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Research Article

Banach-valued Bloch-type functions on the unit ball of a Hilbert space and weak spaces of Bloch-type

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ABSTRACT. In this article, we study the space $\mathcal{B}_\mu(B_X, Y)$ of Y-valued Bloch-type functions on the unit ball B_X of an infinite dimensional Hilbert space X with μ is a normal weight on B_X and Y is a Banach space. We also investigate the characterizations of the space $\mathcal{WB}_\mu(B_X)$ of Y-valued, locally bounded, weakly holomorphic functions associated with the Bloch-type space $\mathcal{B}_\mu(B_X)$ of scalar-valued functions in the sense that $f \in \mathcal{WB}_\mu(B_X)$ if $w \circ f \in \mathcal{B}_\mu(B_X)$ for every $w \in \mathcal{W}$, a separating subspace of the dual Y' of Y.

Keywords: Operators on Hilbert spaces, Bloch spaces, weak holomorphic.

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1. INTRODUCTION

The space of classical Bloch functions on the unit disk $\mathbb{B}_1$ of the complex plane $\mathbb{C}$ was extended to the higher dimension cases. In 1975, using terminology from differential geometry [5], K. T. Hahn introduced the notion of Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$. Further, Bloch functions on bounded homogeneous domains in terms of the Bergman metric was studied by R. M. Timoney in [12, 13]. In [7], S. G. Krantz and D. Ma considered function theoretic and functional analytic properties of Bloch functions on strongly pseudoconvex domain.

Recently, O. Blasco and his colleagues extended the notion to the infinite dimensional setting by considering Bloch functions on the unit ball of an infinite dimensional Hilbert space (see [1, 2, 3]) and, after that, Z. Xu continued the study this topic (see [14]). C. Chu, H. Hamada, T. Honda, G. Kohr generalized the Bloch space to a bounded symmetric domain in a complex Banach space realized as the open unit ball of a JB^*-triple (see [4]). H. Hamada [6] introduced Bloch-type spaces on the unit ball of a complex Banach space.

Motivated by the above results, in this article, the space of Banach-valued Bloch-type functions on the unit ball B_X of an infinite dimensional Hilbert space X with a normal weight (say Bloch-type space) is introduced. We will consider two possible extensions of the classical Bloch space. The first one extends the classical Bloch space by considering the Bloch-type spaces $\mathcal{B}_\mu(B_X, Y)$ of holomorphic functions f on B_X with values in a Banach space Y such that $\sup_{z\in B_X} \mu(z)\|\Diamond f(z)\| < \infty$ where μ is a normal weight on B_X and $\Diamond f$ denotes either the holomorphic gradient ∇f or the radial derivative Rf of f. Basing on the idea in [1] with minor modifications, we give the connection between functions in $\mathcal{B}_\mu(B_X, Y)$ and their restrictions to finite dimensional ones, which leads to the fact that if for a given $m \geq 2$, the restrictions of

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the function to the m-dimensional subspaces have their Bloch-type norms uniformly bounded, then the function is a Bloch-type one and conversely. The second one gives the characterizations of the space $\mathcal{WB}_\mu(B_X)$ of Banach-valued, locally bounded, weakly holomorphic functions associated with the Bloch-type space $\mathcal{B}_\mu(B_X)$ of scalar-valued functions in the sense that $f \in \mathcal{WB}_\mu(B_X)$ if $w \circ f \in \mathcal{B}_\mu(B_X)$ for every $w \in \mathcal{W}$, a separating subspace of the dual Y' of Banach space Y.

Finally, some open problems are proposed at the end of the paper.

2. The Bloch-type spaces on the unit ball of a Hilbert space

Throughout the forthcoming, unless otherwise specified, we shall denote by X a complex Hilbert space with the open unit ball B_X and Y a Banach space. By $\mathscr{H}(B_X, Y)$, we denote the vector space of Y-valued holomorphic functions on B_X. We write $\mathscr{H}(B_X)$ instead of $\mathscr{H}(B_X, \mathbb{C})$. Denote

$$\mathscr{H}^\infty(B_X, Y) = \Big\{ f \in \mathscr{H}(B_X, Y) : \sup_{z \in B_X} \|f(z)\| < \infty \Big\}.$$

It is easy to check that $\mathscr{H}^\infty(B_X, Y)$ is Banach under the sup-norm

$$\|f\|_\infty := \sup_{z \in B_X} \|f(z)\|.$$

Let $(e_k)_{k\in\Gamma}$ be an orthonormal basis of X that we fix at once. Then every $z \in X$ can be written as

$$z = \sum_{k\in\Gamma} z_k e_k, \quad \overline{z} = \sum_{k\in\Gamma} \overline{z_k} e_k.$$

