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Research Article

Analysis of Consistency Indices of Pairwise Comparison Methods

Veysel Çoban [1*]

[1]Industrial Engineering Department, Bilecik Seyh Edebali University, Bilecik, Turkey

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Abstract

The pairwise comparison method is an important tool in the mutual, easy and effective evaluation of factors in the decision-making process. The consistency index (CI) and consistency ratio values determine whether the pairwise comparisons made by the decision makers are performed according to the transitivity and reciprocal properties. Consistency measurement methods in the literature use different computational methods to evaluate the validity of pairwise comparisons. 14 different consistency methods selected from the literature define different consistency index and threshold values to accept the validity of pairwise comparisons. The aim of this study is to observe the behavior and relationships of the consistency indices of 14 different consistency methods in the same and different pairwise comparison matrix (PCM) dimensions. The consistency indices of the methods are compared in all dimensions, and also the random indices of the methods in different dimensions are calculated. According to Saaty's consistency ratio (CR) threshold value (CR≤0.1), threshold values are defined for the consistency indexes of 14 different methods in 8 different dimensions. Thus, decision makers are helped to more easily determine the consistency of pairwise comparisons in different methods and in different dimensions.

Keywords: pairwise comparison matrix, consistency index, random index, consistency ratio, consistency threshold value

İkili Karşılaştırma Yöntemlerinin Tutarlılık İndekslerinin Analizi

Öz

İkili karşılaştırma yöntemi, karar verme sürecinde faktörlerin karşılıklı, kolay ve etkili bir şekilde değerlendirilmesinde önemli bir araçtır. Tutarlılık indeksi ve tutarlılık oran değerleri karar vericiler tarafından yapılan ikili karşılaştırmaların geçişlilik ve karşılıklılık özelliklerine göre yapılıp yapılmadığını belirler. Literatürdeki tutarlılık ölçüm yöntemleri, ikili karşılaştırmaların geçerliliğini değerlendirmek için farklı hesaplama yöntemleri kullanır. Literatürden seçilen 14 farklı tutarlılık yöntemi, ikili karşılaştırmaların geçerliliğini kabul etmek için farklı tutarlılık indeksi ve eşik değerleri tanımlar. Bu çalışma aynı ve farklı ikili karşılaştırma matrisi boyutlarında 14 farklı tutarlılık yönteminin tutarlılık indekslerinin davranışını ve ilişkilerini gözlemlemeyi amaçlar. Yöntemlerin tutarlılık indeksleri tüm boyutlarda karşılaştırılır ve yöntemlerin farklı boyutlardaki rassal indeksleri hesaplanır. Saaty'nin tutarlılık oranı eşik değerine (≤0.1) göre 8 farklı boyutta 14 farklı yöntemin tutarlılık indeksleri için eşik değerler tanımlanır. Böylece karar vericilerin farklı yöntemlerde ve farklı boyutlarda ikili karşılaştırmaların tutarlılığını daha kolay belirlemelerine yardımcı olunur.

Anahtar Kelimeler: ikili karşılaştırma matrisi, tutarlılık indeksi, rassal indeks, tutarlılık oranı, tutarlılık eşik değeri

*Veysel Çoban: veysel.coban@bilecik.edu.tr
Veysel Çoban, https://orcid.org/0000-0002-7885-1935

1. Introduction

Increasing interaction between factors causes decision problems to become more complicated and decision making becomes difficult. Objective approaches to complex problems constitute the most important stage in making correct and permanent decisions. However, in evaluations that cannot be quantitatively measured, subjective judgments are made based on the knowledge and experience of experts [1]. Analytical Hierarchy Process (AHP) and Analytic Network Process (ANP) are important decision methods in solving complex Multi-Criteria Decision Making (MCDM) problems based on subjective evaluation. Pairwise comparison evaluations are the most important and initial step of these MCDM techniques [2]. The matrices developed to reflect the pairwise comparisons are filled according to the relative importance of the factors. Pairwise comparison matrices (PCMs) are evaluated in consistency methods and the priority values of the factors are calculated.

Inconsistency indexes are used to measure the consistency of the evaluations in the PCM, in other words, whether the decision makers make the pairwise evaluations consistently. If the inconsistency values are above the threshold values, it is recommended to make recomparisons or reject the evaluations. Undetected inconsistencies prevent correct prioritization of alternatives and criteria and making correct decisions [3].

First of all, the methods used in determining the inconsistency indices are discussed and the equations that determine the consistency of the PCM are examined. Random PCMs are generated in MS Office Excel program for n=3, 4,..,10 dimensions using the Saaty rating scale ([1/9,9]). Inconsistency values of random PCMs are calculated using the inconsistency methods in the literature, and the relationships between the methods are defined according to the obtained inconsistency indices. Random indices are defined by averaging the inconsistency indices obtained for different dimensions of each method.

The validity of the study is shown by comparing the index values obtained from random PCMs with the index values of Saaty and Crawford and Williams [4, 5]. The accuracy of random indexes for other inconsistency methods is emphasized based on the correlation values of the results. The study contributes to a faster and more efficient evaluation at a common threshold value for methods whose threshold values are not certain (other than CR and geometric consistency ratio - GCR) despite the definition of inconsistency indices. The original contributions made in the content of the study are listed as follows: calculating the consistency indices from 30,000 PCM randomly generated for all methods, defining the relationships between the consistency indices in the correlation matrix, calculating the random indices for all inconsistency index calculation methods, and defining threshold values to decide the consistency of PCMs.

The study consists of the following subsections. In Section 2, studies on consistency indices and random index are examined through literature review. Section 3 covers the formation of pairwise comparisons and calculations of consistency. In addition, 14 different consistency calculation methods are discussed and similarities between them are revealed. In Section 4, consistency indices of different sizes and different methods are calculated and correlations between consistency indices are revealed. Also, random index and consistency threshold values are defined for each method by using the generated consistency indexes. In Section 5, general evaluations are made and suggestions are made for future study.

2. Material and Methods

2.1. Literature Review

The prioritization and ordering of rating scales and alternatives are included in decision making problems. Pairwise mutual evaluations and determining the consistency of these evaluations are critical steps for multi-criteria decision-making methods [4, 7]. PCM provides fast and easy judgments as it evaluates all factors in a single matrix by considering them in pairs and comparatively. Pairwise comparison evaluations made on the Saaty scale or linguistic expressions based on knowledge and experience contain subjective judgments [8, 9]. Identifying possible inconsistencies in judgments and transforming evaluations into usable structures is an important step for the continuation of decision-making methods. Consistency measurements are made to verify the consistency of judgments in pairwise comparisons. Consistency measures check the compliance of pairwise comparisons with the principles of opposition and transitivity [10]. Saaty's proposed consistency scale is based on eigenvector methods [11]. This method indicates that consistent judgments are made when the difference between the eigenvalue of the pairwise comparison matrix and the number of criteria (n) decreases. The Saaty consistency index (CI), which is a measure of consistency, is obtained by dividing the difference by (n-1) [12].

