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RESEARCH ARTICLE

ON WEAK MINIMAL SOLUTIONS OF SET VALUED OPTIMIZATION PROBLEMS AND COMPARISON OF SOME SET ORDER RELATIONS

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ABSTRACT

In this work, some set order relations are compared with each other. In addition, it is shown that every weak minimal solution of a set valued optimization problem with respect to vector optimization criterion, is also a weak minimal solution with respect to set optimization criterion considering some special set orders.

Keywords: Set order relation, Weak minimal element, Set optimization criterion, Vector optimization criterion

1. INTRODUCTION

Set valued optimization theory is based on to solve optimization problems with set valued objective maps. There are several approaches to solve set-valued optimization problems. One of them is vector optimization criterion and is based on comparing values of range of the objective map. The other one is set optimization criterion and presented by Kuroiwa [1-3]. This approach is based on the comparing values of objective map. Therefore, some ordering relations are required for this comparison. Kuroiwa [2,4], Jahn and Ha [5], Young [6], Nishnianidze [7] and Karaman , Soyertem, Atasever Güvenç, Tozkan, Küçük M. and Küçük Y. [8] gave some set order relations. For further researches in this area one can see [9-20].

In [5], Jahn and Ha compared relations $\preccurlyeq_c, \preccurlyeq_{mc}, \preccurlyeq_{mn}, \preccurlyeq_m, \preccurlyeq_s, \preccurlyeq_u, \preccurlyeq_l$. In this study, the relations $\preccurlyeq_c, \preccurlyeq_{mc}, \preccurlyeq_{mn}, \preccurlyeq_m, \preccurlyeq_s, \preccurlyeq_u, \preccurlyeq_l$ are compared with $\preccurlyeq_{m_1}$and $\preccurlyeq_{m_2}$. Then, it is shown that every weak minimal solution (with respect to vector optimization criterion) of a given set valued optimization problem is a weak minimal solution of the problem with respect to $\preccurlyeq_c, \preccurlyeq_{mc}, \preccurlyeq_{mn}, \preccurlyeq_m, \preccurlyeq_s$. For relations $\preccurlyeq_{m_2}$ and $\preccurlyeq_l$, similar results were given by Khushboo and Lalitha [14] and Hernández and Rodríguez-Marín [9], respectively. For $\preccurlyeq_{m_1}$ and $\preccurlyeq_u$ similar results can be obtained.

2. SOME ORDER RELATIONS OF SETS

In this section, we recall the set orders given by Kuroiwa [2,4], Jahn and Ha [5], Young [6], Nishnianidze [7] and Karaman, Soyertem, Atasever Güvenç, Tozkan, Küçük M. and Küçük Y. [8], and some properties of these relations.

Let $(Y,\|\cdot\|)$ be a normed space. A set $K \subset Y$ is called a cone if $\lambda y \in K$ for all $\lambda \geq 0$ and $y \in K$. Throughout this work, $K \subset Y$ is a nonempty, pointed ($K \cap (-K) = \{0_Y\}$), convex cone with nonempty interior. $B(x,r)$ denotes the closed ball with center x and radius r, clS and $intS$ denotes the closure and interior of a set $S \subset Y$, respectively. The algebraic sum and Minkowski (Pontryagin) difference of A and B is defined by $A + B = \{a + b | a \in A, b \in B\}$, $A \dot{-} B = \{ x \in Y \mid x + B \subset A \}$, respectively. A set

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$A \subset Y$ is called K-bounded if for each U neighborhood of 0_Y there exists $t > 0$ such that $A \subset tU + K$.

Relations $\leq_K$ and $<_K$ are defined in the following way:

$$x \leq_K y \iff y - x \in K,$$
$$x <_K y \iff y - x \in int\, K.$$

Also, note that $\leq_K$ is an order relation on Y.

Definition 2.1. Let $\emptyset \neq A \subset Y$.

- **i.** If $A \cap (\bar{x} - K) = \{\bar{x}\}$ then $\bar{x} \in A$ is called a minimal element of A. $\min A$ denotes the set of all minimal elements of A .
- **ii.** If A$\cap$ $(\bar{x} - intK) = \emptyset$ then $\bar{x} \in A$ is called a weak minimal element of A. $w \min A$ denotes the set of all weak minimal elements of A.
- **iii.** If $A \cap (\bar{x} + K) = \{\bar{x}\}$ then $\bar{x} \in A$ is called a maximal element of A. $\max A$ denotes the set of all maximal elements of A .

