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Mulatu Numbers That Are Concatenations of Two Lucas Numbers

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Abstract

In this paper, we find that all Mulatu numbers, which are concatenations of two Lucas numbers are $11,17,73,118$. Let $(M_k)_{k\geq 0}$ and $(L_k)_{k\geq 0}$ be the Mulatu and Lucas sequences. That is, we solve the Diophantine equation $M_k = L_m L_n = 10^d L_m + L_n$ in non-negative integers (k,m,n,d), where d denotes the number of digits of L_n. Solutions of this equation are denoted by $(k,m,n,d) = (4,1,1,1),(5,1,4,1),(8,4,2,1),(9,1,6,2)$. In other words, we have the solutions $M_4 = L_1L_1 = 11$, $M_5 = L_1L_4 = 17$, $M_8 = L_4L_2 = 73$, $M_9 = L_1L_6 = 118$. The proof based on Baker's theory and we used linear forms in logarithms and reduction method to solve of this Diophantine equation.

İki Lucas Sayısının Birleşimi Olan Mulatu Sayıları

Anahtar Kelimeler
Lucas sayıları; Mulatu sayıları; Logaritmalarda lineer formlar; Diophantine denklemleri

Öz

Bu çalışmada iki Lucas sayısının birleşimi olan tüm Mulatu sayılarının $11,17,73,118$ olduğunu buluyoruz. $(M_k)_{k\geq 0}$ ve $(L_k)_{k\geq 0}$ Mulatu ve Lucas dizileri olsun. Yani biz negatif olmayan (k,m,n,d) tam sayılarında $M_k = L_m L_n = 10^d L_m + L_n$ Diyofant denklemini çözüyoruz, burada d, L_n nin basamak sayısını gösterir. Bu denklemin çözümleri $(k,m,n,d) = (4,1,1,1),(5,1,4,1),(8,4,2,1),(9,1,6,2)$ ile ifade edilir. Bir başka deyişle $M_4 = L_1L_1 = 11$, $M_5 = L_1L_4 = 17$, $M_8 = L_4L_2 = 73$, $M_9 = L_1L_6 = 118$ çözümlerine sahibiz. İspat Baker'in teorisine dayanmakta ve biz bu denklemi çözmek için logaritmalarda doğrusal formları ve indirgeme metodunu kullandık.

1. Introduction

Let (M_k) be the sequence of Mulatu numbers defined by $M_0 = 4$, $M_1 = 1$, $M_k = M_{k-1} + M_{k-2}$ for $k \geq 2$. The Mulatu numbers were introduced in (Lemma 2011). For $k \geq 2$, Let (L_k) denotes the Lucas sequence given by the recurrence $L_k = L_{k-1} + L_{k-2}$ with the initial conditions $L_0 = 2, L_1 = 1$. Binet formulas of these numbers are

$$M_k = \frac{(10-\sqrt{5})}{5}\alpha^k + \frac{(10+\sqrt{5})}{5}\beta^k$$

$$L_k = \alpha^k + \beta^k$$

for every $k \geq 0$. The characteristic equation

$$x^2 - x - 1 = 0$$

has roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. It can be verified that

$$\alpha^{k-1} \leq M_k < 4\alpha^k \quad (1)$$

$$\alpha^{k-1} \leq L_k \leq 2\alpha^k \quad (2)$$

by indiction method for $k \geq 0$. Let us give the previous studies for non-negative k, m, n. In (Bank and Luca 2005), let d specifies the number of digits of F_n, authors gave solutions of the equation

$$F_k = 10^d F_m + F_n$$

as $(k,m,n,d) = (7,1,4,1), (7,2,4,1), (8,3,1,1), (8,3,2,1), (10,5,5,1)$. In (Alan 2022), investigator solved the equations

$$F_k = 10^d L_m + L_n$$

$$L_k = 10^d F_m + F_n.$$

Here (F_n) be the sequence of Fibonacci numbers and d indicates the number of digits of L_n and F_n. In (Altassan and Alan 2022), after a short time Altassan and Alan deal with the equations