The accepted and most widely used inconsistency index (CI) is introduced in Saaty's study along with the Analytical Hierarchy Process (AHP) decision making method [12]. Different consistency index methods are suggested by different authors to assess inconsistency in pairwise comparison and prioritization procedures: Crawford and Williams [13]; Koczkodaj [14]; Csató [8]; Harker [15]; Golden and Wang [16]; Shiraishi, Obata and Daigo [17]; Stein - Mizzi [10]; Cavallo - D'Apuzzo [9]; Wedley [18]; Takeda [19]; Salo and Hämäläinen [20]; Fedrizzi and Ferrari [21]; Kou and Lin [22]; Gass and Rapcsák [23]; Kułakowski [24] and Barzilai [25]. Although the Saaty consistency index (CI) is a common and dominant method in the literature, other new inconsistency measurement tools are also used. [4, 5, 8, 26, 27].

The arithmetic mean of the consistency indices of a large number of random PCMs produced using the Satty scale {9, 8,…,1/8,1/9} is defined as the random index [11]. Different Satty RI values are obtained according to different PCMs produced in different numbers and randomly by different authors [4, 26, 28]. The consistency ratio, which is the critical value for the consistency of pairwise comparisons, is obtained from the normalization of the consistency index with the corresponding random index. [5, 28]. 0.1 is defined as the threshold value for the consistency ratios proposed by Saaty (CR) and Crawford and Williams (GCR) of the Saaty school [27,29]. In methods other than these two methods, PCM's consistency decisions are made with special threshold values determined for consistency indexes.

There are also criticisms about the restrictiveness of the CR threshold, due to the increase in the PCM dimension and the 9-point scale [20, 30]. The fact that the consistency ratio is not used in other consistency methods other than CR and GCR increases the importance of correct definition and interpretation of consistency indices. In this study, a large number of random PCMs are produced for 14 different consistency methods and the relationship between the consistency indexes and random indexes of the methods is revealed. Thus, it contributes to the observation of common relations between the methods and the development of consistency ratios that enable decision making at a common threshold values.

Sub-section discusses the creation of pairwise comparison matrices and methods of rating factors from PCMs. The importance of checking the consistency of PCMs is emphasized by taking into account the effects of pairwise comparisons on the decision-making process. The methods of measuring the inconsistencies are mentioned and the relationships between the methods are examined.

2.2. Pairwise Comparison and Consistency

Relative and subjective evaluations are used in the solution process of multiple decision making problems that cannot be defined by quantitative measurements [24, 31]. Multiple alternatives cannot be compared in one dimension according to multiple criteria, and multidimensional evaluations prevent making the right decision [25, 32, 33]. Therefore, solving complex problems by dividing them into sub-problems provides a more accurate assessment and decisions [8, 34, 35]. Considering the factors in pairs and comparing them according to their precedence, it eliminates the confusion in multiple decision making problems. The pairwise comparison matrix P is defined as [4, 21, 34];

$$P = (p_{ij})_{n\times n} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & & p_{2n} \\ \vdots & & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \tag{1}$$

where p_{ij} is greater than zero ($p_{ij} > 0$) and p_{ij} defines the degree of preference of a_i over a_j. The pairwise comparison ratio (p_{ij}) approximates the ratio between the weights (w) of the preferences:

$$p_{ij} \approx \frac{w_i}{w_j}, \forall i, j \tag{2}$$

The ranking of preferences according to the reversible feature is $p_{ji} \approx \frac{w_j}{w_i} = \frac{1}{p_{ij}}$, and the PCM is defined as:

$$P = (p_{ij})_{n\times n} = \begin{pmatrix} 1 & p_{12} & \cdots & p_{1n} \\ 1/p_{12} & 1 & & p_{2n} \\ \vdots & & \ddots & \vdots \\ 1/p_{1n} & 1/p_{2n} & \cdots & 1 \end{pmatrix} \tag{3}$$

Improper execution of the pairwise comparison and prioritization process, which takes place with subjective evaluations, (in other words, inconsistent evaluations are made) causes the decision-making process to proceed on the wrong path and leads to wrong results [33,34,36].

The pairwise comparison matrix entries (p_{ij}) reflecting the gradual relations of the factors with each other consist of normalized weight values (w_i/w_j).

$$P = (p_{ij})_{n\times n} = (w_i/w_j)_{n\times n} = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & & w_2/w_n \\ \vdots & & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix} \quad (4)$$

Although pairwise comparison based on weights eliminates the loss of information, making comparisons without factor weights causes information loss and moving away from absolute evaluations. If the decision maker makes pairwise comparisons directly, it is accepted that the evaluation is made independent of the factor weights. Comparative evaluations that are not established with real factor weights do not meet the priority vector and cause inconsistencies. Therefore, inconsistency indices are developed to determine consistency levels in pairwise comparisons [3, 33].

Eigenvector method and geometric mean method are commonly used to estimate factor weights in pairwise comparison matrix [11,13]. The eigenvector method is based on the Perron-Frobenius eigenvector calculation method [2]. Accordingly, the product of the pairwise comparison matrix (P) and the real weight vector (eigenvector or priority weight, w) is equal to the product of the priority weight vector (w) and the eigenvalue of the matrix (λ) [36, 37].

$$Pw = \lambda w \quad (5)$$

$$\begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & & w_2/w_n \\ \vdots & & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \\ \vdots \\ \lambda w_n \end{pmatrix} \quad (6)$$

where λ denotes the largest eigenvalue satisfying the equality.

The geometric mean method is another widely used method for determining the priority vector [38]. In this method, the geometric mean is calculated in the row plane for the preference element that is located in the comparison center of the PCM. The weighted average values of the elements are normalized to calculate the approximate priority vector [39].

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} p_{ij}}}{\sum_{i=1}^{n}\left(\sqrt[n]{\prod_{j=1}^{n} p_{ij}}\right)} \quad (7)$$

The sum of the normalized weight average values is equal to 1, as follows $\sum_{i=1}^{n} w_i = 1$. The rationality of the decision makers is determined by evaluating the relations between the inputs of the pairwise comparison matrix, p_{ij}. The condition that provides full rationality according to the inputs of the PCM is as follows: [4, 40, 41] :

$$p_{ij} = p_{ik}p_{kj} \tag{8}$$

Pairwise comparison matrices, which consist of non-contradictory evaluations of decision makers, are considered consistent [42]. The increase in the pairwise comparison matrix size makes it difficult to ensure transitivity and consistency among the pairwise evaluations. [35]. Although increasing PCM size is a technical dimension of consistency, inconsistency values in lower dimensional PCM reflect the decision maker's competence in evaluations.

2.3. Inconsistency Indices

All pairwise comparisons in PCM are expected to provide consistent information [7]. Whether the decision maker gives consistent information in the PCM is measured by the degree of meeting the transitivity conditions [41,43]. Inconsistency indices are used to measure transitivity conditions among pairwise comparisons. The inconsistency indexes in the literature are as follows:

1. Saaty consistency index (CI) and consistency ratio (CR): According to Saaty's eigenvector method [2], the evaluations are fully consistent if the maximal eigenvalue (λ_{mak}) of the PCM ($P_{n\times n}$) is equal to the number of elements of the PCM (n). The consistency index (CI) for PCM is calculated as [11,44]:

$$CI_n(P) = \frac{\lambda_{mak} - n}{n-1} \tag{9}$$

$$\lambda_{mak_{n=3}} = 1 + \sqrt[3]{\frac{p_{13}}{p_{12}p_{23}}} + \sqrt[3]{\frac{p_{12}p_{23}}{p_{13}}} \tag{10}$$

The ratio of the consistency index (CI_n) to the random index (RI_n) defined for different dimensions gives the consistency ratio, CR which is a general threshold value. The random index (RI_n) is the arithmetic mean of the consistency indices (CI_n) of a large number of PCMs whose inputs (p_{ij}) are randomly generated on the Saaty scale [1/9,9] [2,42]. The RI_n values generated by Alonso and Lamata [28] with the larger dataset are shown in Table 1.

$$CR = \frac{CI_n}{RI_n} \tag{11}$$

0.1 is the threshold value for CR calculated as the inconsistency ratio of the PCM. If the CR value is 0.1 or less ($CR \leq 0.1$), the decision maker is considered to have made sufficiently consistent assessments [29]. While $CR = 0$ indicates full consistency, $CR = 1$ means the random evaluation. Otherwise, it is expected that the evaluations will be made again by the decision maker. Saaty also defines the CR thresholds as 0.05 and 0.08 for PCMs of sizes 3 and 4, respectively [45].