In this article, $\mathcal{P}(Y)$ denotes the nomempty subsets of Y and

$$\mathcal{M} := \{ A \in \mathcal{P}(Y) \mid \min A \text{ and } \max A \text{ are nonempty} \}.$$

We first recall the set less order relation $\preccurlyeq_s$ defined by Young [6] and Nishnianidze [7], u-type less order relation $\preccurlyeq_u$ and l-type less order relation $\preccurlyeq_l$ given by Kuroiwa [2,4].

Definition 2.2. Let $A, B \in \mathcal{P}(Y)$.

- **i.** $A \preccurlyeq_s B :\iff (\forall\, x \in A\ \exists y \in B : x \leq_K y)$ and $(\forall\, y \in B\ \exists x \in A : x \leq_K y)$
- **ii.** $A \preccurlyeq_u B :\iff (\forall\, x \in A\ \exists y \in B : x \leq_K y)$
- **iii.** $A \preccurlyeq_l B :\iff (\forall\, y \in B\ \exists x \in A : x \leq_K y)$

In the following proposition, characterizations of $\preccurlyeq_u$, $\preccurlyeq_l$ and $\preccurlyeq_s$ are given.

Proposition 2.3.[5] Let $A, B \in \mathcal{P}(Y)$. Then,

$$A \preccurlyeq_l B \iff B \subset A + K$$
$$A \preccurlyeq_u B \iff A \subset B - K$$
$$A \preccurlyeq_s B \iff B \subset A + K \text{ and } A \subset B - K.$$

Definition 2.4. [5] Let $A, B \in \mathcal{P}(Y)$. The relation $\preccurlyeq_c$ is defined as

$$A \preccurlyeq_c B :\iff (A = B)\ or\ (A \neq B \text{ and } \forall x \in A\ \forall y \in B : x \leq_K y)$$

is called certainly less order relation.

Proposition 2.5. [5] Let $A, B \in \mathcal{P}(Y)$. Then we have

$$A \preccurlyeq_c B \iff (A = B)\ or\ (A \neq B \text{ and } B - A \subset K).$$

Now, we recall $\preccurlyeq_m$ minmax less order relation, $\preccurlyeq_{mc}$ minmax certainly less order relation, $\preccurlyeq_{mn}$ the minmax certainly nondominated order relation are defined by Jahn and Ha [5].

Definition 2.6. [5] Let $A, B \in \mathcal{M}$.

- **i.** $A \preccurlyeq_m B :\iff \min A \preccurlyeq_s \min B$ and $\max A \preccurlyeq_s \max B$
- **ii.** $A \preccurlyeq_{mc} B :\iff (A = B)$ or $(A \neq B$ and $\min A \preccurlyeq_c \min B\ and \max A \preccurlyeq_c \max B)$
- **iii.** $A \preccurlyeq_{mn} B :\iff (A = B)$ or $(A \neq B$ and $\max A \preccurlyeq_s \min B)$

Definition 2.7. [5] Let $A \in \mathcal{M}$. If

$$\min A + K = A + K \text{ and } \max A - K = A - K$$

or equivalently

$$A \subset \min A + K \text{ and } A \subset \max A - K$$

then A is said to have quasi domination property.

Throughout this paper, $\mathcal{M}_0$ denotes the family of sets with quasi domination property.

Now, we recall the comparisons of order relations given above.

Proposition 2.8. [5] Let $A, B \in \mathcal{P}(Y)$. Then,

- **i.** $A \preccurlyeq_s B$ implies $A \preccurlyeq_l B$,
- **ii.** $A \preccurlyeq_s B$ implies $A \preccurlyeq_u B$
- **iii.** $A \preccurlyeq_l B$ does not always imply $A \preccurlyeq_u B$ and vice versa.

Proposition 2.9. [5] Let $A, B \in \mathcal{M}_0$ with $A \neq B$. Then,

- **i.** $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_{mc} B \Longrightarrow A \preccurlyeq_m B \Longrightarrow A \preccurlyeq_s B$
- **ii.** $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_{mn} B \Longrightarrow A \preccurlyeq_m B$
- **iii.** $A \preccurlyeq_{mn} B$ does not always imply $A \preccurlyeq_{mc} B$ and vice versa.

The following set order relations $\preccurlyeq_{m_1}$ and $\preccurlyeq_{m_2}$ were introduced by Karaman, Soyertem, Atasever Güvenç, Tozkan, Küçük M. and Küçük Y. [8]

Definition 2.10. Let $A, B \in \mathcal{P}(Y)$.