$$F_n = 10^d F_m + L_k$$

$$F_n = 10^d L_m + F_k.$$

Here d indicates the number of digits of L_k and F_k. In (Erduvan 2023), author showed solutions of the equation

$$L_k = 10^d L_m + L_n$$

as $(k,m,n,d) = (5,1,1,1), (8,3,4,1)$. Let d denotes the number of digits of L_n. In this paper, we tacle of the Diophantine equation

$$M_k = 10^d L_m + L_n. \quad (3)$$

Solutions of the equation (3) are denoted by $(k,m,n,d) = (4,1,1,1), (5,1,4,1), (8,4,2,1), (9,1,6,2)$. The proof depends on lower bounds for linear forms and some tools from Diophantine approximation. For more about Diophantine approximation and Diophantine equations, one can see in (Schmidt 1991, Zannier 2003, Tichy et al. 2008).

2. Tools

Let γ be an algebraic number of degree d over $\mathbb{Q}$ with minimal prmitive polynomial. Then logarithmic height of γ is given

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right),$$

where $c_0 > 0$ and the $\gamma^{(i)}$'s are conjugates of γ.
The following basic properties about logarithmic height was given in (Bugeaud 2018).

$$h(\gamma_1 \mp \gamma_2) \leq h(\gamma_1) + h(\gamma_2) + \log 2, \quad (4)$$

$$h(\gamma_1 \gamma_2^{\pm 1}) \leq h(\gamma_1) + h(\gamma_2), \quad (5)$$

$$h(\gamma_1^{m}) = |m| h(\gamma_1). \quad (6)$$

The following lemma can be found in (Bugeaud et al. 2006).

Lemma 1. Let $\gamma_1, \gamma_2, \dots, \gamma_n$ are positive real algebraic numbers and let $b_1, b_2, \dots, b_n$ be nonzero integers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \gamma_2, \dots, \gamma_n)$ over $\mathbb{Q}$. Let

$$B \geq \max\{|b_1|, |b_2|, \dots, |b_n|\},$$

$$A_i \geq \max\{D \cdot h(\gamma_i), |\log \gamma_i|, (0,16)\}$$

for all $i = 1,2,\dots,n$. If

$$\Gamma := \gamma_1^{b_1} \cdot \gamma_2^{b_2} \cdots \gamma_n^{b_n} - 1 \neq 0$$

then

$$|\Gamma| > \exp(-1.4 \cdot 30^{n+3} \cdot n^{4,5} \cdot D^2 \cdot (1 + \log D) \cdot$$

$$(1 + \log B) \cdot A_1 \cdot A_2 \cdots A_n).$$

The following lemma was given in (Bravo et al. 2016).

Lemma 2. Let τ be irrational number, M be a positive integer and $\frac{p}{q}$ be a convergent of the continued fraction of the τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Put $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no positive integer solution (r,s,t) to the inequality

$$0 < |r\tau - s + \mu| < A \cdot B^{-t}$$

subject to the restrictions that $r \leq M$ and

$$t \geq \frac{log\left({}^{Aq}/_{\varepsilon}\right)}{logB}.$$

The following lemma can be found in (De Weger 1989).

Lemma 3. Let $s, \Gamma \in \mathbb{R}$. If $0 < s < 1$ and $|\Gamma| < s$, then

$|\log(1+\Gamma)| < \frac{-\log(1-s)}{a} \cdot |\Gamma|$

and

$|\Gamma| < \frac{s}{1-e^{-s}} \cdot |e^{\Gamma} - 1|$.

3. Main Theorem

Fistly, we give our auxiliary result. This theorem will be used in the proof of Theorem 5.

Theorem 4. If the equation (3) holds, then the following inequalities are valid.

(a) $\frac{n-1}{5} < d < \frac{n+6}{4}$

(b) $L_n < 10^d < 10L_n$

(c) $n+m-5 < k < n+m+9$

(d) $k-n \geq 1$.

