$

and

$$\|P_l X\|^2 = \sum_{n=1}^{k} g(P_l X, e_n)^2. \tag{25}$$

(M,g) is said to be invariant if $F_l = 0$ and it is said to be totally real if $P_l = 0$ for each $l \in \{1,2,3\}$. Furthermore, (M,g) becomes $3-$slant if for each $l \in \{1,2,3\}$, the angle θ between $\varphi_l X$ and the tangent space T_pM is constant for every p in M and every $X \neq 0$ which is not linearly dependent by ξ_l [12].

We remark that a $3-$slant submanifold becomes invariant when $\theta = 0$ and it becomes totally real if $\theta = \frac{\pi}{2}$. Furthermore, the following classification could be stated:

Definition 3. [12] A submanifold (M,g) is said to be a $3-$semi-slant submanifold if we have three orthogonal distributions D_1, D_2, D_3, where $D_3=\mathrm{Span}\{\xi_1,\xi_2,\xi_3\}$ and the following cases occur:

i) $TM=D_1\oplus D_2\oplus D_3$,

ii) $\varphi_i(D_1)\subset D_1,\ \forall l\in\{1,2,3\}$,

iii) D_2 is $3-$slant with $\theta\neq 0$.

It is clear that (M,g) is $3-$slant if $D_1=0$ and it becomes an invariant submanifold if $\theta=0$.

Example 1. Let us consider $11-$dimensional Euclidean space E^{11}. If we define

$$\varphi_1((x_i)_{i\in\{1,\ldots,11\}})=(-x_2,x_1,-x_3,x_4,-x_7,-x_8,x_5,x_6,0,-x_{11},x_{10})$$
$$\varphi_2((x_i)_{i\in\{1,\ldots,11\}})=(-x_4,-x_3,x_1,x_2,-x_7,-x_8,x_5,x_6,x_{11},0,x_9),$$
$$\varphi_2((xi)_{i\in\{1,\ldots,11\}})=(x_2,-x_1,x_3,-x_4,-x_7,-x_8,x_5,x_6,-x_{10},x_9,0)$$

such that $\xi_1=\partial x_9$, $\xi_2=\partial x_{10}$, $\xi_3=\partial x_{11}$ and η_1, η_2, η_3 are duals of ξ_1, ξ_2, ξ_3, respectively. We find $(E^{11},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$ is an almost contact $3-$structure manifold.

Let us define the following submanifold of $(E^{11},\xi_l,\eta_l,\varphi_l)_{l\in\{1,2,3\}}$:

$$M=\{(u_1,u_2,u_3,u_4,u_5 cos\alpha,u_5 sin\alpha,u_6 cos\beta,u_6 sin\beta,u_7,u_8,u_9)\},$$

where $\alpha,\beta\in[0,\frac{\pi}{2})$. In this case, we obtain

$$Y_1=\partial x_1,\quad Y_2=\partial x_2,\quad Y_3=\partial x_3,\quad Y_4=\partial x_4,$$
$$Y_5=cos\alpha\ \partial x_5+sin\alpha\ \partial x_6,\quad Y_6=cos\beta\ \partial x_7+sin\beta\ \partial x_8,$$
$$\xi_1=\partial x_9,\quad \xi_2=\partial x_{10},\quad \xi_3=\partial x_{11}$$

and

$$N_1=-sin\alpha\ \partial x_5+cos\alpha\ \partial x_6,\quad N_2=-sin\beta\ \partial x_7+cos\beta\ \partial x_8,$$

where $T_pM = \mathrm{Span}\{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, \xi_1, \xi_2, \xi_3\}$, $T_p^{\perp}M = \mathrm{Span}\{N_1, N_2\}$ and $\{\partial x_1, \ldots, \partial x_{11}\}$ is the natural basis of E^{11}. If we put $\mathrm{D}_1 = \mathrm{Span}\{Y_1, Y_2, Y_3, Y_4\}$, $\mathrm{D}_2 = \mathrm{Span}\{Y_5, Y_6\}$ and $\mathrm{D}_3 = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, then M becomes $3-$semi invariant with $\theta = |\alpha - \beta|$.