Table 1. RI_n values by Alonso and Lamata.

n	1	2	3	4	5	6	7	8	9	10

RI_n	0	0	0.5247	0.8816	1.1086	1.2479	1.3417	1.4057	1.4499	1.4854

2. Geometric Consistency Index (GCI) and Consistency Ratio (GCR): The relationship between the priority values (w_i, w_j) obtained from the PCM and the pairwise comparison inputs, (p_{ij}) is evaluated. The GCI obtained using the logarithmic least squares method is defined as follows [13,46]:

$$GCI_n(P) = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\ln \left(p_{ij} \frac{w_j}{w_i} \right) \right)^2 \quad (12)$$

where $p_{ij} \frac{w_j}{w_i}$ represents the error rate, and on consistent evaluation, $p_{ij} \frac{w_j}{w_i}$ is calculated as 1 and $\ln(1)$ as 0. Also, the threshold values defined for GCIs of different PCM sizes are as follows: GCI_3 (P)≤0.3147, GCI_4 (P)≤0.3526, GCI_5 (P)≤0.3717, GCI_6 (P)≤0.3755, GCI_7 (P)≤0.3755, GCI_8 (P)≤0.3744, GCI_9 (P)≤0.3733. and GCI_{10} (P)≤0.3709 [34]. The Saaty CR threshold value is also accepted for the geometric consistency ratio (GCR) obtained from normalization of GCI with corrected RI_n (k_n). The threshold value for the consistency measure (GCR), which is likened to Saaty's CR model, is similarly assumed to be 0.1 [27].

$$GCR = \frac{GCI_n(P)}{k_n} \quad (13)$$

where k_n is the normalization coefficient derived according to the Saaty RI_n values.

$$k_n = \frac{2n}{(n-2)} RI_n \quad (14)$$

There are 14 different inconsistency index calculation methods selected in the literature to determine the consistency of the PCM. These calculations are based on the Row geometric mean method (RGMM), the eigenvector method (EVM), the arithmetic mean method (AMM), singular value decomposition method (SVDM), Chi squares method (X^2M), the least squares methods (LSM) and the logarithmic least squares methods (LLSM) [47]. Among these methods, only Saaty and Crawford and Williams methods improve the consistency ratio based on random inconsistency indexes. The 0.1 threshold defined for the consistency ratios enables decision makers to determine the consistency of the evaluations in PCMs more easily and quickly. Table 2 presents the chronological order of the inconsistency index calculation methods in the literature.

The methods generally measure consistency based on transitivity in the PCM. The size parameter n is also included in the calculations, as the increase in PCM size causes evaluation uncertainty [28,47]. Methods other than Saaty and Crawford et al. evaluate the consistency of the PCM according to the consistency index. For example, the Golden and Wang [16] method states that the PCM is consistent in the case of $GW(P) \cong 0$ while the consistency index values are greater than or equal to zero ($GW(P) \geq 0$). However, the method does not explain how much approximation to zero indicates how much consistency or inconsistency.

Table 2. Inconsistency indices in chronological order.

Inconsistency Methods	Measurement Formulas	Consistency Requirements
Consistency Index and Consistency Ratio (EVM) (Saaty, 1977) [11] (**M1**)	$CI(P)=\frac{\lambda_{max}-n}{n-1}$ $CR(P)=\frac{CI(P)}{RI_n}$	$CR(P)\leq 0.1$
Geometric Consistency Index (GCI) and Geometric Consistency Ratio (GCR) (LLSM) (Crawford and Williams, 1985) [38] (**M2**)	$GCI(P)=\frac{2}{(n-1)(n-2)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\ln\left(p_{ij}\frac{w_j}{w_i}\right)\right)^2$ $GCR(P)=\frac{GCI(P)}{k(n)}$	$k(n)=\frac{2n}{(n-2)}RI_n$ $GCR(P)\leq 0.1$
Golden and Wang (1989) [16] (**M3**)	$GW(P)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\|p_{ij}^{norm}-w_i\right\|$	p_{ij}^{norm}, column normalized input GW(P) ≥ 0 If $GW(P)\cong 0$, P is consistent
Takeda (1993) [19] (**M4**)	$MC(P)=\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\prod_{k=1}^{n}p_{ij}p_{jk}p_{ki}\right)^{\frac{1}{n}}$	$MC(P)\geq 1$ If $MC(P)=1$, P is consistent
Duszak and Koczkodaj (1994) [48] (**M5**)	$K(P)=\max_{i<j<k} min\left\{\left\|1-\frac{p_{ik}}{p_{ij}p_{jk}}\right\|,\left\|1-\frac{p_{ij}p_{jk}}{p_{ik}}\right\|\right\}$	$K(P)\geq 0$ If $K(P)=0$, P is consistent
Inconsistency index (Salo and Hämäläinen, 1995) [20] (**M6**)	$AI(P)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{max\,R_{ij}-min\,R_{ij}}{(1+max\,R_{ij})\times(1+min\,R_{ij})}$	$R_{ij}=\{p_{ik}p_{kj}\|k=1,2,\dots,n\}$ $AI(P)\geq 0$ If $AI(P)=0$, P is consistent
Shiraishi, Obata, and Daigo, (1998) [17] (**M7**)	$c_3(P)=\frac{1}{\binom{n}{3}}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(2-\frac{p_{ik}}{p_{ij}p_{jk}}-\frac{p_{ij}p_{jk}}{p_{ik}}\right)$	$c_3\leq 0$ If $c_3=0$, P is consistent
Barzilai (1998) [25] (**M8**)	$RE(P)=\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\log p_{ij}-\log\frac{w_i}{w_j}\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\log p_{ij}\right)^2}$	w_i, weight obtained by geometric mean method $RE(P)\in[0,1]$ If $RE(P)=0$, P is consistent
Harmonic Consistency Index (Stein and Mizzi, 2007) [10] (**M9**)	$\mathrm{HCI(P)}=\frac{(h(P)-n)(n+1)}{n(n-1)}$	$h(P)=\frac{n}{\sum_{i=1}^{n}s_j^{-1}}$ (harmonic mean) HCI(P) ≥ 0 If HCI(P) $=0$, P is consistent
Cavallo and D'Apuzzo, (2009) [9] (**M10**)	$I_{CD}(P)=\prod_{i=1}^{n-2}\prod_{j=i+1}^{n-1}\prod_{k=j+1}^{n}\left(mak\left\{\frac{p_{ik}}{p_{ij}p_{jk}},\frac{p_{ij}p_{jk}}{p_{ik}}\right\}\right)^{1/\binom{n}{3}}$	$I_{CD}(P)\geq 1$ If $I_{CD}(P)=1$, P is consistent

Table 2. Inconsistency indices in chronological order (cont.).