Proof: a) Since d is the number of digits of L_n, we can write $d = \lfloor \log_{10} L_n \rfloor + 1$. From here, we find

$$d = \lfloor \log_{10} L_n \rfloor + 1 \leq \log_{10} L_n + 1$$

$$\leq \log_{10} 2\alpha^n + 1 < \frac{n+6}{4}$$

and

$$d = \lfloor \log_{10} L_n \rfloor + 1 > \log_{10} L_n$$

$$\geq \log_{10} \alpha^{n-2} > \frac{n-1}{5}.$$

So, we obtain $\frac{n-1}{5} < d < \frac{n+6}{4}$.

b) d is the number of digits of L_n and so we can write $d = \lfloor \log_{10} L_n \rfloor + 1$. Then, we get

$$L_n = 10^{\log_{10} L_n} < 10^d \leq 10^{\log_{10} L_n + 1} < 10L_n.$$

c) When we consider Theorem 4(b) and the inequalities (1) and (2) together, we can write

$$\alpha^{k-1} \leq M_k = 10^d L_m + L_n < 10L_nL_m + L_nL_m$$

$$= 11L_nL_m < \alpha^{n+m+8}$$

and

$$\alpha^{k+3} > 4\alpha^k \geq M_k = 10^d L_m + L_n$$

$$> L_nL_m > \alpha^{n+m-2}.$$

Thus, we obtain $n+m-5 < k < n+m+9$.

d) Since

$$M_k = 10^d L_m + L_n > L_nL_m + L_n \geq 2L_n,$$

it is obvious that the case $k-n \geq 1$.

Now, we can give our main result.

Theorem 5. Let $d \geq 1$, $k \geq 4$ and $m, n \geq 0$. Here d indicates the number of digits of L_n. If $M_k = 10^d L_m + L_n$, then

$$(k, M_k, L_m, L_n) \in$$

$$\{(4,11,1,1), (5,17,1,17), (8,73,7,3), (9,118,1,18)\}.$$

Proof. We start to our proof by taking $4 \leq k \leq 109$ under the condition that the equation (3) is valid. Then, we get

$$(k, m, n, d) =$$

$$(4,1,1,1), (5,1,4,1), (8,4,2,1), (9,1,6,2)$$

by using a computer program. After this we will take $k \geq 110$. Now, we design the equation (3) as

$$\left(\frac{10-\sqrt{5}}{5}\right)\alpha^k - 10^d \alpha^m =$$

$$-\left(\frac{10+\sqrt{5}}{5}\right)\beta^k + 10^d \beta^m + L_n$$

i.e.,

$$(2\sqrt{5}-1)\alpha^k - 10^d\sqrt{5}\alpha^m =$$

$$-(2\sqrt{5}+1)\beta^k + 10^d\sqrt{5}\beta^m + \sqrt{5}L_n.$$

If we do the necessary mathematical process, we find

$$\left|\frac{(2\sqrt{5}-1)\cdot\alpha^{k-m}}{\sqrt{5}\cdot 10^d} - 1\right| \leq \frac{2\sqrt{5}+1}{10^d\cdot\sqrt{5}\cdot\alpha^{k+m}} + \frac{1}{\alpha^{2m}} + \frac{\sqrt{5}\cdot L_n}{10^d\cdot\alpha^m}$$

$$\leq \frac{1}{\alpha^m}\left(\frac{2\sqrt{5}+1}{10^d\cdot\sqrt{5}\cdot\alpha^k} + \frac{1}{\alpha^m} + 1\right),$$

i.e.,

$$\left|\frac{(2\sqrt{5}-1)\cdot\alpha^{k-m}}{\sqrt{5}\cdot 10^d}-1\right| \leq \frac{2.01}{\alpha^m}. \quad (7)$$