Given $f \in \mathscr{H}(B_X, Y)$ and $z \in B_X$. We will denote, as usual, by $\nabla f(z)$ the gradient of f at z; that is, the unique element representing the linear operator $f'(z) \in L(X, Y)$. We can write

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_k}(z) \Big)_{k\in\Gamma}$$

and hence

$$f'(z)(x) = \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(x_k e_k) \quad \forall x \in X.$$

We define the radial derivative of f at $z \in B_X$ as follows:

$$Rf(z) := \sum_{k\in\Gamma} \frac{\partial f}{\partial z_k}(z)(z_k e_k) = f'(z)(z).$$

It is obvious that

$$\|Rf(z)\| \le \|\nabla f(z)\| \|z\| \quad \forall z \in B_X$$

and

$$\|\nabla f(z)\| := \sup_{u \in Y', \|u\|=1} \|\nabla(u \circ f)(z)\|, \quad \|Rf(z)\| := \sup_{u \in Y', \|u\|=1} |R(u \circ f)(z)|.$$

Definition 2.1. *A positive, continuous function μ on the interval $[0,1)$ is called normal if there are three constants $0 \le \delta < 1$ and $0 < a < b < \infty$ such that*

$$(W_1) \qquad \frac{\mu(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\mu(t)}{(1-t)^a} = 0,$$

$$(W_2) \qquad \frac{\mu(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\mu(t)}{(1-t)^b} = \infty.$$

If we say that a function $\mu : B_X \to [0,\infty)$ *is normal, we also assume that it is radial, that is,* $\mu(z) = \mu(\|z\|)$ *for every* $z \in B_X$.

Then, it follows from (W_1) that a normal function μ is strictly decreasing on $[\delta, 1)$ and $\mu(t) \to 0$ as $t \to 1$. Note that, for every non-increasing, normal weight μ,

$$S_\mu := \sup_{t\in[0,1)} \frac{(1-t)^b}{\mu(t)} < \infty. \tag{2.1}$$

Throughout this paper, a weight always is assumed to be normal. For a normal weight μ on B_X, we denote

$$I_\mu(z) := \int_0^{\|z\|} \frac{dt}{\mu(t)} \quad \forall z \in B_X.$$

In the sequel, when no confusion can arise, we will use the symbol $\Diamond$ to denote either ∇ or R. We define Bloch-type spaces on the unit ball B_X as follows:

$$\mathcal{B}^{\Diamond}_{\mu}(B_X, Y) := \Big\{ f \in \mathscr{H}(B_X, Y) : \ \|f\|_{s\mathcal{B}^{\Diamond}_{\mu}(B_X,Y)} := \sup_{z\in B_X} \mu(z)\|\Diamond f(z)\| < \infty \Big\}.$$

It is easy to check $\|\cdot\|_{s\mathcal{B}^{\Diamond}_{\mu}(B_X,Y)}$ is a semi-norm on $\mathcal{B}^{\Diamond}_{\mu}(B_X, Y)$ and this space is Banach under the sup-norm

$$\|f\|_{\mathcal{B}^{\Diamond}_{\mu}(B_X,Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}^{\Diamond}_{\mu}(B_X,Y)}.$$

We also define little Bloch-type spaces on the unit ball B_X as follows:

$$\mathcal{B}^{\Diamond}_{\mu,0}(B_X, Y) := \Big\{ f \in \mathcal{B}^{\Diamond}_{\mu}(B_X, Y) : \ \lim_{\|z\|\to 1} \mu(z)\|\Diamond f(z)\| = 0 \Big\}$$

endowed with the norm induced by $\mathcal{B}^{\Diamond}_{\mu}(B_X, Y)$. In the case $Y = \mathbb{C}$, we write $\mathcal{B}^{\Diamond}_{\mu}(B_X)$, $\mathcal{B}^{\Diamond}_{\mu,0}(B_X)$ instead of the respective notations. For $\mu(z) = 1 - \|z\|^2$, we write $\mathcal{B}^{\Diamond}(B_X, Y)$ instead of $\mathcal{B}^{\Diamond}_{\mu}(B_X, Y)$ and when $\dim X = m$, $Y = \mathbb{C}$ we obtain correspondingly the classical Bloch space $\mathcal{B}^{\Diamond}(\mathbb{B}_m)$. We will show below that the study of Bloch-type spaces on the unit ball can be reduced to studying functions defined on finite dimensional subspaces.