Inconsistency Methods	**Measurement Formulas**	**Consistency Requirements**
Cosine Consistency Index (Kou and Lin, 2014) [22] (**M11**)	$CCI(P) = \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}\right)^2}$	$b_{ij} = \frac{p_{ij}}{\sqrt{\sum_{k=1}^{n} p_{kj}^2}}$ $CCI(P) \leq n$ If $h(P) = n$, P is consistent
Kułakowski (2015) [24] (**M12**)	$E(P) = \max_{i,j}\left\{p_{ij}\frac{w_j}{w_i} - 1\right\}$	w_i, weight from any method $E(P) \geq 0$ If $E(P) = 0$, P is consistent
Grzybowski (2016) [49] (**M13**)	$\text{ATI(P)} = \frac{1}{\binom{n}{3}}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} min\left\{\left\|1 - \frac{p_{ik}}{p_{ij}p_{jk}}\right\|, \left\|1 - \frac{p_{ij}p_{jk}}{p_{ik}}\right\|\right\}$	$\text{ATI(P)} \geq 0$ If $ATI(P) = 0$, P is consistent
Chi-square (χ^2) Inconsistency Index (X^2M) (Fedrizzi and Ferrari, 2018) [21] (**M14**)	$\chi^2(P) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(p_{ij} - e_{ij})^2}{e_{ij}}$	$e_{ij} = \frac{(\sum_{k=1}^{n} p_{ik})(\sum_{s=1}^{n} p_{sj})}{\sum_{s=1}^{n}\sum_{k=1}^{n} p_{sk}}$ $\chi^2(P) \geq 0$ If $\chi^2(P) = 0$, P is consistent

3. Results and Discussions

3.1. Analysis of Consistency Indices

This section examines the relationships between the consistency indices of different models. Common threshold values depending on the consistency value are defined for models with high correlation. Consistency indices and random indices are important tools in reflecting the relationship between 14 different consistency calculation methods. The random index (RI_n) represents the mean value of the inconsistency values obtained from randomly generated pairwise comparison values for n-dimensional PCMs. The number of PCMs that can be generated using the Saaty index is calculated as $17^{\frac{n*(n-1)}{2}}$, where n defines the number of factors in the PCM.

Table 3. Correlation between inconsistency indices for $n = 3$.

	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1													
$M2$	0,613	1												
$M3$	0,998	0,64 (0,641)	1											
$M4$	-0,951 (-0.95)	-0,693 (-0.694)	-0,962 (-0.961)	1										
$M5$	0,917 (0,915)	0,393 (0,39)	0,91 (0,907)	-0,78 (-0.775)	1									
$M6$	0,975 (0,974)	0,498 (0,497)	0,977	-0,917 (-0.915)	0,95 (0,949)	1								
$M7$	0,889 (0,885)	0,672 (0,673)	0,908 (0,906)	-0,976 (-0.975)	0,711 (0,704)	0,873 (0,87)	1							
$M8$	-0,916 (-0.914)	-0,39 (-0.387)	-0,909 (-0,906)	0,779 (0,774)	-1	-0,95 (-0.948)	-0,71 (-0.703)	1						
$M9$	0,91 (0,908)	0,817 (0,818)	0,926 (0,925)	-0,978	0,706 (0,701)	0,848 (0,847)	0,963 (0,962)	-0,705 (-0.699)	1					
$M10$	0,613	1	0,64 (0,641)	-0,693 (-0,694)	0,393 (0,39)	0,498 (0,497)	0,672 (0,673)	-0,39 (-0.387)	0,817 (0,818)	1				
$M11$	0,997	0,577	0,997 (0,996)	-0,946 (-0.944)	0,933 (0,931)	0,989	0,89 (0,887)	-0,932 (-0.93)	0,895 (0,893)	0,577	1			
$M12$	0,935 (0,933)	0,791 (0,792)	0,95 (0,949)	-0,988	0,75 (0,745)	0,884 (0,882)	0,963 (0,962)	-0,749 (-0.744)	0,995	0,791 (0,792)	0,923 (0,922)	1		
$M13$	0,997	0,619	0,999	-0,967 (-0.966)	0,901 (0,898)	0,978	0,916 (0,913)	-0,9 (-0.897)	0,925 (0,924)	0,619	0,996	0,949 (0,948)	1	
$M14$	0,997 (0,998)	0,577 (0,638)	0,997 (0,995)	-0,946 (-0.949)	0,933 (0,9)	0,989 (0,961)	0,89 (0,882)	-0,932 (-0.899)	0,895 (0,916)	0,577 (0,638)	1 (0,991)	0,923 (0,937)	0,996 (0,994)	1

First of all, $n = 3$ dimensional PCMs are defined and evaluated because all combinations are easily accessible. All combinations of 4913 different PCMs ($17^3 = 4913$) that can be generated for n=3 are identified and the consistency indexes and random indexes of these PCMs are calculated for 14 different methods (Table 3). The correlation values obtained from the actual values differ from the 30000 randomly generated PCMs for n=3 that are shown in parentheses. Since the number of randomly generated PCMs (30,000) is sufficiently large and inclusive than the actual number of PCMs (4913), there is no significant difference between the consistency indices of the different models. The following evaluations are made from the relationships among the consistency methods of all PCM combinations of n=3.

- The Duszak - Koczkodaj and Grzybowski models, which have the lowest relationship (0.577) with the Saaty model, and Saaty and Takedo models show the highest (1.000) inconsistency relationship as they evaluate consistency using the proportional transitivity method. Also, the highest negative correlation (-1) is observed between Shiraishi et al. and Cavallo - D'Apuzzo models.
- The inconsistency ratio relationships of the Duszak - Koczkodaj and Grzybowski models with the Shiraishi et al. and Cavallo - D'Apuzzo models reflect the lowest correlation values as -0.390 and 0.393, respectively.
- Kulakowski model has the highest similarity with other models with an average correlation value of 0.915.
- The Duszak - Koczkodaj and Grzybowski models have the lowest similarity with an average correlation of 0.663 among the other models.
- Shiraishi et al. and Kou - Lin models show an inverse relationship with other models.

There are no high differences (0-0.061) between the randomly derived models and the full combination (4913) models.

Table 4. Correlation between consistent indices ($CR \leq 0.1$) for $n = 3$.

	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1													
$M2$	0.973	1												
$M3$	0.868	0.937	1											
$M4$	0.973	1.000	0.937	1										
$M5$	0.910	0.915	0.947	0.914	1									
$M6$	0.899	0.945	0.964	0.945	0.957	1								
$M7$	-0.971	-1.000	-0.934	-1.000	-0.907	-0.942	1							
$M8$	0.460	0.572	0.578	0.573	0.485	0.661	-0.578	1						
$M9$	0.769	0.880	0.873	0.881	0.763	0.907	-0.885	0.813	1					
$M10$	0.969	0.990	0.960	0.989	0.963	0.969	-0.987	0.555	0.858	1				
$M11$	-0.717	-0.853	-0.903	-0.853	-0.739	-0.832	0.857	-0.662	-0.907	-0.831	1			
$M12$	0.958	0.973	0.963	0.972	0.984	0.972	-0.968	0.533	0.831	0.996	-0.805	1		
$M13$	0.910	0.915	0.947	0.914	1.000	0.957	-0.907	0.485	0.763	0.963	-0.739	0.984	1	
$M14$	0.983	0.995	0.926	0.995	0.916	0.923	-0.994	0.509	0.832	0.987	-0.829	0.971	0.916	1

The $n = 3$ evaluations on real PCMs ($17^3 = 4913$) are an important template to evaluate and compare the different inconsistency models for $n > 3$ dimensions. Also, when only consistent PCMs are examined, different correlation values are reached among consistency methods (Table 4). While the correlation of the M8 method with all other methods decreases, the highest increases occur in the correlations of the M5 and M10 methods and the M7 and M5/M13 methods as 0.571 and 0.517, respectively. Therefore, defining threshold inconsistency indices for highly variable M5, M7, M8, M10, and M13 methods may cause the evaluation risk. Only the correlation between the M5 and M13 methods does not change and remains the same as 1.