4. Inequalities Involving Ricci Curvatures

Let us indicate the set of all unit vectors in T_pM by T_p^1M.

Theorem 1. [27] Let M be a $k-$dimensional submanifold of $(\tilde{M}, \tilde{g})$. The following cases hold:

i) For any $X \in T_p^1M$, we get

$$\mathrm{Ric}(X) \le \frac{1}{4}k^2 \|\hbar\|^2 + \tilde{R}ic_{T_pM}(X). \tag{26}$$

Here $\tilde{R}ic_{T_pM}(X)$ is the $k-$Ricci curvature of $X \in T_p^1M$.

ii) The equality case of (26) occurs for $X \in T_p^1M$ if and only if

$$\begin{cases} \sigma(X,Z) = 0, & \text{for each } Z \perp X, \\ 2\sigma(X,X) = k\hbar(p). \end{cases}$$

iii) The equality case of (26) occurs for each $X \in T_p{}^1M$ if and only if either p is a totally geodesic point or p is a totally umbilical point for $k = 2$.

From Theorem 1, we can state:

Corollary 1. [28] For any Riemannian submanifold, any two of the below three cases refer to the other one:

i) X satisfies the equality case of (26).

ii) $\hbar(p) = 0$.

iii) $X \in N_p$.

Now, we assume that $\{\xi_1, \xi_2, \xi_3\}$ is tangent to M and $X \in T_p^1 M$ throughout this paper.

Lemma 1. For any $k-$dimensional submanifold of $\tilde{M}(c)$. We find

$$\tilde{K}(e_l \wedge e_j) = \frac{c}{4}\left\{1 + \sum_{n=1}^{3}[3g(P_n e_l, e_j)^2 - \eta_n^2(e_j) - \eta_n^2(e_l)]\right\}, \tag{27}$$

$$\tilde{R}ic_{T_pM}(X) = \frac{c}{4}\left\{(n-4) + \sum_{n=1}^{3}[3\|P_n X\|^2 + (2-k)\eta_n^2(X)]\right\}, \tag{28}$$

$$\tilde{\tau}_{T_pM}(p) = \frac{c}{8}\left\{(k-1)(k-6) + 3\sum_{n=1}^{3}\|P_n\|^2\right\}. \tag{29}$$

Proof. From (22), we have

$$\begin{aligned}\tilde{g}(\tilde{R}(e_l, e_j)e_j, e_l) = \frac{c}{4}\{&g(e_l, e_l)g(e_j, e_j) - g(e_l, e_j)g(e_j, e_l)\\ &+\sum_{n=1}^{3}\Big[g(e_l, \varphi_n e_l)g(e_j, \varphi_n e_j) - g(e_l, \varphi_n e_j)g(e_j, \varphi_n e_l)\\ &-2g(e_l, \varphi_n e_j)g(e_j, \varphi_n e_l) - g(e_l, e_l)\eta_n(e_j)\eta_n(e_j)\\ &+g(e_l, e_j)\eta_n(e_j)\eta_n(e_l) - g(e_j, e_j)\eta_n(e_l)\eta_n(e_l)\\ &+g(e_j, e_l)\eta_n(e_l)\eta_n(e_j)\Big]\Big\},\end{aligned}$$

which is equivalent to (27). In view of (1) and (27), we find

$$\tilde{R}ic_{T_pM}(e_1) = \frac{c}{4}\left\{(k-1) + \sum_{n=1}^{3}\left[3\sum_{j=1}^{k}g(P_n e_1, e_j)^2 + (2-k)\sum_{j=1}^{k}\eta_n^2(e_1)\right]\right\}.$$

Putting $e_1 = X$ and using (25) in the last equation, we obtain (28). From (2) and (28), we get

$$\tilde{\tau}_{T_pM}(p) = \frac{c}{8}\left\{k(k-4) + \sum_{l=1}^{k}\sum_{n=1}^{3}\Big[3\|P_n e_l\|^2 + (2-k)\eta_n^2(e_l)\Big]\right\}.$$

Considering (24) in the last equation, we obtain (29).