Now, for each finite subset $F \subset \Gamma$, in symbol $|F| = m < \infty$, we denote by $\mathbb{B}_{[F]}$ the unit ball of $\operatorname{span}\{e_k,\ k \in F\}$. Without loss of generality we may assume that $F = \{1, \dots, m\}$, and hence $\mathbb{B}_{[F]} = \mathbb{B}_m$. For each $m \in \mathbb{N}$, we denote

$$z_{[m]} := (z_1, \dots, z_m) \in \mathbb{B}_m.$$

For $m \geq 2$ by

$$OS_m := \{x = (x_1, \dots, x_m),\ x_k \in X, \langle x_k, x_j\rangle = \delta_{kj}\},$$

we denote the family of orthonormal systems of order m. It is clear that OS_1 is the unit sphere of X. For every $x \in OS_m$, $f \in \mathscr{H}(B_X, Y)$, we define

$$f_x(z_{[m]}) = f\Big(\sum_{k=1}^m z_k x_k\Big).$$

Then

$$\Big\|\nabla f_x(z_{[m]})\Big\| = \Big\|\nabla f\Big(\sum_{k=1}^m z_k x_k\Big)\Big\|. \tag{2.2}$$

Definition 2.2. *Let $\mathbb{B}_1$ be the open unit ball in $\mathbb{C}$ and $f \in \mathscr{H}(B_X, Y)$. We define an affine semi-norm as follows*

$$\|f\|_{s\mathcal{B}_\mu^{\text{aff}}(B_X,Y)} := \sup_{\|x\|=1} \|f(\cdot x)\|_{s\mathcal{B}_\mu(\mathbb{B}_1,Y)},$$

where $f(\cdot x) : \mathbb{B}_1 \to Y$ given by $f(\cdot x)(\lambda) = f(\lambda x)$ for every $\lambda \in \mathbb{B}_1$, and

$$\|f(\cdot x)\|_{s\mathcal{B}_\mu^R(\mathbb{B}_1,Y)} = \sup_{\lambda \in \mathbb{B}_1} \mu(\lambda x)\|f'(\cdot x)(\lambda)\|.$$

It is easy to see that $\|\cdot\|_{s\mathcal{B}_\mu^{\text{aff}}(B_X,Y)}$ is a semi-norm on $\mathcal{B}_\mu(B_X, Y)$. We denote

$$\mathcal{B}_\mu^{\text{aff}}(B_X, Y) := \{f \in \mathcal{B}_\mu(B_X, Y) : \|f\|_{s\mathcal{B}_\mu^{\text{aff}}(B_X,Y)} < \infty\}.$$

It is also easy to check that $\mathcal{B}_\mu^{\text{aff}}(B_X, Y)$ is Banach under the norm

$$\|f\|_{\mathcal{B}_\mu^{\text{aff}}(B_X,Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}_\mu^{\text{aff}}(B_X,Y)}.$$

We also define little affine Bloch-type spaces on the unit ball B_X as follows:

$$\mathcal{B}_{\mu,0}^{\text{aff}}(B_X, Y) := \Big\{f \in \mathcal{B}_\mu^{\text{aff}}(B_X, Y) : \lim_{|\lambda|\to 1} \sup_{\|x\|=1} \mu(\lambda x)\|f'(\cdot x)(\lambda)\| = 0\Big\}.$$

As the above, for $\mu(z) = 1 - \|z\|^2$ we use notation $\mathcal{B}$ and $\mathcal{B}_0$ instead of $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$, respectively.

Proposition 2.1. *Let $f \in \mathscr{H}(B_X, Y)$. The following are equivalent:*

(1) $f \in \mathcal{B}_\mu^\nabla(B_X, Y)$;
(2) $\sup_{x \in OS_m} \|f_x\|_{\mathcal{B}_\mu^\nabla(\mathbb{B}_m,Y)} < \infty$ *for every $m \geq 2$;*
(3) *There exists $m \geq 2$ such that* $\sup_{x \in OS_m} \|f_x\|_{\mathcal{B}_\mu^\nabla(\mathbb{B}_m,Y)} < \infty$.