Here, we kept in view that $k \geq 110$, $m \geq 0$, $d \geq 1$ and $L_n < 10^d$ from Theorem 4(b). Now, we are ready to apply Lemma 1 with $(\gamma_1, b_1) := (\alpha, k-m)$, $(\gamma_2, b_2) := (10, -d)$ and $(\gamma_3, b_3) := \left(\frac{(2\sqrt{5}-1)}{\sqrt{5}}, 1\right)$. Furthermore, $D = 2$. Put

$$\Gamma_1 := \frac{(2\sqrt{5}-1)\cdot\alpha^{k-m}}{\sqrt{5}\cdot 10^d}-1.$$

Now, we suppose that $\Gamma_1 = 0$. Then, we get $\alpha^{k-m} = \frac{(10+\sqrt{5})10^d}{19}$. If we conjugate in $\mathbb{Q}(\sqrt{5})$, then we obtain $\beta^{k-m} = \frac{(10-\sqrt{5})10^d}{19}$. From this, it can be seen that $F_{k-m} = \frac{2\cdot 10^d}{19}$. This is not possible. Furtheremore, we can say

$$h(\gamma_1) = h(\alpha) = \frac{\log\alpha}{2} < 0.49,$$

$$h(\gamma_2) = h(10) = \log 10 < 2.31,$$

$$h(\gamma_3) = h\left(\frac{2\sqrt{5}-1}{\sqrt{5}}\right) \leq 2\log 2 + \log 5 = \log 20 < 3$$

by the inequalities (4), (5) and (6). Thence, we can choose $A_1 := 0.98$, $A_2 := 4.62$ and $A_3 := 6$. We can take $B := k+3$. Because

$$d < \frac{n+6}{4} < \frac{(k-m+5)+6}{4} < k-m+3 < k+3 \quad (8)$$

from Theorem 4(a),(c). When we regard the inequality (7) and use Lemma 1, it can be seen that

$$2.01 \times \alpha^{-m} > |\Gamma_1| > \exp(A\cdot(1+\log(\mathrm{k}+3))\cdot$$

$$(0.98\cdot 4.62\cdot 6).$$

Here, $A = -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)$. Last inequality allude to

$$m\log\alpha - \log(2.01) <$$

$$2.64\times 10^{13}\times(1+\log(k+3)). \quad (9)$$

We redesign the equation (3) as

$$\alpha^k\left(\frac{10-\sqrt{5}}{5}-\alpha^{n-k}\right)-10^d\cdot L_m =$$

$$-\left(\frac{10+\sqrt{5}}{5}\right)\beta^k-\beta^n,$$

i.e.,

$$\alpha^k\left(\left(2\sqrt{5}-1\right)-\sqrt{5}\cdot\alpha^{n-k}\right)-10^d\sqrt{5}L_m =$$

$$-\left(2\sqrt{5}+1\right)\beta^k-\sqrt{5}\beta^n. \quad (10)$$

When the mathematical operations are made to the equality (10) we get

$$\left|1-\frac{10^d\cdot\sqrt{5}\cdot L_m}{\alpha^k\left((2\sqrt{5}-1)-\sqrt{5}\alpha^{n-k}\right)}\right| \leq$$

$$\frac{1}{\alpha^k}\left|\frac{1}{(2\sqrt{5}-1)-\sqrt{5}\alpha^{n-k}}\right|\left(\frac{2\sqrt{5}+1}{\alpha^k}+\frac{\sqrt{5}}{\alpha^n}\right),$$

i.e.,

$$\left|1-\frac{10^d\cdot\sqrt{5}\cdot L_m}{\alpha^k\left((2\sqrt{5}-1)-\sqrt{5}\alpha^{n-k}\right)}\right| \leq \frac{1.07}{\alpha^k}, \quad (11)$$

where we kept in view that $k-n \geq 1$ from Theorem 4(d), $k \geq 110$, and $n \geq 0$. Let $(\gamma_1, b_1) := (\alpha, -k)$, $(\gamma_2, b_2) := (10, d)$ and

$$(\gamma_3, b_3) := \left(\sqrt{5}L_m\left(\left(2\sqrt{5}-1\right)-\sqrt{5}\alpha^{n-k}\right)^{-1}, 1\right).$$