- **i.** $A \preccurlyeq_{m_1} B :\Leftrightarrow (B \dot{-} A) \cap K \neq \emptyset$,
- **ii.** $A \preccurlyeq_{m_2} B :\Leftrightarrow (A \dot{-} B) \cap (-K) \neq \emptyset$.
- **iii.** $A \prec_{m_1} B :\Leftrightarrow (B \dot{-} A) \cap (intK) \neq \emptyset$,
- **iv.** $A \prec_{m_2} B :\Leftrightarrow (A \dot{-} B) \cap (-intK) \neq \emptyset$,

Note that $\preccurlyeq_{m_1}$ and $\preccurlyeq_{m_2}$ are partial order relations on $\mathcal{B}(Y) = \{A \in \mathcal{P}(Y) \mid A \text{ is bounded}\}$.

Strict order relations $\prec_u, \prec_l, \prec_s$ and $\prec_c$ are defined in the following way [9]:

Definition 2.11. Let $A, B \in \mathcal{P}(Y)$.

- **i.** $A \prec_u B :\Leftrightarrow A \subset B - int(K)$
- **ii.** $A \prec_l B :\Leftrightarrow B \subset A + int(K)$
- **iii.** $A \prec_s B :\Leftrightarrow A \prec_u B$ and $A \prec_l B$
- **iv.** $A \prec_c B :\Leftrightarrow B - A \subset int(K)$

In addition, $\prec_m, \prec_{mc}$ and $\prec_{mn}$ are defined in the following way:

Definition 2.12. Let $A, B \in \mathcal{M}$.

- **i.** $A \prec_m B :\Leftrightarrow \min A \prec_s \min B$ and $\max A \prec_s \max B$
- **ii.** $A \prec_{mc} B :\Leftrightarrow \min A \prec_c \min B$ and $\max A \prec_c \max B$
- **iii.** $A \prec_{mn} B :\Leftrightarrow \max A \prec_s \min B$.

Proposition 2.13. Let $A, B \in \mathcal{M}$. Then the following relations are satisfied:

- **i.** $A \prec_c B$ implies $A \prec_* B$ where $* \in \{mc, mn, m, s, u, l\}$,
- **ii.** If $B \dot{-} A \neq \emptyset$ then $A \prec_c B$ implies $A \prec_{m_1} B$,
- **iii.** If $A \dot{-} B \neq \emptyset$ then $A \prec_c B$ implies $A \prec_{m_2} B$,
- **iv.** $A \prec_{mc} B$ implies $A \prec_m B$,
- **v.** If $A, B \in \mathcal{M}_0$ then $A \prec_{mn} B$ implies $A \prec_m B, and\ A \prec_m B$ implies $A \prec_s B$.

Proof: The proofs of each relation above may easily be carried out similar to the proof of Proposition 2.9 (Proposition 3.10 in [5]) and Proposition 3.1 considering $K + int\ K \subset int\ K$ and therefore omitted here.

Definition 2.14. [5,8,9] Let $\mathcal{S} \subset \mathcal{P}(Y)$ and $* \in \{u, l, s, m, c, mn, mc, m_1, m_2\}$.

- **i.** $A \in \mathcal{S}$ is called a $*$-minimal element of $\mathcal{S}$ if for any $B \in \mathcal{S}$ such that $B \preccurlyeq_* A$ implies $A \preccurlyeq_* B$. The family of $*$-minimal elements of $\mathcal{S}$ is denoted by $* - \min \mathcal{S}$.
- **ii.** $A \in \mathcal{S}$ is called a weak $*$-minimal element of $\mathcal{S}$ if for any $B \in \mathcal{S}$ such that $B \prec_* A$ implies $A \prec_* B$. The family of weak $*$-minimal elements of $\mathcal{S}$ is denoted by $* - w \min \mathcal{S}$.

Let $X \neq \emptyset, F: X \rightrightarrows Y$ be a set valued map. The set valued optimization problem is defined as

$$(P) \begin{cases} \min F(x) \\ s.t. x \in X. \end{cases}$$

There are three solution concepts for this optimization problem: Vector optimization, set optimization and lattice criteria.

In vector optimization criterion, x_0 is called a minimal solution (weak minimal solution) of (P) if there exists $y_0 \in F(x_0)$ such that $y_0 \in \min F(X)$ $(y_0 \in w \min F(X))$ where $F(X) = \bigcup_{x \in X} F(x)$. (x_0, y_0) is called a minimizer (weak minimizer) of (P). The set of all minimal (weak minimal) solutions of (P) is denoted by $Eff(F, X)$ $(wEff(F, X))$.

Set optimization criterion is based on finding efficient sets of the family $\mathcal{F}(X) = \{F(x) \mid x \in X\}$ with respect to given set order relation. In set optimization criterion, we denote problem (P) as $(* - P)$ where $\preccurlyeq_*$is the set order relation considered in the problem. In this criterion, x_0 is called a $*$-minimal solution (weak $*-$ minimal solution) of $(* - P)$ if $F(x_0) \in * - \min \mathcal{F}(X)$ $(F(x_0) \in * - \mathrm{w} \min \mathcal{F}(X))$. The set of all $*-$minimal (weak $*-$ minimal) solutions of $(* - P)$ is denoted by $* - Eff(F, X)$ $(* - wEff(F, X))$.