In view of Theorem 1 and (28), we obtain

Theorem 2. For any $k-$dimensional submanifold of $\tilde{M}(c)$, we have the following cases:

i) For any $X \in T_p^1 M$, we get

$$Ric(X) \le \frac{1}{4}k^2 \|\hbar\|^2 + \frac{c}{4}\left\{(k-4) + \sum_{n=1}^{3}\left[3\|P_n X\|^2 + (2-k)\eta_n^2(X)\right]\right\}. \tag{30}$$

ii) The equality case of (30) occurs for $X \in T_p^1 M$ if and only if

$$\begin{cases} \sigma(X,Z) = 0, & \text{for each } Z \perp X, \\ \sigma(X,X) = \frac{k}{2}\hbar(p). \end{cases}$$

iii) The equality case of (30) occurs for each $X \in T_p^1 M$ if and only if p is a totally geodesic point.

From Theorem 2, we immediately have

Corollary 3. For $k-$dimensional submanifold of $\tilde{M}(c)$, any two of the below three cases refer to the other one:

i) X satisfies the equality case of (30).

ii) $\hbar(p) = 0$.

iii) $X \in N_p$.

Definition 4. Let D be a distribution on M.

i) If $\sigma(X,Z) = 0$ is satisfied for all $X, Z \in \mathrm{D}$, then M is said to be $\mathrm{D}-$geodesic.

ii) If there exists a smooth function λ on M satisfying $\sigma(X,Z) = \lambda g(X,Z)$ for each $X, Z \in \mathrm{D}$, then M is called $\mathrm{D}-$umbilical.

Theorem 3. For any $k-$dimensional $3-$semi-slant submanifold, the following cases occur:

i) For every unit $X \in \mathrm{D}_1$, we get

$$\mathrm{Ric}(X) \le \frac{1}{4}k^2 \|\hbar\|^2 + \frac{c}{4}(k+5). \tag{31}$$

ii) The equality case of (31) is true for each $X \in \mathrm{D}_1$ at $p \in M$ if and only if M is D_1- geodesic.

iii) For every unit $Y \in \mathrm{D}_2$, we get

$$\mathrm{Ric}(Y) \leq \frac{1}{4}k^2\|\hbar\|^2 + \frac{c}{4}\{(k-4)+9\cos^2\theta\}. \tag{32}$$

iv) The equality case of (32) is true for all $X \in \mathrm{D}_2$ at $p \in M$ if and only if M is D_2- geodesic.

Proof. If $X \in \mathrm{D}_1$, we obtain

$$\|P_n X\|^2 = 1,\ \eta_n(X) = 0 \ \text{ and } \ \sum_{n=1}^{3}\sum_{j=1}^{k}\eta_n(e_j) = 3.$$

Using these facts in (28), we obtain (31). The equality case of (31) occurs for each $X \in \mathrm{D}_1$ if and only if $\sigma(X,Z)=0$ for all $X, Z \in \mathrm{D}_1$. This implies that M is D_1-geodesic.

If X belongs to D_1, we obtain

$$\sum_{n=1}^{3}\|P_n X\|^2 = 3\cos^2\theta,\ \eta_n(X) = 0 \ \text{ and } \ \sum_{n=1}^{3}\sum_{j=1}^{k}\eta_n(e_j) = 3.$$

Using these facts in (29), we obtain (32). The equality case of (32) occurs for each $Y \in \mathrm{D}_2$ if and only if $\sigma(Y,Z)=0$ for all $Y, Z \in \mathrm{D}_2$. This implies that M is D_2-geodesic.