Moreover, for each $m \geq 2$

$$\|f\|_{s\mathcal{B}_\mu^\nabla(B_X,Y)} = \sup_{x \in OS_m} \|f_x\|_{s\mathcal{B}_\mu^\nabla(\mathbb{B}_m,Y)}. \tag{2.3}$$

Proof. (1) ⇒ (2): Let $m \geq 2$ and $z_{[m]} \in \mathbb{B}_m$. According to (2.2)

$$\|\nabla f_x(z_{[m]})\| = \left\|\nabla f\left(\sum_{j=1}^m z_j e_j\right)\right\|.$$

Denote $\mu^{[m]} = \mu|_{\mathbb{B}_m}$. Since $\|\sum_{j=1}^m z_j e_j\| = \|z_{[m]}\|$ we get

$$\begin{aligned} \|f_x\|_{s\mathcal{B}_{\mu^{[m]}}^\nabla(\mathbb{B}_m,Y)} &= \sup_{z_{[m]} \in \mathbb{B}_m} \mu^{[m]}(z_{[m]})\|\nabla f_x(z_{[m]})\| \\ &\leq \sup_{z \in B_X} \mu^{[m]}(z_{[m]})\left\|\nabla f\left(\sum_{j=1}^m z_j e_j\right)\right\| \\ &\leq \|f\|_{s\mathcal{B}_\mu^\nabla(B_X,Y)}. \end{aligned} \tag{2.4}$$

In particular, we obtain (2).

(2) $\Rightarrow$ (1): Let $z = \sum_{k\in\Gamma} z_k e_k$. We denote the partial sums of this series by s_n. Because f is holomorphic, $\frac{\partial f}{\partial z_j}$ are continuous. Then with $e_{[m]} := (e_1, \ldots, e_m)$ we have

$$\begin{aligned}\|\nabla f(z)\| &= \sup_{u\in Y', \|u\|=1} \|\nabla(u\circ f)(z)\| \\ &= \sup_{u\in Y', \|u\|=1} \lim_{n\to\infty} \|\nabla(u\circ f)(s_n)\| \\ &\leq \sup_{u\in Y', \|u\|=1} \sup_{m\geq 2} \|\nabla(u\circ f_{e_{[m]}})(z_{[m]})\| \\ &= \sup_{x\in OS_m, m\geq 2} \|\nabla f_x(z_{[m]})\|.\end{aligned}$$

Then, it follows from the assumption (2) and $\|z_{[m]}\| \leq \|z\|$, that

$$\begin{aligned}\mu^{[m]}(z_{[m]})\|\nabla f(z)\| &\leq \mu^{[m]}(z_{[m]})\|\nabla f(z)\| \\ &\leq \sup_{x\in OS_m, m\geq 2} \mu^{[m]}(z_{[m]})\|\nabla f_x(z_{[m]})\| < \infty.\end{aligned} \tag{2.5}$$

Thus $f \in \mathcal{B}^{\nabla}_{\mu}(B_X, Y)$.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (1): Assume that there exists $m \geq 2$ such that $\sup_{x\in OS_m} \|f_x\|_{\mathcal{B}_\mu(B_X,Y)} < \infty$. We fix $z \in B_X$, $z \neq 0$. Consider $x = (\frac{z}{\|z\|}, x_2, \ldots, x_m) \in OS_m$ and put $z_{[m]} := (\|z\|, 0, \ldots, 0) \in \mathbb{B}_m$. Then $\|z_{[m]}\| = \|z\|$ and

$$\left\|\nabla f_x(z_{[m]})\right\| = \left\|\nabla f\Big(\sum_{k=1}^{m} z_k x_k\Big)\right\| = \|\nabla f(z)\|. \tag{2.6}$$

This implies that

$$\begin{aligned}\|f\|_{s\mathcal{B}^{\nabla}_{\mu}(B_X,Y)} &= \sup_{z\in\mathcal{B}_X} \mu(z)\|\nabla f(z)\| \\ &\leq \sup_{z\in\mathcal{B}_X} \mu(z_{[m]})\|\nabla f_x(z_{[m]})\| \\ &\leq \sup_{x\in OS_m} \|f_x\|_{\mathcal{B}_\mu(\mathbb{B}_m,Y)} < \infty.\end{aligned} \tag{2.7}$$

Thus $f \in \mathcal{B}^{\nabla}_{\mu}(B_X, Y)$. On the other hand, it is obvious that

$$\sup_{x\in OS_m} \|f_x\|_{\mathcal{B}^{\nabla}_{\mu}(\mathbb{B}_m,Y)} \leq \|f\|_{s\mathcal{B}^{\nabla}_{\mu}(B_X,Y)} \quad \forall m \geq 2. \tag{2.8}$$

Hence, we obtain (2.3) from (2.4), (2.5), (2.7) and (2.8). $\square$

Remark 2.1. *The proposition is not true for the case $m = 1$. Indeed, let X be a Hilbert space with the orthonormal basis $\{e_n\}_{n\geq 1}$. Consider $f : B_X \to \mathbb{C}$ given by*

$$f(z) := \sum_{n=1}^{\infty} \frac{\langle z, e_n\rangle}{\sqrt{n}} \quad \forall z \in B_X.$$