3. COMPARISON OF SET ORDER RELATIONS

In this section, relations $\preccurlyeq_c$, $\preccurlyeq_{mc}$, $\preccurlyeq_{mn}$, $\preccurlyeq_m$, $\preccurlyeq_s$, $\preccurlyeq_u$, $\preccurlyeq_l$ are compared with $\preccurlyeq_{m_1}$and $\preccurlyeq_{m_2}$.

Proposition 3.1. Let $A, B \in \mathcal{P}(Y)$.

- **i.** If $B \dot{-} A \neq \emptyset$ and $A \preccurlyeq_c B$ then $A \preccurlyeq_{m_1} B$ and if $A \dot{-} B \neq \emptyset$ and $A \preccurlyeq_c B$ then $A \preccurlyeq_{m_2} B$,
- **ii.** $A \preccurlyeq_{m_1} B$ does not always imply $A \preccurlyeq_c B$ and $A \preccurlyeq_{m_2} B$ does not imply $A \preccurlyeq_c B$.

Proof: i. Let $A \preccurlyeq_c B$. If $A = B$ then it is obvious that $A \preccurlyeq_{m_1} B$. If $A \neq B$ then, by Proposition 2.5, $B - A \subset K$. Since $B \dot{-} A \subset B - A$ we have $B \dot{-} A \subset K$ which completes the proof.
ii. See Example 3.2.

Example 3.2. Let $Y = \mathbb{R}^2, K = \mathbb{R}^2_+, C = B\big((0,0), 1\big)$ and $D = B\big((0,0), 2\big)$. Then $D \dot{-} C = C$. Hence, $(D \dot{-} C) \cap K \neq \emptyset$ and $(D \dot{-} C) \cap (-K) \neq \emptyset$. Then, we have $C \preccurlyeq_{m_1} D$ and $D \preccurlyeq_{m_2} C$, respectively. However, $C - D = D - C = B\big((0,0), 3\big) \not\subset K$ which means $D \not\preccurlyeq_c C$ and $C \not\preccurlyeq_c D$.

Proposition 3.3.[8] Let $A, B \in \mathcal{P}(Y)$. Then,

- **i.** $A \preccurlyeq_{m_1} B \implies A \preccurlyeq_u B$ and $A \preccurlyeq_{m_2} B \implies A \preccurlyeq_l B$,
- **ii.** $A \preccurlyeq_u B$ does not imply $A \preccurlyeq_{m_1} B$ and $A \preccurlyeq_l B$ does not imply $A \preccurlyeq_{m_2} B$.

Proposition 3.4. Let $A, B \in \mathcal{P}(Y)$. Then, the following relations are satisfied:

- **i.** If $A \preccurlyeq_l B$, $B \dot{-} A \neq \emptyset$ and A is K-bounded then $A \preccurlyeq_{m_1} B$.
- **ii.** If $A \preccurlyeq_u B$, $A \dot{-} B \neq \emptyset$ and B is $-K$-bounded then $A \preccurlyeq_{m_2} B$.
- **iii.** $A \preccurlyeq_{m_1} B$ does not imply $A \preccurlyeq_l B$ and $A \preccurlyeq_{m_2} B$ does not imply $A \preccurlyeq_u B$.

Proof: i. Let $A \preccurlyeq_l B$ and $y \in B \dot{-} A$. Then $B \subset A + K$ and $y + A \subset B$. This implies

$$y + A \subset B \subset A + K. \tag{3.1}$$

Let a be a fixed arbitrary element of A. From (3.1) there exists $a_1 \in A$ such that $y + a \in a_1 + K$. For $a_1 \in A$ there exists $a_2 \in A$ such that $y + a_1 \in a_2 + K$. By this way, we construct a sequence $\{a_n\}$ in A which satisfies $y + a_{n-1} \in a_n + K$ for all $n \in \mathbb{N}$. Since K is a convex cone we have

$$ny + a + a_1 + a_2 + \cdots + a_{n-1} \in a_1 + a_2 + \cdots + a_n + K$$

for all $n \in \mathbb{N}$. Hence, $y \in \frac{a_n - a}{n} + K$ for all $n \in \mathbb{N}$. Since A is K-bounded so is $A - a$. Thus, for closed unit ball $B(0_Y, 1)$ there exists $t \in \mathbb{R}$ such that

$$y \in \frac{a_n - a}{n} + K \subset \frac{A - a}{n} + K \subset \frac{t}{n} B(0_Y, 1) + K. \tag{3.2}$$

As K is closed we obtain $y \in K$ by taking limit $n \to \infty$ in (3.2). Hence, $B \dot{-} A \subset K$ which implies $A \preccurlyeq_{m_1} B$.

ii. It can be proved similar to (i).

iii. See Example 3.5.