In view of Theorem 3, we find

Theorem 4. For any $k-$dimensional submanifold of $\tilde{M}(c)$, we find the following cases:

i) For the Ricci tensor S of M, we have the following table:

Table 1:

	M	Inequality
(1)	$3-$ slant	$S \leq \left(\frac{1}{4}k^2\|\hbar\|^2 + \frac{c}{4}\left\{(k-4)+9\cos^2\theta\right\}\right)g.$
(2)	invariant	$S \leq \left(\frac{1}{4}k^2\|\hbar\|^2 + \frac{c}{4}(k+5)\right)g.$

(3)	totally real	$S \leq \left(\frac{1}{4}k^2\|\hbar\|^2 + \frac{c}{4}(k-1)\right)g.$

ii) The equality case of $(1)-(3)$ occurs if and only if M is a totally geodesic submanifold.

5. Inequalities Involving Scalar Curvatures

Lemma 2. [29] If $a_1,\ldots,a_k\,(k>1)$ are real numbers, then

$$\frac{1}{k}\left(\sum_{l=1}^{k}a_l\right)^2 \leq \sum_{l=1}^{k}a_l^2 \tag{33}$$

is satisfied. The equality case of (33) occurs if and only if $a_1 = a_2 = \cdots = a_k$.

Theorem 5. For any $k-$dimensional submanifold of $\tilde{M}(c)$. Then

$$\tau(p) \leq \frac{k(k-1)}{2}\|\hbar\|^2 + \frac{c}{8}\left\{(k-1)(k-6) + 3\sum_{n=1}^{3}\|P_n\|^2\right\} \tag{34}$$

is satisfied. The equality case of (34) is true for p in M if and only if p is a totally umbilical point.

Proof. Assume that e_{k+1} is parallel to $\hbar(p)$ and $e_1,\ldots,e_k$ diagonalize $A_{e_{k+1}}$. In this case, we can write

$$A_{e_{k+1}} = \operatorname{diag}\left(\sigma_{11}^{k+1}, \sigma_{22}^{k+1}, \ldots, \sigma_{kk}^{k+1}\right) \tag{35}$$

and

$$A_{e_s} = \left(\sigma_{lj}^{s}\right), \quad \operatorname{trace} A_{e_s} = \sum_{l=1}^{k}\sigma_{ll}^{s} = 0 \tag{36}$$

for each $l, j = 1,\ldots,k$ and $s = k+2,\ldots,m$. From (12), (35) and (36), we get

$$2\tau(p) = \frac{c}{4}\left\{(k-1)(k-6) + 3\sum_{n=1}^{3}\|P_n\|^2\right\} + k^2\|\hbar\|^2 - \sum_{l=1}^{k}(\sigma_{ll}^{k+1})^2 - \sum_{s=k+2}^{m}\sum_{l,j=1}^{k}(\sigma_{lj}^{s})^2. \tag{37}$$

Considering Lemma 2, we arrive at

$$k\|\hbar\|^2 \leq \sum_{l=1}^{k}(\sigma_{ll}^{k+1})^2. \tag{38}$$

From (37) and (38), the eq. (34) could be obtained. If the equality situation of (34) occurs, from Lemma 2, we find

$$\sigma_{11}^{k+1} = \sigma_{22}^{k+1} = \cdots = \sigma_{kk}^{k+1} \quad \text{and} \quad A_{e_s} = 0.$$

The last equation implies that p is a totally umbilical point. The other direction of proof is easy to follow.

For any $k-$dimensional $3-$semi-slant submanifold of $\tilde{M}(c)$, we put $\dim D_1 = s_1$, $\dim D_2 = s_2$ and $k = s_1 + s_2 + 3$. Then, we have the following:

Theorem 6. For any $k-$dimensional $3-$semi-slant submanifold of $\tilde{M}(c)$, we find

$$\tau(p) \leq \frac{k(k-1)}{2}\|\hbar\|^2 + \frac{c}{8}\{(k-1)(k-6) + 9(s_1 + 2 + s_2\cos^2\theta)\}. \tag{39}$$

The equality case of (39) is true for p in M if and only if p is a totally umbilical point.

Proof. If M is $3-$semi-slant, it can be found

$$\sum_{n=1}^{3}\|P_n\|^2 = 3s_1 + 6 + 3s_2\cos^2\theta. \tag{40}$$

Considering (40) in Theorem 5, the proof is easy to follow.