As the factor size increases ($n > 3$), it becomes more difficult to identify all combinations of PCMs and to arrive at their exact CI and RI values. Because the number of actual PCMs with factor size greater than 3 is very high (e.g. $17^6 = 24{,}137{,}569$ for n=4), 30000 random PCMs are generated to make calculations and decisions for each size. 14 different inconsistency methods are run for the same PCMs and the relationships between the methods' inconsistency indices are shown in tables (Table 5). Table 5 shows the correlation between the inconsistency indices of 14 different methods of randomly generated pairwise comparison matrices for $n = 3,4,\dots,10$ dimensions.

Table 5.a Correlation values of inconsistency indexes of 14 methods.

$n=4$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,993	1,000												
$M3$	0,918	0,925	1,000											
$M4$	0,989	0,993	0,891	1,000										
$M5$	0,596	0,599	0,736	0,542	1,000									
$M6$	0,912	0,929	0,963	0,886	0,784	1,000								
$M7$	-0,914	-0,904	-0,745	-0,941	-0,397	-0,727	1,000							
$M8$	0,891	0,920	0,931	0,890	0,638	0,947	-0,747	1,000						
$M9$	0,907	0,911	0,875	0,921	0,457	0,802	-0,864	0,828	1,000					
$M10$	0,918	0,923	0,804	0,937	0,448	0,803	-0,856	0,797	0,878	1,000				
$M11$	-0,927	-0,941	-0,985	-0,912	-0,652	-0,952	0,768	-0,953	-0,906	-0,838	1,000			
$M12$	0,832	0,836	0,751	0,842	0,478	0,749	-0,797	0,751	0,773	0,775	-0,767	1,000		
$M13$	0,716	0,711	0,804	0,659	0,811	0,854	-0,492	0,692	0,576	0,654	-0,740	0,547	1,000	
$M14$	0,992	0,995	0,914	0,991	0,584	0,912	-0,906	0,898	0,907	0,939	-0,930	0,832	0,702	1,000

$n=5$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,992	1,000												
$M3$	0,922	0,931	1,000											
$M4$	0,988	0,987	0,890	1,000										
$M5$	0,601	0,604	0,741	0,554	1,000									
$M6$	0,912	0,926	0,972	0,884	0,783	1,000								
$M7$	-0,922	-0,908	-0,755	-0,941	-0,414	-0,732	1,000							
$M8$	0,908	0,935	0,940	0,900	0,648	0,944	-0,775	1,000						
$M9$	0,897	0,899	0,862	0,906	0,464	0,793	-0,851	0,831	1,000					
$M10$	0,934	0,947	0,838	0,935	0,480	0,825	-0,866	0,847	0,872	1,000				
$M11$	-0,926	-0,938	-0,983	-0,908	-0,664	-0,952	0,772	-0,955	-0,899	-0,852	1,000			
$M12$	0,699	0,683	0,616	0,722	0,437	0,621	-0,673	0,621	0,632	0,608	-0,635	1,000		
$M13$	0,801	0,803	0,893	0,748	0,784	0,927	-0,584	0,789	0,656	0,738	-0,833	0,503	1,000	
$M14$	0,983	0,982	0,916	0,986	0,597	0,911	-0,897	0,909	0,895	0,925	-0,932	0,713	0,786	1,000

Table 5.b Correlation values of inconsistency indexes of 14 methods.

$n=6$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,993	1,000												
$M3$	0,935	0,944	1,000											
$M4$	0,990	0,987	0,909	1,000										
$M5$	0,641	0,643	0,761	0,605	1,000									
$M6$	0,924	0,934	0,983	0,900	0,793	1,000								
$M7$	-0,930	-0,918	-0,781	-0,944	-0,469	-0,756	1,000							
$M8$	0,926	0,948	0,957	0,915	0,691	0,954	-0,802	1,000						
$M9$	0,909	0,909	0,869	0,916	0,522	0,816	-0,868	0,852	1,000					
$M10$	0,954	0,965	0,870	0,952	0,542	0,855	-0,898	0,885	0,892	1,000				
$M11$	-0,934	-0,943	-0,986	-0,918	-0,708	-0,965	0,790	-0,965	-0,898	-0,874	1,000			
$M12$	0,638	0,613	0,572	0,667	0,454	0,578	-0,611	0,562	0,584	0,553	-0,588	1,000		
$M13$	0,868	0,872	0,943	0,830	0,770	0,965	-0,673	0,870	0,741	0,810	-0,904	0,516	1,000	
$M14$	0,980	0,976	0,933	0,984	0,647	0,925	-0,897	0,924	0,904	0,930	-0,945	0,662	0,860	1,000

$n=7$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,995	1,000												
$M3$	0,951	0,957	1,000											
$M4$	0,993	0,989	0,931	1,000										
$M5$	0,676	0,677	0,769	0,651	1,000									
$M6$	0,941	0,947	0,989	0,922	0,795	1,000								
$M7$	-0,939	-0,929	-0,815	-0,949	-0,527	-0,792	1,000							

$M8$	0,945	0,961	0,971	0,933	0,720	0,965	-0,834	1,000						
$M9$	0,926	0,927	0,889	0,932	0,580	0,851	-0,888	0,881	1,000					
$M10$	0,927	0,936	0,866	0,924	0,580	0,853	-0,877	0,878	0,876	1,000				
$M11$	-0,947	-0,953	-0,990	-0,933	-0,736	-0,976	0,816	-0,974	-0,908	-0,864	1,000			
$M12$	0,610	0,584	0,569	0,637	0,486	0,577	-0,574	0,548	0,567	0,502	-0,579	1,000		
$M13$	0,912	0,915	0,969	0,885	0,764	0,982	-0,745	0,918	0,810	0,834	-0,944	0,541	1,000	
$M14$	0,980	0,976	0,951	0,984	0,687	0,943	-0,902	0,940	0,920	0,901	-0,960	0,641	0,909	1,000

$n=8$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,995	1,000												
$M3$	0,955	0,961	1,000											
$M4$	0,995	0,991	0,939	1,000										
$M5$	0,686	0,686	0,768	0,667	1,000									
$M6$	0,946	0,951	0,992	0,931	0,790	1,000								
$M7$	-0,943	-0,934	-0,827	-0,951	-0,550	-0,806	1,000							
$M8$	0,949	0,964	0,976	0,939	0,727	0,969	-0,843	1,000						
$M9$	0,938	0,939	0,900	0,943	0,609	0,871	-0,901	0,895	1,000					
$M10$	0,972	0,980	0,909	0,971	0,618	0,894	-0,929	0,923	0,930	1,000				
$M11$	-0,949	-0,955	-0,992	-0,938	-0,745	-0,982	0,823	-0,978	-0,910	-0,904	1,000			
$M12$	0,579	0,552	0,545	0,605	0,484	0,556	-0,541	0,518	0,544	0,508	-0,554	1,000		
$M13$	0,929	0,932	0,978	0,910	0,756	0,988	-0,777	0,939	0,846	0,884	-0,961	0,530	1,000	
$M14$	0,980	0,976	0,958	0,983	0,701	0,951	-0,903	0,946	0,930	0,942	-0,965	0,612	0,930	1,000

Table 5.c Correlation values of inconsistency indexes of 14 methods.