Moreover, $D = 2$. Let

$$\Gamma_2 := 1-\frac{10^d\cdot\sqrt{5}\cdot L_m}{\alpha^k\left((2\sqrt{5}-1)-\sqrt{5}\alpha^{n-k}\right)}.$$

If $\Gamma_2 = 0$, then we can write

$$10^d L_m = \left(2-\frac{\sqrt{5}}{5}\right)\alpha^k-\alpha^n. \quad (12)$$

If we conjugate in $\mathbb{Q}(\sqrt{5})$, we get

$$10^d L_m = \left(2+\frac{\sqrt{5}}{5}\right)\beta^k-\beta^n. \quad (13)$$

Therefore, from the equalities (12) and (13), we get $2\cdot 10^d\cdot L_m = M_k - L_n$. This case is not possible. Because $M_k = 10^d L_m + L_n$. Since,

$$h(\gamma_3) \leq h(L_m) + 2h(\sqrt{5}) +$$

$$h(2\sqrt{5}-1)+(k-n)h(\alpha)+\log 2$$

$$\leq m\frac{\log\alpha}{2}+2\log 5+(m+9)\frac{\log\alpha}{2}+2\log 2$$

$$=\frac{2\log 40+9\log\alpha}{2}+m\log\alpha$$

we can choose $A_1 \coloneqq \log\alpha$, $A_2 \coloneqq \log 100$ and $A_3 \coloneqq 11.8+2m\log\alpha$. Moreover, we can say $B \coloneqq k+3$ from the inequality (8) and $m \geq 0$. If we apply Lemma 1 and regard the inequality (11) then we get

$$k\log\alpha < 2.15\times 10^{12}\cdot(1+\log(k+3)\cdot$$

$$(11.8+2m\log\alpha)+\log 1.07. \tag{14}$$

From the inequalities (9) and (14), we obtain $k < 1.17\times 10^{30}$. Now, we only need to lower bound on k. Put

$$z_1 \coloneqq (k-m)\log\alpha - d\log 10+\log\left(\frac{2\sqrt{5}-1}{\sqrt{5}}\right)$$

and $\Gamma_1 \coloneqq e^{z_1}-1$. From (7), we have

$$|\Gamma_1| \coloneqq |e^{z_1}-1| < \frac{2.01}{\alpha^m} < 0.8$$

for $m \geq 2$. We can choose $s \coloneqq 0.8$, according to Lemma 3, and so we get

$$|z_1| \coloneqq \left|(k-m)\log\alpha - d\log 10+\log\left(\frac{2\sqrt{5}-1}{\sqrt{5}}\right)\right|$$

$$< -\frac{\log 0.2}{0.8}\cdot\frac{2.01}{\alpha^m} < 4.05\times\alpha^{-m}$$

i.e.,

$$0 < \left|(k-m)\frac{\log\alpha}{\log 10}-d+\frac{\log\left(\frac{2\sqrt{5}-1}{\sqrt{5}}\right)}{\log 10}\right| <$$

$$1.76\times\alpha^{-m}. \tag{15}$$

Take $\tau \coloneqq \frac{\log\alpha}{\log 10} \notin \mathbb{Q}$, $\mu \coloneqq \left(\frac{\log\left(\frac{2\sqrt{5}-1}{\sqrt{5}}\right)}{\log 10}\right)$, $A \coloneqq 1.76$, $B \coloneqq \alpha$, $t \coloneqq m$ and $M \coloneqq 1.17\times 10^{30}$. We found that $q_{61} > 6M$ for τ. Moreover

$$0.43 < \varepsilon \coloneqq \|\mu q_{61}\| - M\|\tau q_{61}\| < 0.44.$$

According to Lemma 2, If the inequality (15) has a solution then

$$m \leq \frac{\log\left(\frac{Aq_{61}}{\varepsilon}\right)}{\log B} \leq 151.69,$$

and so $m \leq 151$. Combining $m \leq 151$ and the inequality (14), we get $k < 2.76\times 10^{16}$. Now, put

$$z_2 \coloneqq d\log 10 - k\log\alpha+\log\left(\frac{\sqrt{5}\cdot L_m}{(2\sqrt{5}-1)-\sqrt{5}\cdot\alpha^{n-k}}\right).$$