Example 3.5. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B\big((0,0), 1\big)$, $B = [-1,1] \times [-1,1]$. Since $B \dot{-} A = \{(0,0)\}$, we have $(B \dot{-} A) \cap K \neq \emptyset$ and $(B \dot{-} A) \cap (-K) \neq \emptyset$. So, $A \preccurlyeq_{m_1} B$ and $B \preccurlyeq_{m_2} A$, respectively. In addition, since $B \not\subset A + K$ and $B \not\subset A - K$, it follows that $A \not\preccurlyeq_l B$ and $B \not\preccurlyeq_u A$, respectively.

Observe that Proposition 3.4 (i) may not be true without K-boundedness of A or the condition $B \dot{-} A \neq \emptyset$ as shown in the following example.

Example 3.6. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B\big((0,0), 2\big)$, $C = B\big((2,2), 1\big)$. It is obvious that $A \preccurlyeq_l C$ and $C \dot{-} A = \emptyset$. So, we have $A \not\preccurlyeq_{m_1} C$.

In addition, let $C = \{0\} \times (-\infty, 0]$, $D = (0, \infty) \times (-\infty, 0)$. It can be seen that C is not K-bounded and $C \preccurlyeq_l D$. Since, $D \dot{-} C = D$ and $D \cap K = \emptyset$ we obtain $C \not\preccurlyeq_{m_1} D$.

The following corollary is obtained directly from the definition of $\preccurlyeq_s$, Proposition 3.3 and Proposition 3.4.

Corollary 3.7. Let $A, B \in \mathcal{P}(Y)$, A be K-bounded, B be $-K$-bounded, $B \dot{-} A \neq \emptyset$, $A \dot{-} B \neq \emptyset$ and K be closed. Then,

$$A \preccurlyeq_s B \Leftrightarrow A \preccurlyeq_{m_1} B \text{ and } A \preccurlyeq_{m_2} B.$$

The following example shows that $A \preccurlyeq_s B$ does not always imply $A \preccurlyeq_{m_1} B$ and $A \preccurlyeq_{m_2} B$.

Example 3.8. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$.

- **i.** If $A = \{(-1,0)\} \cup int\, K$, $B = \{(0,0)\} \cup int\, K$, then we have $A \preccurlyeq_s B$. Since $A \dot{-} B = int\, K$ and $(A \dot{-} B) \cap (-K) = \emptyset$ we get $A \not\preccurlyeq_{m_2} B$.
- **ii.** For $A = \{(0,0)\} \cup (-intK)$, $B = \{(1,0)\} \cup (-intK)$ one can see that $B \dot{-} A = -int\, K$ and hence $A \not\preccurlyeq_{m_1} B$. Furthermore we have $A \preccurlyeq_s B$.

Remark 3.9. In Example 3.8 (i), as $B \dot{-} A = \{(1,0)\} + int\, K$ we have $A \preccurlyeq_{m_1} B$. Therefore, $A \preccurlyeq_{m_1} B$ does not imply $A \preccurlyeq_{m_2} B$. In addition in Example 3.8 (ii), since $A \dot{-} B = \{(-1,0)\} - int\, K$ we obtain $A \preccurlyeq_{m_2} B$. Hence, $A \preccurlyeq_{m_1} B$ does not necessarily follow from $A \preccurlyeq_{m_2} B$.

Corollary 3.10. Let $A, B \in \mathcal{M}_0$, $B \dot{-} A \neq \emptyset$, A be K-bounded and K be closed. Then, the following are satisfied:

- **i.** If $A \preccurlyeq_m B$ then $A \preccurlyeq_{m_1} B$,
- **ii.** If $A \preccurlyeq_{mn} B$ then $A \preccurlyeq_{m_1} B$,

iii. $A \preccurlyeq_{m_1} B$ does not imply $A \preccurlyeq_m B$ and $A \preccurlyeq_{mn} B$.

Proof: (i) and (ii) are proved by using Proposition 2.9 and Proposition 3.4. For (iii) see Example 3.12.

Corollary 3.11. Let $A, B \in \mathcal{M}_0$, $A \dot{-} B \neq \emptyset$, B be $-K$-bounded and K be closed. Then, the following are satisfied:

i. If $A \preccurlyeq_m B$ then $A \preccurlyeq_{m_2} B$,

ii. If $A \preccurlyeq_{mn} B$ then $A \preccurlyeq_{m_2} B$,

iii. $A \preccurlyeq_{m_2} B$ does not imply $A \preccurlyeq_m B$ and $A \preccurlyeq_{mn} B$.