$n=9$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,996	1,000												
$M3$	0,964	0,969	1,000											
$M4$	0,996	0,993	0,953	1,000										
$M5$	0,698	0,698	0,763	0,686	1,000									
$M6$	0,956	0,960	0,994	0,946	0,783	1,000								
$M7$	-0,953	-0,945	-0,856	-0,958	-0,585	-0,838	1,000							
$M8$	0,960	0,972	0,984	0,952	0,735	0,976	-0,869	1,000						
$M9$	0,955	0,956	0,922	0,959	0,642	0,901	-0,923	0,920	1,000					
$M10$	0,979	0,985	0,926	0,977	0,643	0,912	-0,946	0,938	0,948	1,000				
$M11$	-0,958	-0,963	-0,994	-0,950	-0,749	-0,987	0,850	-0,984	-0,925	-0,919	1,000			
$M12$	0,624	0,600	0,603	0,645	0,496	0,616	-0,578	0,572	0,593	0,561	-0,609	1,000		
$M13$	0,948	0,951	0,986	0,935	0,748	0,992	-0,823	0,958	0,888	0,911	-0,974	0,605	1,000	
$M14$	0,983	0,979	0,969	0,985	0,715	0,963	-0,916	0,958	0,947	0,951	-0,974	0,656	0,952	1,000

$n=10$	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,868	1,000												
$M3$	0,823	0,975	1,000											
$M4$	0,876	0,994	0,963	1,000										
$M5$	0,589	0,700	0,754	0,691	1,000									
$M6$	0,817	0,967	0,995	0,956	0,773	1,000								
$M7$	-0,864	-0,954	-0,878	-0,964	-0,606	-0,863	1,000							
$M8$	0,809	0,977	0,988	0,961	0,731	0,982	-0,888	1,000						
$M9$	0,845	0,968	0,938	0,970	0,658	0,922	-0,938	0,938	1,000					
$M10$	0,867	0,988	0,940	0,983	0,656	0,928	-0,957	0,949	0,962	1,000				
$M11$	-0,813	-0,969	-0,996	-0,959	-0,743	-0,991	0,871	-0,988	-0,938	-0,932	1,000			
$M12$	0,566	0,619	0,627	0,657	0,514	0,641	-0,589	0,597	0,615	0,585	-0,633	1,000		
$M13$	0,821	0,963	0,990	0,952	0,740	0,995	-0,856	0,971	0,917	0,930	-0,983	0,633	1,000	
$M14$	0,855	0,983	0,977	0,987	0,716	0,972	-0,926	0,967	0,959	0,959	-0,980	0,671	0,965	1,000

The correlation values in the tables highlight the following results.

- Duszak et al. and Kulakowski methods have the weakest correlation with other methods. Although the Grzybowski method has a high correlation with the Golden et al. and Duszak et al. methods, it shows a low correlation with the other methods. The mean lowest correlation is seen between the Duszak et al. method and the Shiraishi et al. and Kulakowski methods as -0.492 and 0.499, respectively (Table 6).
- Shiraishi et al. and Kou et al. models have inverse correlation with other models apart from each other.

- Crawford et al., Takedo and Fedrizzi et al. methods have the highest mean correlation with other methods. The mean highest correlation is seen between Crawford et al. and Takedo methods as 0.991 (Table 6).
- There are no method pairs with 100% correlation among the inconsistency indices at $n > 4$. Methods other than Duszak et al. and Kulakowski generally show more than 80% correlation in all dimensions.
- Correlation tables show that threshold values based on inconsistency indexes can be defined among highly correlated methods. The increasing PCM size with the increase in the number of factors causes to increase the inconsistencies in the randomly assigned pairwise evaluation values and the deviations in the consistency calculations.

Table 6. Mean correlation values of inconsistency indexes of 14 methods.

	$M1$	$M2$	$M3$	$M4$	$M5$	$M6$	$M7$	$M8$	$M9$	$M10$	$M11$	$M12$	$M13$	$M14$
$M1$	1,000													
$M2$	0,979	1,000												
$M3$	0,924	0,952	1,000											
$M4$	0,978	0,991	0,925	1,000										
$M5$	0,633	0,653	0,761	0,622	1,000									
$M6$	0,913	0,942	0,985	0,915	0,790	1,000								
$M7$	-0,925	-0,924	-0,801	-0,947	-0,492	-0,777	1,000							
$M8$	0,910	0,949	0,964	0,923	0,695	0,962	-0,808	1,000						
$M9$	0,921	0,936	0,892	0,942	0,554	0,850	-0,898	0,877	1,000					
$M10$	0,936	0,953	0,863	0,951	0,545	0,847	-0,916	0,866	0,914	1,000				
$M11$	-0,925	-0,954	-0,989	-0,933	-0,711	-0,973	0,809	-0,972	-0,913	-0,870	1,000			
$M12$	0,693	0,686	0,654	0,721	0,499	0,658	-0,659	0,635	0,661	0,625	-0,666	1,000		
$M13$	0,821	0,846	0,919	0,812	0,797	0,940	-0,667	0,851	0,741	0,769	-0,879	0,564	1,000	
$M14$	0,969	0,983	0,944	0,987	0,658	0,936	-0,908	0,929	0,930	0,933	-0,955	0,723	0,840	1,000

3.2. Analysis of Random Indices

Random inconsistency indices (RI_n) are defined for each method by taking the average of the inconsistency indices generated from 30,000 randomly generated PCMs for each dimension (n) (Table 7). There is a 92.34% correlation between the RI_n values (Table 7) recommended for Saaty's inconsistency indices in the literature and the average inconsistency indices calculated in this study. The differences between the values obtained in the literature and this study arise from the difference in the randomly generated PCM. An increase in the size (n) reduces the probability of an exact match of the randomly selected groups of 30,000 and the probability of a perfect match with the RI_n values in different studies.

Table 7. Random indexes for inconsistency methods.