Then $f \in \mathscr{H}(B_X)$ because

$$\sum_{n=1}^{\infty} \frac{|\langle z, e_n\rangle|^2}{n} \leq \sum_{n=1}^{\infty} |\langle z, e_n\rangle|^2 = \|z\|^2 < 1.$$

For each $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \in OS_1$ *and for every* $z_{[1]} := z_1 \in \mathbb{B}_1$*, we have*

$$\nabla f_x(z_{[1]}) = \nabla f(z_1 x_1) = \nabla f\bigg(\sum_{n=1}^{\infty} \frac{\langle z_1 x_1, e_n \rangle}{\sqrt{n}}\bigg),$$

and thus, since $\|\nabla f_x(z_{[1]})\|^2 = |x_1|^2 \leq 1$ *we get*

$$\sup_{x \in OS_1} \|f_x(z_{[1]})\|_{\mathcal{B}_\nabla(\mathbb{B}_1)} = \sup_{x \in OS_1} (1 - \|z_{[1]}\|^2)\|\nabla f_x(z_{[1]})\| \leq 1.$$

However, $f \notin \mathcal{B}^\nabla(B_X)$ *because for every* $z \in B_X$*, we have*

$$\|\nabla f(z)\|^2 = \sum_{n=1}^{\infty} \Big|\frac{\partial f}{\partial z_n}(z)\Big|^2 = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Proposition 2.2. *Let* $f \in \mathscr{H}(B_X, Y)$*. The following are equivalent:*

(1) $f \in \mathcal{B}^\nabla_{\mu,0}(B_X, Y)$;
(2) $\forall \varepsilon > 0\ \exists \varrho > 0\ \forall z \in B_X$ *with* $\|z_{[m]}\| > \varrho$ *for every* $m \geq 2$

$$\sup_{m \geq 2} \sup_{x \in OS_m} \mu(z_{[m]})\|\nabla f_x(z_{[m]})\| < \varepsilon;$$

(3) $\exists m \geq 2\ \forall \varepsilon > 0\ \exists \varrho > 0\ \forall z \in B_X$ *with* $\|z_{[m]}\| > \varrho$

$$\sup_{x \in OS_m} \mu(z_{[m]})\|\nabla f_x(z_{[m]})\| < \varepsilon.$$

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1): The proof is straight-forward by putting $x \in OS_m$ and $z_{[m]} \in \mathbb{B}_m$ as in the proof of (3) $\Rightarrow$ (1) in Proposition 2.1 for each $z \in B_X$ with $\|z\| > \varrho$. □

In the next proofs below we need the following lemma.

Lemma 2.1. *For every* $f \in \mathcal{B}^\nabla_\mu(B_X, Y)$ *and* $x \in X$ *with* $\|x\| = 1$*, we have*

(2.9) $$Rf(\lambda x) = \lambda f'(\cdot x)(\lambda) \quad \forall \lambda \in \mathbb{B}_1$$

and

(2.10) $$f'(\cdot x)(\lambda)(\mu) = f'(\lambda x)(\mu x) \quad \forall \lambda, \mu \in \mathbb{B}_1.$$

Proof. First, it follows from the Bessel inequality that every $x \in X$ has only a countable number of non-zero Fourier coefficients $\langle x, e_j \rangle$. Indeed, for every $\varepsilon > 0$ the set $\{j \in \Gamma : |\langle x, e_j \rangle| > \varepsilon\}$ is finite. Then we still have $x = \sum_{j \in \Gamma} \langle x, e_j \rangle e_j = \sum_{j \in \Gamma} x_j e_j$ where the sum is in fact a countable one, and it is independent of the particular enumeration of the countable number of non-zero summands. Hence, we can write $x = \sum_{j=1}^{\infty} x_j e_j$. Then, by the definitions of $f(\cdot x)$

and $f'(\cdot\, x)(\lambda)$, we have

$$\begin{aligned}&\Big\|\frac{1}{t}\sum_{k=1}^{\infty}\Big(f\Big(\sum_{j=1}^{k}\lambda x_je_j+t\lambda x_ke_k+\sum_{j=k+1}^{\infty}(\lambda+t\lambda)x_je_j\Big)\\&\qquad-f\Big(\sum_{j=1}^{k}\lambda x_je_j+\sum_{j=k+1}^{\infty}(\lambda+t\lambda)x_je_j\Big)\Big)-\lambda f'(\cdot\, x)(\lambda)\Big\|\\&=\Big\|\frac{f((\lambda+t\lambda)x)-f(\lambda x)}{h}-\lambda f'(\cdot\, x)(\lambda)\Big\|\\&=\Big\|\frac{f(\cdot\, x)(\lambda+t\lambda)-f(\cdot\, x)(\lambda)}{t}-\lambda f'(\cdot\, x)(\lambda)\Big\|\to 0\quad\text{as } t\to 0.\end{aligned}$$

Hence (2.9) is proved.