Proof: (i) and (ii) can be proved similar with Corollary 3.10 (i,ii). For (iii) see Example 3.12.

Example 3.12. Let $B = [-2,1] \times [-1,1]$ and $A = B\big((0,0),1\big)$. As $B \dot{-} A = [-1,0] \times \{0\}$ we have $A \preccurlyeq_{m_1} B$. Since $\min A = \{(x,y) \mid x^2 + y^2 = 1, x, y \leq 0\}, \min B = \{(-2,-1)\}$ we have $\min B \not\subset \min A + K$. So, $\min A \not\preccurlyeq_s \min B$ i.e. $A \not\preccurlyeq_m B$. In addition, $\max A = \{(x,y) \mid x^2 + y^2 = 1,\ x, y \geq 0\}$. Since $\min B \not\subset \max A + K$, $\max A \not\preccurlyeq_s \min B$. Hence, we get $A \not\preccurlyeq_{mn} B$.

Also, we have $B \preccurlyeq_{m_2} A$. Since $\max B \not\preccurlyeq_s \max A$ and $\max B \not\preccurlyeq_s \min A$ it follows that $B \not\preccurlyeq_m A$ and $B \not\preccurlyeq_{mn} A$,
respectively.

Remark 3.13: In Corollary 3.10 and Corollary 3.11 quasi domination property cannot be omitted. For example, if $K = \mathbb{R}^2_+$, $A = \{0\} \times [0,1]$ and $B = \{(0,1)\} \cup \big((0,\infty) \times (-\infty,0)\big)$, then it is obvious that $B \notin \mathcal{M}_0$, $\min A = \{(0,0)\}$,$\min B = \max B = \max A = \{(0,1)\}$, $B \dot{-} A = (0,\infty) \times (-\infty,-1)$. So, $A \preccurlyeq_m B$, $A \preccurlyeq_{mn} B$ and $A \not\preccurlyeq_{m_1} B$. A similar example can be given for Corollary 3.11.

Theorem 3.14. Let $A, B \in \mathcal{M}_0$ and $A \preccurlyeq_{mc} B$. The following relations are satisfied:

i. If $B \dot{-} A \neq \emptyset$ then $A \preccurlyeq_{m_1} B$,

ii. If $A \dot{-} B \neq \emptyset$ then $A \preccurlyeq_{m_2} B$,

iii. Neither $A \preccurlyeq_{m_1} B$ nor $A \preccurlyeq_{m_2} B$ imply $A \preccurlyeq_{mc} B$.

Proof: i. $A \preccurlyeq_{mc} B$ means $A = B$ or $A \neq B$ and

$$\min A \preccurlyeq_c \min B, \quad \max A \preccurlyeq_c \max B. \tag{3.3}$$

If $A = B$, the proof is completed. Consider the case $A \neq B$. Let $y \in B \dot{-} A$, i.e. $y + A \subset B$. Since $B \in \mathcal{M}_0$ and $\min A \subset A$ we have

$$y + \min A \subset y + A \subset B \subset \min B + K.$$

Then, for arbitrary $a \in \min A$ there exists $b \in \min B$ and $k \in K$ such that $y + a = b + k$. Hence, $y = b - a + k$. From (3.3) $a \leq_K b$, i.e. $b - a \in K$. In addition, by convexity of K we obtain $y = b - a + k \in K + K \subset K$. So, $B \dot{-} A \subset K$ which gives $A \preccurlyeq_{m_1} B$.

ii. It can be proved similar to (i).

iii. See Example 3.15.

Example 3.15. Let $K = \mathbb{R}^2_+$, $A = B\big((0,0),1\big), C = B((1,0),1)$. It is clear that $C \dot{-} A = \{(1,0)\}$ and $A \preccurlyeq_{m_1} C$. Also $\min A = \{(x,y) \mid x^2 + y^2 = 1, x, y \leq 0\}$, $\min C = \{(x,y) \mid (x-1)^2 + y^2 = 1, x \leq 1, y \leq 0\}$. Thus, $(-1,0) \in \min A$, $(1,-1) \in \min C$ and $(-1,0) \not\leq_K (1,-1)$. Therefore, we have $\min A \not\preccurlyeq_c \min C$ which implies A $\not\preccurlyeq_{mc} C$.

Remark 3.16. The example given in Remark 3.13 shows that quasi domination property of B in Theorem 3.14 (ii) is necessary. In this example, B does not have quasi domination property. In addition, $A \preccurlyeq_{mc} B$, A $\not\preccurlyeq_{m_1} B$. Furthermore, since $A\dot{-}B = \emptyset$ we have A $\not\preccurlyeq_{m_2} B$.