Inconsistency methods	$n = 3$ (4913 all)	$n = 3$ (30000 random)	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
Saaty (1977)	0,524457	0,572	0,890	1,058	1,170	1,238	1,407	1,455	1,440
Crawford et al. (1985)	2,629	2,603	2,442	2,419	2,420	2,406	2,622	2,630	2,203
Golden et al.(1989)	0,144	0,144	0,143	0,132	0,118	0,106	0,104	0,094	0,073
Takedo(1993)	1,524	1,518	1,802	1,994	2,126	2,207	2,386	2,444	2,243
Duszak et al. (1994)	0,774	0,773	0,925	0,959	0,966	0,967	0,997	0,998	0,942
Salo et al. (1995)	0,380	0,379	0,542	0,641	0,705	0,745	0,840	0,866	0,754
Shiraishi et al. (1998)	-39,805	-39,083	-36,581	-36,291	-36,543	-36,259	-39,573	-39,904	-33,253
Barzilai (1998)	0,333	0,332	0,467	0,556	0,617	0,657	0,750	0,778	0,675
Stein et al. (2007)	0,548	0,540	0,787	0,968	1,105	1,189	1,378	1,440	1,238

Cavallo et al. (2009)	41,579	40,856	13,035	10,799	10,144	8,643	10,461	10,399	8,807
Kou et al. (2014)	2,690	2,692	3,360	4,007	4,658	5,326	5,817	6,466	7,573
Kulakowski (2005)	1,528	1,515	4,111	6,527	8,742	10,321	12,493	13,734	12,555
Grzybowski (2016)	0,774	0,773	0,734	0,724	0,717	0,716	0,773	0,774	0,652
Fedrizzi et al. (2018)	6,999	6,924	20,912	42,639	71,657	106,785	161,933	214,911	230,569

The random index values are normalized over the maximum value for each method (Figure 1). The variation of the normalized random index values is examined according to the n dimension. An increase in n causes different behavior in RI values. While the M11 and M14 methods increase continuously with the increase of n, the M1, M4, M6, M9 and M12 methods increase up to n=9 and decrease at n=10. In addition, sharper increases occur in the M12 and M14 methods. The RI values show a continuous decrease with the increase of n only in the M3 method. Although n increases in M2, M5, M7 and M13 methods, RI values get closer to each other. In addition, RI values are more stable at n>3 in the M10 method. In general, similar threshold consistency values can be defined for methods that exhibit similar RI behavior.

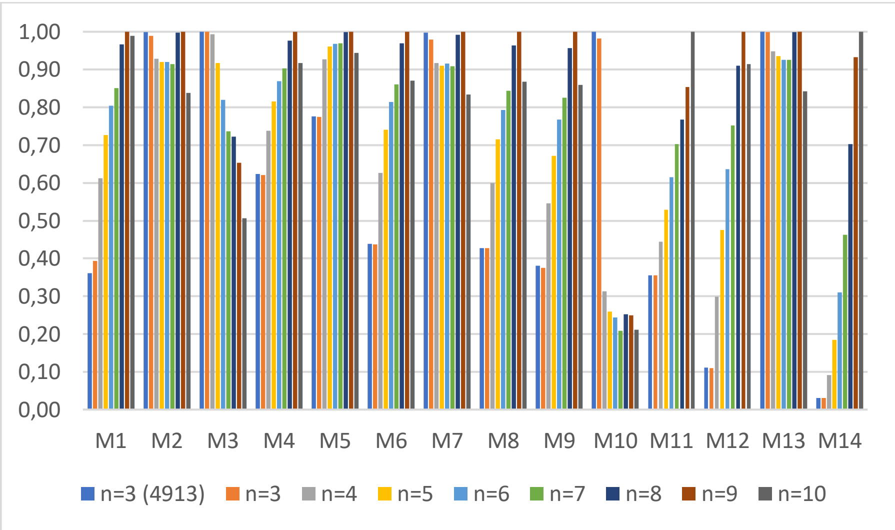

Figure 1. Change of random index values according to PCM size.

The high similarity between the Saaty RI values in the literature and the Saaty RI value obtained from this study (Table 8), indicates that the RI values of other consistency methods described in this study may be valid. The differences in the RI values in Table 8 are due to the different randomly generated PCMs and the applied different methods. Accordingly, increasing the size of n causes an increase in the differences and deviations between the RI values.

Table 8. Saaty RI_n values [28].

		n							
	Iteration	3	4	5	6	7	8	9	10
This study	30000	0.572	0.89	1.058	1.17	1.238	1.407	1.455	1.44
Saaty [50]	500	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49
Lane et al. [30]	2500	0.52	0.87	1.1	1.25	1.34	1.40	1.45	1.49
Golden Wang [16]	1000	0.58	0.89	1.12	1.24	1.33	1.40	1.45	1.49
Noble [51]	500	0.49	0.82	1.03	1.16	1.25	1.31	1.36	1.39
Forman [52]		0.52 (77487)	0.89 (63822)	1.11 (41645)	1.25	1.34			
Tummala, Wan [44]		0.5 (470000)	0.83 (122000)	1.05 (73000)	1.18	1.27	1.33	1.37	1.41

Aguaron et al. [46]	100000	0.53	0.88	1.12	1.25	1.34	1.40	1.45	1.48
Alonso et al.[53]	100000	0.53	0.88	1.11	1.25	1.34	1.41	1.45	1.49
Alonso et al. [28]	500000	0.53	0.88	1.11	1.25	1.34	1.41	1.45	1.49

3.3. Threshold Values Based on Consistency Indices

In this section, threshold values are determined for the consistency indices of the methods that have a high correlation with the Saaty consistency ratio (CR). First, the CR values of the n=3 dimension, of which all PCM combinations can be defined, are calculated and combined with the CI values of the other methods. The CI values of all methods are ordered according to the order of the CR values from smallest to largest. The compatibility of the consistency index values of the inconsistency methods, which are ranked up to the threshold value of $CR \leq 0.1$, with the CR values is examined and threshold values are defined for the methods with high correlation. The M2 ($R^2 = 1$), M4 ($R^2 = 1$), M7 ($R^2 = 0.9994$), M10 ($R^2 = 0.9873$), M12 ($R^2 = 0.9557$), and M14 ($R^2 = 0.9903$) methods have very high fitness with the consistent Saaty method at $n = 3$, whereas M3 ($R^2 = 0.8416$), M5 ($R^2 = 0.8343$), M6 ($R^2 = 0.8732$), M9 ($R^2 = 0.7148$), M11 ($R^2 = 0.6448$) and M13 ($R^2 = 0.8343$) represent the high fittness and M8 ($R^2 = 0.3435$) have low fitness.

Table 9. Threshold values of inconsistency indexes for methods.

Inconsistency Methods	Consistency Requirements	Threshold values for consistency indices							
		3	4	5	6	7	8	9	10
M1	$CI(P) \geq 0$ If $CI(P) = 0$, P is consistent	0.0522	0,088	0,111	0,125	0.134	0.140	0.145	0.148
M2	$GCI(P) \geq 0$ If $GCI(P) = 0$, P is consistent	0.311	0.263	0.259	0.252	0.255	0.242	0.214	0.241
M3	$\text{GW(P)} \geq 0$ If $GW(P) \cong 0$, P is consistent	0.065	0.045	0.056	0.016	0.016	0.027	0.026	0.020
M4	$MC(P) \geq 1$ If $MC(P) = 1$, P is consistent	1.052	1.067	1.082	1.093	1.108	1.115	1.112	1.153
M5	$K(P) \geq 0$ If $K(P) = 0$, P is consistent	0.619	0.733	0.8	0.857	0.901	0.926	0.938	0.963
M6	$AI(P) \geq 0$ If $AI(P) = 0$, P is consistent	0.143	0.227	0.219	0.177	0.191	0.211	0.191	0.224
M7	$c_3 \leq 0$ If $c_3 = 0$, P is consistent	-1.006	-0.888	-0.96	-1.028	-1.175	-1.240	-1.172	-1.669
M8	$RE(P) \in [0,1]$ If $RE(P) = 0$, P is consistent	0.033	0.099	0.199	0.083	0.127	0.229	0.119	0.108
M9	$\text{HCI(P)} \geq 0$ If $\text{HCI(P)} = 0$, P is consistent	0.021	0.033	0.103	0.0106	0.016	0.101	0.148	0.112
M10	$I_{CD}(P) \geq 1$ If $I_{CD}(P) = 1$, P is consistent	2.625	2.121	1.621	1.476	1.392	1.322	1.260	1.246
M11	$CCI(P) \leq n$ If $CCI(P) = n$, P is consistent	2.947	3.924	4.758	5.914	6.817	7.505	8.410	9.476
M12	$E(P) \geq 0$ If $E(P) = 0$, P is consistent	0.380	0.540	1.627	2,659	4.226	6.043	0.361	0.390
M13	$\text{ATI(P)} \geq 0$ If $ATI(P) = 0$, P is consistent	0.619	0.475	0.240	0.171	0.129	0.099	0.078	0.064
M14	$\chi^2(P) \geq 0$ If $\chi^2(P) = 0$, P is consistent	0.837	1.941	4.210	7.417	13.357	21.916	27.400	35.191