As a result of Theorem 6, we also have the following:

Corollary 4. For any $k-$dimensional submanifold M of $\tilde{M}(c)$,

i) we have the following table:

Table 2:

	M	Inequality
(1)	$3-$ slant	$\tau(p)\leq\frac{k(k-1)}{2}\|\hbar\|^2+\frac{c}{8}\{(k-1)(k-6)+9((s_1+s_2)\cos^2\theta+2)\}.$
(2)	invariant	$\tau(p)\leq\frac{k(k-1)}{2}\|\hbar\|^2+\frac{c}{8}\{(k-1)(k+3)\}.$
(3)	totally real	$\tau(p)\leq\frac{k(k-1)}{2}\|\hbar\|^2+\frac{c}{8}\{k^2-7k+24\}.$

ii) the equality case of (1)-(3) for each case is satisfied if and only if p is a totally umbilical point.

Proof. If M is $3-$slant, then it can be obtained

$$\sum_{n=1}^{3}\|P_n\|^2=3(s_1+s_2)\cos^2\theta+6. \tag{41}$$

Putting (41) in (34), we get the first case of Table 2.

Consider the fact that $\varphi_l\xi_j=\xi_n$, if M is invariant, then we find

$$\sum_{n=1}^{3}\|P_n\|^2=3(s_1+s_2)+6=3(k-1). \tag{42}$$

Putting (42) in (34), we get the second case of Table 2.

Considering the fact that $\varphi_l\xi_j=\xi_n$, if M is totally real, then we find

$$\sum_{n=1}^{3}\|P_n\|^2=6. \tag{43}$$

Putting (43) in (34), we get the third case of Table 2.

The proof of ii) is easy to follow from Theorem 6.

Theorem 7. For any $k-$dimensional submanifold of $\tilde{M}(c)$, we have

$$\tau(p) \leq \frac{1}{2}k^2 \|\hbar\|^2 + \frac{c}{8}\left\{(k-1)(k-6) + 3\sum_{n=1}^{3}\|P_n\|^2\right\}. \tag{44}$$

The equality case of (44) occurs for p in M if and only if p is a totally geodesic point.

Proof. The proof is easy to follow by (12) and (29).

As a result of Theorem 7, we find the following:

Corollary 5. For any $k-$dimensional $3-$semi-slant submanifold of $\tilde{M}(c)$, we have

$$\tau(p) \leq \frac{1}{2}k^2 \|\hbar\|^2 + \frac{c}{8}\{(k-1)(k-6) + 9(s_1 + 2 + s_2\cos^2\theta)\}. \tag{45}$$

The equality case of (45) occurs for p in M if and only if p is a totally geodesic point.

Corollary 6. For any $k-$dimensional submanifold of $\tilde{M}(c)$,

i) we have the following table:

Table 3:

	M	Inequality
(1)	$3-$slant	$\tau(p) \leq \frac{1}{2}k^2 \|\hbar\|^2 + \frac{c}{8}\{(k-1)(k-6) + 9((s_1 + s_2)\cos^2\theta + 2)\}.$
(2)	invariant	$\tau(p) \leq \frac{1}{2}k^2 \|\hbar\|^2 + \frac{c}{8}\{(k-1)(k+3)\}.$
(3)	totally real	$\tau(p) \leq \frac{1}{2}k^2 \|\hbar\|^2 + \frac{c}{8}\{k^2 - 7k + 24\}.$

ii) The equality case of (1)-(3) occurs if and only if p is a totally geodesic point.