For $\lambda,\eta\in\mathbb{B}_1$ we have

$$\begin{aligned}&\|\eta f'(\cdot\, x)(\lambda)-f'(\lambda x)(\eta x)\|\\&=\Big\|\frac{f(\cdot\, x)(\lambda+t\eta)-f(\cdot\, x)(\lambda)}{t}-\eta f'(\cdot\, x)(\lambda)-\frac{f(\lambda x+t\eta x)-f(\lambda x)}{t}+f'(\lambda x)(\eta x)\Big\|\\&\le\Big\|\frac{f(\cdot\, x)(\lambda+t\eta)-f(\cdot\, x)(\lambda)}{t}-\eta f'(\cdot\, x)(\lambda)\Big\|+\Big\|\frac{f(\lambda x+t\eta x)-f(\lambda x)}{t}+f'(\lambda x)(\eta x)\Big\|\\&\to 0\quad\text{as}\quad t\to 0.\end{aligned}$$

Then $f'(\cdot\, x)(\lambda)(\eta)=\eta f'(\cdot\, x)(\lambda)=f'(\lambda x)(\eta x)$, and (2.10) is proved. □

Proposition 2.3. (1) *The spaces $\mathcal{B}^R_\mu(B_X,Y)$ and $\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)$ coincide. Moreover,*

$$\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}\le\|f\|_{s\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)}\lesssim\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}\quad\forall f\in\mathcal{B}^R_\mu(B_X,Y).$$

(2) *The spaces $\mathcal{B}^R_{\mu,0}(B_X,Y)$ and $\mathcal{B}^{\mathrm{aff}}_{\mu,0}(B_X,Y)$ coincide.*

Proof. (1) Let $f\in\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)$. In order to prove $f\in\mathcal{B}_R(B_X,Y)$ it suffices to show that

$$Rf(z)=\|z\|f'\Big(\cdot\,\frac{z}{\|z\|}\Big)(\|z\|)\quad\forall z\in B_X\setminus\{0\}.\tag{2.11}$$

It is easy to see that (2.11) follows immediately from (2.9) for $y=\frac{z}{\|z\|}$ and $\lambda=\|z\|$ for every $z\in B_X\setminus\{0\}$. Moreover, it follows from (2.11) that

$$\|f\|_{s\mathcal{B}^R_\mu(B_X,Y)}\le\|f\|_{s\mathcal{B}^{\mathrm{aff}}_\mu(B_X,Y)}.$$

Thus, the first inequality in (1) is proved. Now, let $f\in\mathcal{B}^R_\mu(B_X,Y)$ and $x\in X$ be such that $\|x\|=1$. Since f is holomorphic at $0\in B_X$, its derivative $f':B_X\to L(X,Y)$ is also holomorphic, and thus there are $r\in(0,1)$ and $M>0$ such that

$$\|f'(z)\|_{L(X,Y)}\le M\quad\forall z\in\overline{B}(0,r):=\{u\in X:\ \|u\|\le r\}.$$

Then, by (2.10) we have

$$\begin{aligned}\sup_{|\lambda|\le r}\mu(\lambda x)\|f'(\cdot x)(\lambda)\| &= \sup_{|\lambda|\le r}\mu(\lambda x)\sup_{|\eta|\le 1}\|f'(\cdot x)(\lambda)(\eta)\|\\ &= \sup_{|\lambda|\le r}\mu(\lambda x)\sup_{|\eta|\le 1}\|f'(\lambda x)(\eta x)\|\\ &\le \sup_{|\lambda|\le r}\mu(\lambda x)\|f'(\lambda x)\|\le M.\end{aligned}$$

For the case where $\|z\| > r$, by (2.9), (2.10) and the increasing monotony of the function $\frac{1-t}{t}$, similar calculation to [1, Proposition 2.4], we have

$$\mu(\lambda x)|f'(\cdot x)(\lambda)| \le \Big(\mu(\lambda x)\frac{1-r}{r} + \mu(\lambda x)\Big)\|Rf(\lambda x)\|. \tag{2.12}$$