As the PCM size increases, the coefficient of fitness among the Saaty CR values and the consistency indices of the methods decreases. The threshold values according to the consistency values of the methods with the Saaty CR value are shown in Table 9.

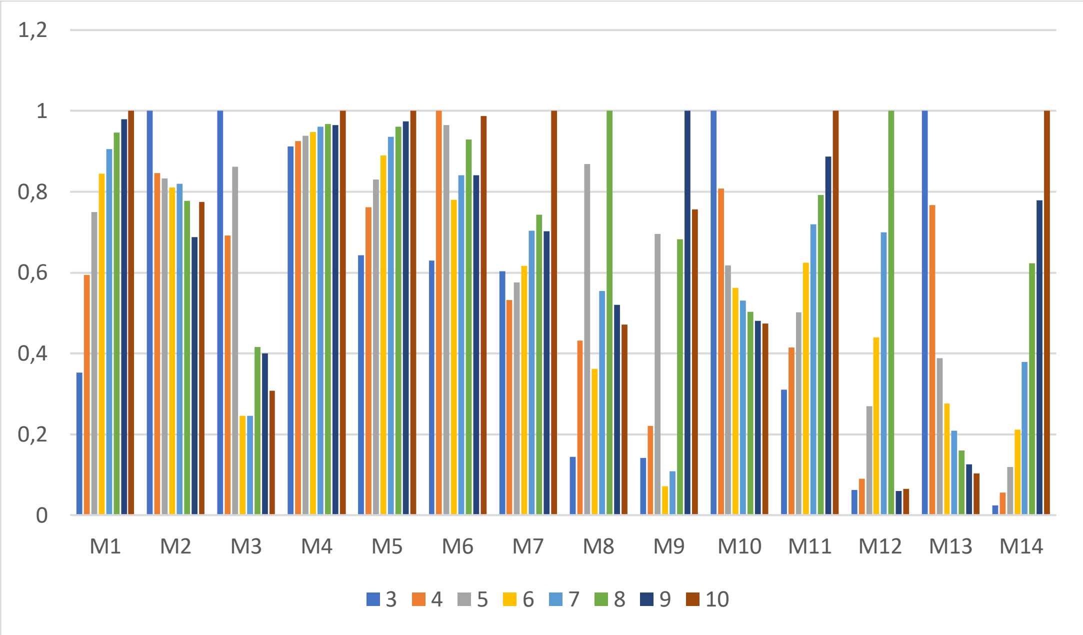

Figure 2. Change of threshold values according to PCM size.

When the changes in the normalized threshold values according to the size of the PCM are analysed, the following results are obtained (Figure 2);

- The threshold values tend to increase with the increase in PCM size in the M1, M4, M5, M11 and M14 methods
- The threshold values decrease continuously with the increase of the PCM size in the M10 and M13 methods
- The change in the threshold values of the M4 method is very small with the increase of PCM size.
- As the PCM values increase, the threshold values of the M3, M6, M7, M8, M9 and M12 methods show unstable changes.

The variation and differences in the threshold values of the methods are based on the computational characteristics of the methods and the increased variability caused by the increasing PCM size.

4. Conclusions

Pairwise comparison operations are an important process step in multi-criteria decision making problems. PCMs play a critical role in defining the positions of the problem elements relative to each other and determining their priority values. Checking whether the decision makers make consistent evaluations enables the decision-making process to proceed more accurately and to make more accurate decisions. The inconsistency index criterion is a value that determines whether the decision makers make their evaluations consistently in PCMs. The inconsistency index defines a consistency value for the decision maker by measuring the transitivity among the elements in the PCM. Threshold values for inconsistency are an important tool that helps decision makers in determining the adequacy of consistency index values.

Although there are different inconsistency index calculation methods in the literature, other methods, except Saaty (M1) and Crawford and Williams (M2) methods, did not specify a threshold value for inconsistency. In this study, 14 different inconsistency calculation methods are discussed and inconsistency indexes of randomly derived PCMs are calculated according to 14 different inconsistency methods.

The similarities between the methods and the relative validity of the methods are identified based on the relationships between the inconsistency indices. M2 (0.911), M4 (0.904) and M14 (0.907) methods have a high correlation with the consistency indices of other methods, while the correlations of M5 (0.672) and M12 (0.674) methods with other methods are weaker according to the correlation tables (Table 3, 4, 5 and 6). The random index values of the methods in different dimensions are calculated by averaging the inconsistency indices of 30000 randomly derived PCMs (Table 7). The obtained random index values can be used as an important tool in calculating the consistency ratios, as in the calculations of CR and GCR. Comparisons are made with common Saaty RI_n values and the validity of the results is shown (Table 8) in order to prove the accuracy of the obtained random index values. Threshold values of consistency indices are defined for 14 different methods at different PCM dimensions using the common Saaty CR threshold ($CR \leq 0.1$). Thresholds for inconsistencies can be used as an important tool in deciding the consistency of PCMs for methods without threshold values, except for the M1 (CR) and M2 (GCR) methods.

While the random indices show regular changes with the increase in PCM size, irregular movements are encountered in the changes in the threshold values (M3, M6, M7, M8, M9, M12) (Fig. 1 and Fig. 2). The increase in size and differences in methods (RGMM, EVM, AMM, SVDM, X^2M, LSM and LLSM) are the main causes of variability. Increasing the number of randomly derived PCMs provides to increase the number of random inconsistency indexes. The use of a more comprehensive inconsistency index helps to obtain more valid random indices and more consistent threshold values based on inconsistency indexes.

Thus, different inconsistency methods are more compatible with each other in making decisions about PCM according to the threshold values of consistency indexes. However, increasing the size of the PCM weakens the power of randomly derived evaluations to reflect true random index values. For example, $17^6 = 24.137.569$ is the number of matrices that can be derived for n=4, and $17^{15} \cong 2,9 \times 10^{18}$ is the number of matrices that can be derived for n=6, of which only any 30,000 random portion is considered and the inconsistency indexes are calculated. In future studies, more accurate random indices and threshold values can be defined by increasing the number of randomly generated PCMs. In addition, consistency ratios based on random indices and threshold values can be developed for methods other than Saaty and Crawford and Williams, and the consistency ratio of 0.1 can be accepted as a general threshold value in deciding the consistency.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

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