We need the following lemma for later uses:

Lemma 3. Let $a_1, \ldots, a_k, a$ $(k > 2)$ be real numbers satisfying

$$\left(\sum_{l=1}^{k} a_l\right)^2 = (k-1)\left(\sum_{l=1}^{k} a_l^2 + a\right). \tag{46}$$

Then

$$2a_1a_2 \geq a$$

is satisfied if and only if we find

$$a_1 + a_2 = a_3 = \cdots = a_k.$$

Let $\{e_1,\ldots,e_k\}$ be an orthonormal basis and $\Pi = \mathrm{Span}\{e_1,e_2\}$. We define

$$\left\| P_n|_{\pi^\perp} \right\|^2 = \sum_{j,t=3}^{k} g(P_n e_t, e_j)^2. \tag{47}$$

Then we have

Theorem 8. Let M be a $k-$dimensional $(k \geq 3)$ submanifold of $\tilde{M}(c)$. Then, for each point $p \in M$ and each φ_l-plane section $\Pi = \mathrm{Span}\{e_1,e_2\}$ such that $\varphi_l e_1 = e_2$, we have

$$\tau(p) - K(e_1 \wedge e_2) \leq \frac{k^2(k-2)}{2(k-1)}\|\hbar\|^2 + \frac{c}{8}\left\{(k^2 - 7k + 4) + 3\left\|P_n|_{\pi^\perp}\right\|^2\right\}. \tag{48}$$

The equality case (48) occurs at p in M if and only if there exists an orthonormal basis $\{e_{k+1},\ldots,e_m\}$ of $T_p^\perp M$ such that the shape operators A_{e_s} take the following forms:

$$A_{e_{k+1}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{k-2} \end{pmatrix}, \tag{49}$$

$$A_{e_s} = \begin{pmatrix} c_s & d_s & 0 \\ d_s & -c_s & 0 \\ 0 & 0 & 0_{k-2} \end{pmatrix}, \qquad s \in \{k+2,\ldots,m\}. \tag{50}$$

Proof. Assume that $\hbar(p)$ is in the direction of e_{k+1} and $e_1,\ldots,e_k$ diagonalize $A_{e_{k+1}}$. In this case, A_{e_s} take the forms (35) and (36). Thus, we can write

$$\left(\sum_{l=1}^{k}\sigma_{ll}^{k+1}\right)^2=(k-1)\left(\sum_{l=1}^{k}(\sigma_{ll}^{k+1})^2+\sum_{l\neq j=1}^{k}(\sigma_{lj}^{k+1})^2+\sum_{s=k+2}^{m}\sum_{l,j=1}^{k}(\sigma_{lj}^{s})^2+\omega\right) \tag{51}$$

such that

$$\omega=2\tau(p)-\frac{c}{8}\left\{(k-1)(k-6)+3\left\|P_n\right\|^2\right\}-\frac{k^2(k-2)}{k-1}\left\|\hbar\right\|^2. \tag{52}$$

Applying Lemma 3 to (51), we find

$$2\sigma_{11}^{k+1}\sigma_{22}^{k+1}\geq\omega+\sum_{l\neq j=1}^{k}(\sigma_{lj}^{k+1})^2+\sum_{s=k+2}^{m}\sum_{l,j=1}^{k}(\sigma_{lj}^{s})^2. \tag{53}$$

Using (53) in (27), it also follows that

$$\begin{aligned}K(e_1\wedge e_2)\geq&\frac{c}{4}\left\{1+\sum_{n=1}^{3}[3g(\varphi_n e_1,e_2)^2-\eta_n^2(e_1)-\eta_n^2(e_2)]\right\}\\&+\frac{1}{2}\omega+\sum_{s=k+2}^{m}\sum_{j>2}\{(\sigma_{1j}^{s})^2+(\sigma_{2j}^{s})^2\}+\frac{1}{2}\sum_{s=k+2}^{m}(\sigma_{11}^{s}+\sigma_{22}^{s})^2\\&+\frac{1}{2}\sum_{s=k+2}^{m}\sum_{l,j>2}(\sigma_{lj}^{s})^2\end{aligned} \tag{54}$$

or we have

$$K(e_1\wedge e_2)\geq\frac{c}{4}\left\{1+\sum_{n=1}^{3}[3g(\varphi_n e_1,e_2)^2-\eta_n^2(e_1)-\eta_n^2(e_2)]\right\}+\frac{1}{2}\omega. \tag{55}$$

In view of (52) and (55), we get (48).