4. RELATIONS BETWEEN WEAK MINIMAL ELEMENTS WITH RESPECT TO VECTOR AND SET OPTIMIZATON CRITERIA

In this section, set of weak minimal elements of a set optimization problem with respect to vector and set optimization criteria are compared.

Lemma 4.1. Let $\mathcal{S} \subset \mathcal{P}(Y)$. Then $A \in c - w \min\ \mathcal{S}$ if and only if there is no $B \in\ \mathcal{S}$ such that $B \prec_c A$.

Proof: Suppose that $B \in\ \mathcal{S}$ and $B \prec_c A$. Hence, $b <_K a$ i.e. $a - b \in int\ K$ for all $a \in A$ and $b \in B$. As $A \in c - \min\ \mathcal{S}$ we have $A \prec_c B$ which means $a <_K b$ i.e. $a - b \in -int\ K$ for all $a \in A$ and $b \in B$. This contradicts with pointedness of K.

Lemma 4.2. Let $\mathcal{S} \subset \mathcal{M}$. Then $A \in mc - w \min\ \mathcal{S}$ if and only if there is no $B \in\ \mathcal{S}$ such that $B \prec_{mc} A$.

Proof: Let $A \in mc - \min\ \mathcal{S}$. Suppose there exists $B \in \mathcal{S}$ such that $B \prec_{mc} A$. Since $A \in mc - w \min\ \mathcal{S}$, we get $A \prec_{mc} B$. Hence, we obtain

$$\min A \prec_c \min B, \quad \max A \prec_c \max B$$
$$\min B \prec_c \min A, \quad \max B \prec_c \max A.$$

So, we have $\min A \prec_c \min A$ and $\max A \prec_c \max A$ which is a contradiction.

The converse is obtained directly from the definition of weak mc-minimal element.

Lemma 4.3. Let $\mathcal{S} \subset \mathcal{M}_0$. Then $A \in mn -$ w min $\mathcal{S}$ if and only if there is no $B \in\ \mathcal{S}$ such that $B \prec_{mn} A$.

Proof: Let $A \in mn -$ w min $\mathcal{S}$ and there exist $B \in\ \mathcal{S}$ such that $B \prec_{mn} A$. Then,

$$\min A \subset \max B + int(K). \tag{4.1}$$

As $A \in mn -$ w min $\mathcal{S}$ we have $A \prec_{mn} B$ which implies

$$\min B \subset \max A + int(K). \tag{4.2}$$

From (4.1), (4.2), convexity of K and since $A, B \in \mathcal{M}_0$ we get

$$\min A \subset \max B + int(K) \subset B + int\ K \subset \min B + intK \subset \max A + int(K).$$

Hence, for arbitrary $y \in \min A$ there exist $\tilde{y} \in \max A$ and $k \in int(K)$ such that $y = \tilde{y} + k$. We, therefore, obtain $\tilde{y} <_K y$ that contradicts to $y \in \min A$.

The definition of weak mn-minimal element implies the converse statement.

Lemma 4.4. Let $\mathcal{S} \subset \mathcal{M}$. Then $A \in m -$ w min $\mathcal{S}$ if and only if there is no $B \in\ \mathcal{S}$ such that $B \prec_m A$.

Proof: Let $A \in m -$ w min $\mathcal{S}$. Suppose that there exist $B \in \mathcal{S}$ satisfying $B \prec_m A$. Since $A \in m -$ w min $\mathcal{S}$ we get $A \prec_m B$. So, we have $\min B \subset \min A + int(K)$ and $\min A \subset \min B + int(K)$.
Since K is convex $\min B + intK \subset \min A + int(K) + int\ (K) \subset \min A + int\ K$. Hence,
$\min A \subset \min A + int(K)$. Then, for any $y \in \min A$ there exists $\tilde{y} \in \min A$ such that $\tilde{y} <_K y$ which contradicts to $y \in \min A$.
The converse can be proved by using the definition of weak m-minimal element.

Lemma 4.5: Let $\mathcal{S} \subset \mathcal{P}(Y), A \in \mathcal{S},\ w \min A \neq \emptyset$ or $w \max A \neq \emptyset$. Then $A \in s - w \min A$ if and only if there is not any $B \in \mathcal{S}$ such that $B \prec_s A$.