This implies that

$$\sup_{|\lambda|>r}\mu(\lambda x)|f'(\cdot x)(\lambda)| \le \frac{1}{r}\sup_{z\in B_X}\mu(z)\|Rf(z)\|.$$

Therefore, $f \in \mathcal{B}_\mu^{\mathrm{aff}}(B_X, Y)$, and we also obtain $\|f\|_{s\mathcal{B}_\mu{}^{\mathrm{aff}}(B_X,Y)} \le \frac{1}{r}\|f\|_{s\mathcal{B}_\mu^R(B_X,Y)}$. Hence, the second inequality in (1) is proved

(2) Let $f \in \mathcal{B}_{\mu,0}^{\mathrm{aff}}(B_X, Y)$. Then, using (2.11) it is easy to see that $f \in \mathcal{B}_{\mu,0}^R(B_X, Y)$. In the converse direction, it follows from (2.12) that $f \in \mathcal{B}_{\mu,0}^{\mathrm{aff}}(B_X, Y)$ if $f \in \mathcal{B}_{\mu,0}^R(B_X, Y)$. □

Next, we will compare the spaces $\mathcal{B}_\mu^\nabla(B_X, Y)$ and $\mathcal{B}_\mu^R(B_X, Y)$. We need a vector-valued version of Lemma 4.11 in [12]. First we note that

$$f \in \mathcal{B}_\mu(\mathbb{B}_1, Y) \quad \text{if and only if} \quad u\circ f \in \mathcal{B}_\mu(\mathbb{B}_1) \text{ for all } u \in Y' \tag{2.13}$$

and, interchanging the suprema, we have

$$\|f\|_{\mathcal{B}_\mu^\nabla(\mathbb{B}_1,Y)} \asymp \sup_{\|u\|=1}\|u\circ f\|_{\mathcal{B}_\mu^\nabla(\mathbb{B}_1)}. \tag{2.14}$$

Lemma 2.2. *Let $f \in \mathcal{B}^{\mathrm{aff}}(\mathbb{B}_2, Y)$. If there exists $M > 0$ such that $\|f(\cdot x)\|_{s\mathcal{B}^{\mathrm{aff}}(\mathbb{B}_1,Y)} \le M$ for any $x = (x_1, x_2) \in \mathbb{B}_2$, then*

$$\mu((x_1,0))\|\nabla f(x_1,0)\| \le 2\sqrt{2}MR_\mu \quad \forall x_1 \in \mathbb{C}, |x_1| < 1, \tag{2.15}$$

where $R_\mu := 1 + \max_{t\in[0,\delta]}\mu(t)I_\mu(\delta)$.

Proof. We modify the proof of Lemma 4.11 in [12]. Fix $u \in Y'$ with $\|u\| = 1$. By the hypothesis, $f(\cdot x) \in \mathcal{B}(\mathbb{B}_1, Y)$. Then it follows from (2.13) that $u\circ f(\cdot x) \in \mathcal{B}_\mu(\mathbb{B}_1)$.

$$\|u\circ f(\cdot x)\|_{s\mathcal{B}_\mu^\nabla} \le \|u\|\|f(\cdot x)\|_{s\mathcal{B}_\mu^{\mathrm{aff}}} \le M.$$

First of all, the hypotheses imply that

$$\mu((x_1,0))\Big|\frac{\partial(u\circ f)}{\partial x_1}(x_1,0)\Big| \le M,$$

and so it is sufficient to show that

$$\mu((x_1,0))\Big|\frac{\partial(u\circ f)}{\partial x_2}(x_1,0)\Big| \le 2\sqrt{2}M.$$

Indeed, from the hypotheses, we have

$$|f(z)-f(0)| = \Big|\int_0^1\langle\nabla f(tz),\overline{z}\rangle dt\Big| \le M\int_0^{\|z\|}\frac{dt}{\mu(t)} = MI_\mu(z).$$