If the equality case of (48) occurs, then we find

$$\begin{cases}\sigma_{1j}^{k+1}=\sigma_{2j}^{k+1}=0, & j=n+1,\ldots,k,\\ \sigma_{lj}^{s}=0, & l,j=n+1,\ldots,k,\\ \sigma_{11}^{s}+\sigma_{22}^{s}=0\end{cases} \tag{56}$$

for $s=k+2,\ldots,m$. From Lemma 3, it can be found

$$\sigma_{11}^{k+1} + \sigma_{22}^{k+1} = \sigma_{33}^{k+1} = \cdots = \sigma_{kk}^{k+1}, \tag{57}$$

which shows that A_{e_s} becomes as in (49) and (50).

In view of Theorem 8, we get

Corollary 7. Let M be a $k-$dimensional $3-$semi-slant submanifold of $\tilde{M}(c)$. For each $\varphi_l -$plane section $\Pi = \text{Span}\{e_1, e_2\}$, we have

$$\tau(p) - K(e_1 \wedge e_2) \leq \frac{k^2(k-2)}{2(k-1)}\|\hbar\|^2 + \frac{c}{8}\{k^2 - 7k + 14 + 9(s_1 + s_2\cos^2\theta)\}. \tag{58}$$

The equality case of (58) is satisfied if and only if A_{e_s} becomes as in (49) and (50).

Proof. Under this assumption, we find

$$\left\|P_n\big|_{\pi^\perp}\right\|^2 = 3(s_1 + s_2\cos^2\theta). \tag{59}$$

Using (59) in (48), the proof could be obtained.

Corollary 8. Let M be a $k-$dimensional submanifold of $\tilde{M}(c)$ and $\Pi = \text{Span}\{e_1, e_2\}$ be a $\varphi_l -$section.

i) We get the below table:

Table 4:

	M	Inequality
(1)	invariant	$\tau(p) - K(e_1 \wedge e_2) \leq \frac{k^2(k-2)}{2(k-1)}\|\hbar\|^2 + \frac{c}{8}\{k^2 + 2k - 15\}$
(2)	totally real	$\tau(p) - K(e_1 \wedge e_2) \leq \frac{k^2(k-2)}{2(k-1)}\|\hbar\|^2 + \frac{c}{4}\{k^2 - 7k + 32\}.$

ii) The equality case of (1)-(2) is satisfied if and only if A_{e_s} becomes as in (49) and (50).

Proof. Assume that M is invariant. In this case, we find

$$\left\| P_n \big|_{\pi^\perp} \right\|^2 = 3(s_1 + s_2) = 3(k-3). \tag{60}$$

Using (60) in (48), we obtain the first case of Table 4.

If M is totally real, then we have

$$\left\| P_n \big|_{\pi^\perp} \right\|^2 = 6. \tag{61}$$

Using (61) in (48), we obtain the second case of Table 4.

The proof of ii) is straightforward from Theorem 8.

References

[1] Kuo, Y., *On almost contact 3-structure*, Tôhoku Mathematical Journal (2) 22, 325–332, 1970.

[2] Udrişte, C., *Structure presque coquaternioniennes*, Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie, Nouvelle Série, 13, 487–507, 1969.

[3] Balgeshir, M.B.K., *Point-wise slant submanifolds in almost contact geometry*, Turkish Journal of Mathematics, 40, 657-664, 2006.

[4] Boyer, C. P., Galicki, K., *3-Sasakian manifolds*, arXiv preprint hep-th/9810250, 1998.

[5] Galicki, K., Salamon, S., *Betti numbers of 3-Sasakian manifold,* Geometriae Dedicata, 63, 45-68, 1996.

[6] Gibbons, G.W., Rychenkova, P., *Cones, tri-Sasakian structures and superconformal invariance,* Physics Letters B, 443, 138-142, 1998.

[7] Kashiwada, T., *On a contact 3-structure,* Mathematische Zeitschrift, 238, 829-832, 2001.

[8] Mann, B., Galicki, K., Boyer, C.P.*, The geometry and topology of 3-Sasakian manifolds*, Journal für die. reine und angewandte Mathematik, 455, 183-220, 1994.