Proof: Let $A \in s - w \min A$ and $w \min A \neq \emptyset$. Assume that there exists $B \in \mathcal{S}$ such that $B \prec_s A$. Since $A \in\ s - w \min A$ we have $A \prec_s B$. So, we obtain $A \subset B + int(K), A \subset B - int(K), B \subset A + int(K)$ and $B \subset A - int(K)$. Hence,

$$A \subset A + int(K) \tag{4.3}$$
$$A \subset A - int(K). \tag{4.4}$$

Therefore, $w \min A \subset A \subset A + int\ K$. So, for any $a \in w \min A$ there exists $\tilde{a} \in A$ such that $\tilde{a} <_K a$ which contradicts with $a \in w \min A \neq \emptyset$.

If we suppose $w \max A \neq \emptyset$ then (4.4) contradicts with $w \max A \neq \emptyset$.

The converse is obtained directly from the definition of weak s-minimal element.

Lemma 4.6 [9]: Let $\mathcal{S} \subset \mathcal{P}(Y), A \in \mathcal{S},\ w \min A \neq \emptyset$. Then $A \in l - w \min A$ if and only if there is not any $B \in \mathcal{S}$ such that $B \prec_l A$.

Following theorems state that every weak minimal solution of the set valued optimization problem (P) with respect to vector optimization criterion is also a weak minimal solution of $(* - P)$ where $* \in \{c, mc, mn, m, s, l\}$.

Theorem 4.7: $wEff(F,X) \subset c - wEff(F,X)$.

Proof: Let $x_0 \in wEff(F,X)$ and (x_0, y_0) be a weak minimizer of (P). Assume that $x_0 \notin c - wEff(F,X)$. So there is $\tilde{x} \in X$ such that $F(\tilde{x}) \prec_c F(x_0)$. So we have $y <_K\ y_0$ for all $y \in F(\tilde{x})$. This contradicts with $y_0 \in w \min F(X)$.

Theorem 4.8. [9] If $x_0 \in X$ is a weak minimal solution of (P) then x_0 is weak minimal solution of $(l - P)$.

Theorem 4.9. $wEff(F,X) \subset s - wEff(F,X)$.
Proof: Let $x_0 \in wEff(F,X)$. Theorem 4.8 gives that x_0 is weak l-minimal solution. Suppose that $x_0 \notin s - wEff(F,X)$. Then from Lemma 4.5 there exists $\tilde{x} \in X$ satisfying $F(\tilde{x}) \prec_s F(x_0)$. So, we get $F(\tilde{x}) \prec_l F(x_0)$ that contradicts with weak l −minimality of x_0 by Lemma 4.6.

The following theorem is a result of Theorem 4.9, Proposition 2.13 and Lemma 4.2-4.5.

Theorem 4.10. Let F has quasi domination property (i.e. $F(x) \in \mathcal{M}_0$ for all $x \in X$). Then $wEff(F,X) \subset mn - wEff(F,X),\ wEff(F,X) \subset mc - wEff(F,X), wEff(F,X) \subset m - wEff(F,X)$.

Proof: Suppose $x_0 \in wEff(F,X)$ but $x_0 \notin mn - wEff(F,X)$. Then from Lemma 4.3 there exists $\tilde{x} \in X$ satisfying $F(\tilde{x}) \prec_{mn} F(x_0)$. So, from Proposition 2.13 we obtain $F(\tilde{x}) \prec_s F(x_0)$. However, Theorem 4.9 gives that x_0 is a weak s- minimal solution which contradicts the inequality $F(\tilde{x}) \prec_s F(x_0)$.

Proof can be done similar for the problems $(mc - P)$ and $(m - P)$.

Remark 4.11. Quasi domination property is necessary in Theorem 4.10. For example, if the set valued map $F:\{1,2\} \to \mathbb{R}^2$ is defined as $F(1) = \{(-1,1)\} \cup \{\,(0,y) \mid y < 0\,\}$ and $F(2) = \{(-2,0)\} \cup \{\,(0,y) \mid y < 0\,\}$, then it is obvious that $F(1)$ and $F(2)$ don't have quasi domination property. As $w\min\big(F(1) \cup F(2)\big) = \{(-2,0)\} \cup \{\,(0,y) \mid y < 0\,\}$ we obtain $\big(1,(0,-1)\big)$ and $\big(2,(0,-1)\big)$ is a weak minimizer of (P). Hence, $x_0 = 1$ and $x_0 = 2$ weak minimal solutions of (P).

In addition, since $\min F(1) = \max F(1) = \{(-1,1)\}$, $\min F(2) = \max F(2) = \{(-2,0)\}$ we have $F(2) \prec_{mc} F(1), F(2) \prec_{mn} F(1), F(2) \prec_{m} F(1)$. So, $x_0 = 1$ is the unique weak minimal solution of $(mc - P), (mn - P)$ and $(m - P)$.

A similar result for relation $\preccurlyeq_{m_2}$ was obtained by Khushboo and Lalitha in [14].

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CONFLICT OF INTEREST

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