From (11), it is seen that

$$|\Gamma_2| \coloneqq |1-e^{z_2}| < 1.07\cdot\alpha^{-k} < 0.01$$

for $k \geq 110$. Therefore, choosing $s \coloneqq 0.01$ in Lemma 3, we obtain

$$0 < \left|d\log 10 - k\log\alpha+\log\left(\frac{\sqrt{5}\cdot L_m}{(2\sqrt{5}-1)-\sqrt{5}\cdot\alpha^{n-k}}\right)\right|$$

$$< \frac{\log\left(\frac{100}{99}\right)}{0.01}\cdot\frac{1.07}{\alpha^k} < 1.08\times\alpha^{-k}$$

i.e.,

$$0 < \left|d\frac{\log 10}{\log\alpha}-k+\frac{\log\left(\frac{\sqrt{5}\cdot L_m}{(2\sqrt{5}-1)-\sqrt{5}\cdot\alpha^{n-k}}\right)}{\log\alpha}\right|$$

$$< 2.25\times\alpha^{-k}. \tag{16}$$

Put $\tau \coloneqq \frac{\log 10}{\log\alpha}$, $\mu \coloneqq \frac{\log\left(\frac{\sqrt{5}\cdot L_m}{(2\sqrt{5}-1)-\sqrt{5}\cdot\alpha^{n-k}}\right)}{\log\alpha}$, $A \coloneqq 2.25$, $B \coloneqq \alpha$, $t \coloneqq k$ and $M \coloneqq 2.76\times 10^{16}$. We found that $q_{41} > 6M$ for τ. Thus, we can say

$$0.0009 < \varepsilon \coloneqq \|\mu q_{41}\| - M\|\tau q_{41}\| < 0.4993$$

for $2 \leq m \leq 151$ and $1 \leq k-n \leq m+9$. Thence, the inequality (16) has a solution, for

$$k \leq \frac{\log\left(\frac{Aq_{41}}{\varepsilon}\right)}{\log B} \leq 107.9$$

from Lemma 2. Hence, $k \leq 107$ contradicts our presumption that $k \geq 110$. When we take into account the cases $m=0$ and $m=1$ for $k \geq 110$ we get,

$$\alpha^k\left(\frac{10-\sqrt{5}}{5}-\alpha^{n-k}\right)-x\cdot 10^d =$$

$-\left(\frac{10+\sqrt{5}}{5}\right)\beta^k+\beta^m.$

Here, $x=1$ or $x=2$. If we consider above equation and use the same arguments again, we can say that $k\leq 91$. This is a contradiction. Thus, the proof of our theorem is finished.

4. Conclusion and Conjecture

In this paper, we deal with the Diophantine equation

$$M_k=10^d L_m+L_n.$$

This implies that $M_4=L_1L_1=11$, $M_5=L_1L_4=17$, $M_8=L_4L_2=73$, $M_9=L_1L_6=118$. In recent years, many investigators have been using different integer seqences such as Fermat sequence Mersenne sequence, k-Generalized Fibonacci and Lucas sequence to solve exponantial Diophantine equations. More information about these integer sequences can be seen in (Annouk and Özer 2022, Badidja et al. 2021, Deza 2021, Kılıç and Taşçı 2006). Our study can be done using these integer sequences. Let k,m,n,d in non-negative integers and d denotes the number of digits of L_n. Furthermore, we think that the only Mulatu number, which is concatenations of three Lucas numbers is 2118. The Diophantine equation which is a generalization of our result is expressed as

$$M_k=10^d L_m+10^l L_n+L_r.$$

The solution of this equation will be of the form $M_{15}=2118=L_0L_1L_6$. Here k,m,n,r are non-negative integers and d,l represent the number of digits of the L_n and L_r, respectively. We give this proof to readers as a problem.

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