[9] Montano, B. C., De Nicola, A., *3-Sasakian manifolds, 3-cosymplectic manifolds and Darboux theorem*, Journal of Geometry and Physics, 57, 2509-2520, 2007.

[10] Chen, B.-Y., *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, 1990.

[11] Malek, F., Balgeshir, M.B.K., *Slant submanifolds of almost contact metric 3-structure manifolds*, Mediterranean Journal of Mathematics, 10, 1023-1033, 2013.

[12] Malek, F., Balgeshir, M.B.K., *Semi-slant and bi-slant submanifolds of almost contact metric 3-structure manifolds*, Turkish Journal of Mathematics, 37, 1030-1039, 2013.

[13] Chen B.-Y., *Mean curvature and shape operator of isometric immersions in real space forms*, Glasgow Mathematical Journal, 38, 87-97, 1996.

[14] Chen B.-Y., *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*, Glasgow Mathematical Journal, 41, 33-41, 1999.

[15] Chen B.-Y., *Riemannian DNA, inequalities and their applications*, Tamkang Journal of Science and Engineering, 3, 123-130, 2000.

[16] Chen B.-Y., *Pseudo-Riemannian geometry, $\delta-$invariants and applications*, World Scientific Publishing, 2011.

[17] Akgün, A., Gülbahar, M., *Bi-slant submanifolds of an S manifold,* Arab Journal of Mathematical Sciences, 2021.

[18] Li, Y., Khatri, M., Singh, J.P., Chaubey, S.K., *Improved Chen's inequalities for submanifolds of generalized Sasakian-space-forms*, Axioms, 11, 324, 2022.

[19] Özgür, C., *Chen inequalities for submanifolds of a cosymplectic space form with a semi-symmetric metric connection*, Analele Stiintifice Ale Universitatii Al. İ. Cuza Din İasi-Serie Noua-Matematica, 58, 2012.

[20] Poyraz, N.Ö., *Chen inequalities on spacelike hypersurfaces of a GRW spacetime*, Differential Geometry and its Applications, 81, 101863, 2022.

[21] Poyraz, N.Ö., Doğan, B., Yaşar, E., *Chen inequalities on lightlike hypersurface of a Lorentzian manifold with semi-symmetric metric connection*, International Electronic Journal of Geometry, 10, 1-14, 2017.

[22] Slesar, V., Şahin, B., Vilcu, G.E., *Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms*, Journal of Inequalities and Applications, 2014, 1-10, 2014.

[23] Siddiqui, A.N., Chen, B.Y., Bahadir, O., *Statistical solitons and inequalities for statistical warped product submanifolds*. Mathematics, 7, 797, 2019.

[24] Tripathi, M.M., Gülbahar, M., Kılıç, E., Keleş, S., *Inequalities for scalar curvature of pseudo-Riemannian submanifolds*, Journal of Geometry and Physics, 112, 74-84, 2017.

[25] Chen, B.-Y., *On Ricci curvature of isotropic and Langrangian submanifolds in complex space forms*, Archiv der Mathematik, 74, 154-160, 2000.

[26] Chen B.-Y., *A Riemannian invariant for submanifolds in space forms and its applications* in: Geometry and Topology of Submanifolds, World Scientific Publishing, River Edge, 6, 58-81, 1994.

[27] Hong, S., Tripathi, M. M., *On Ricci curvature of submanifolds*, International Journal Pure and Applied Mathematical Sciences, 2, 227-245, 2005.

[28] Hong, S., Matsumoto, K., Tripathi, M.M., *Certain basic inequalities for submanifolds of locally conformal Kaehler space forms*, Sut Journal of Mathematics, 41, 75-94, 2005.

[29] Tripathi, M.M., *Certain basic inequalities for submanifolds in $(\kappa,\mu)-$spaces,* In: Recent Advances in Riemannian and Lorentzian Geometries (Baltimore, MD, 2003), Contemporary Mathematics 337, American Mathematical Society, 187—202, 2003.