Then, using the Cauchy integral formula and a simple estimate, we obtain

$$\begin{aligned}&\mu((x_1,0))\Big|\frac{\partial(u\circ f)}{\partial x_2}(x_1,0)\Big|\\ \le&\mu((x_1,0))\frac{1}{2\pi}\int_{|w|=1/\sqrt[4]{2}}\frac{\|u\||f(x_1,w)-f(0)+f(0)-f(x_1,0)|}{|w|^2}dw\\ \le&\mu((x_1,0))\frac{2MI_\mu(x_1)}{2\pi}\int_{|w|=1/\sqrt[4]{2}}\frac{dw}{w^2}\le 2\sqrt{2}MR_\mu\end{aligned}$$

as required. □

Theorem 2.1. (1) *The spaces* $\mathcal{B}^\nabla_\mu(B_X,Y)$ *and* $\mathcal{B}^R_\mu(B_X,Y)$ *coincide. Moreover,*

$$\|f\|_{\mathcal{B}^R_\mu(B_X,Y)}\asymp\|f\|_{\mathcal{B}^\nabla_\mu(B_X,Y)}.$$

(2) *The spaces* $\mathcal{B}^\nabla_{\mu,0}(B_X,Y)$ *and* $\mathcal{B}^R_{\mu,0}(B_X,Y)$ *coincide.*

Proof. We prove this theorem by modifying the method of Timoney which was used in [12].

(1) Let us show that $\|f\|_{s\mathcal{B}^\nabla_\mu(B_X,Y)}\le 2\sqrt{2}R_\mu\|f\|_{s\mathcal{B}^{\text{aff}}_\mu(B_X,Y)}$ and the result follows using Pr-position 2.3. Fix $u\in Y'$ with $\|u\|=1$. Let $z\in B_X$ and $v\in X$ with $\|v\|=1$ be fixed. We may assume that $\dim X\ge 2$. Then there exist orthonormal unit vectors $e_1,e_2\in X$ and $s,t_1,t_2\in\mathbb{C}$ with $|s|<1$ and $|t_1|^2+|t_2|^2=1$ such that $z=se_1$, $v=t_1e_1+t_2e_2$. For $f\in\mathcal{B}^R_\mu(B_X,Y)$ put

$$F(z_1,z_2)=(u\circ f)(z_1e_1+z_2e_2),\quad (z_2,z_2)\in\mathbb{B}_2.$$

Then $F\in H(B_X)$ and it is easy to check that F satisfies the assumptions of Lemma 2.2. Then

$$\mu(z)|\nabla(u\circ f)(z)|=\mu(s)|\nabla(u\circ f)(se_1)|=\mu(s,0)|\nabla F(s,0)|\le 2\sqrt{2}MR_\mu,$$

hence, $\|f\|_{s\mathcal{B}^\nabla_\mu(B_X,Y)}\le 2\sqrt{2}R_\mu\|f\|_{s\mathcal{B}^{\text{aff}}_\mu(B_X,Y)}$ as required.

(2) Because $\|Rf(z)\|<\|\nabla f(z)\|$ for every $z\in B_X$, it suffices to show that $\mathcal{B}^R_{\mu,0}(B_X,Y)\subset\mathcal{B}^\nabla_{\mu,0}(B_X,Y)$. Let $f\in\mathcal{B}^R_{\mu,0}(B_X,Y)$ and consider the function $F(z_1,z_2)$ defined in the proof of the part (1). In exactly the same estimates in [6, Theorem 2.8(i)] we obtain that

$$\Big|\frac{\partial F}{\partial z_2}(z_1,0)\Big|\le\frac{\pi|z_1|}{2\mu(|z_1|)\delta}\sup_{r_0\le\|z\|<1}\mu(z)|Rf(z)|\quad\text{for } |z_1|\ge r_0\tag{2.16}$$

and

$$\Big|\frac{\partial F}{\partial z_1}(z_1,0)\Big|=\Big|\frac{Rf(z_1e_1)}{z_1}\Big|\le\frac{1}{\mu(|z_1|)\delta}\sup_{r_0\le\|z\|<1}\mu(z)|Rf(z)|\quad\text{for } |z_1|\ge r_0.\tag{2.17}$$

From (2.16) and (2.17), we obtain

$$\begin{aligned}\mu(z)|\langle\nabla f(z),v\rangle|&=\mu(s)|\langle\nabla f(se_1),t_1e_1+t_2e_2\rangle|\\&=\mu(s)\Big|t_1\frac{\partial F}{\partial z_1}(s,0)+t_2\frac{\partial F}{\partial z_2}(s,0)\Big|\\&\le\mu(s)\bigg(\Big|\frac{\partial F}{\partial z_1}(s,0)\Big|^2+\Big|\frac{\partial F}{\partial z_2}(s,0)\Big|^2\bigg)^{1/2}\\&\le\frac{\pi}{\sqrt{2}\delta}\sup_{r_0\le\|z\|<1}\mu(z)|Rf(z)|,\quad \|z\|\ge r_0,\|v\|=1.\end{aligned}\tag{2.18}$$

Now, by the hypothesis, for every $\varepsilon>0$ we can find $r_0\in(\delta,1)$ such that $\mu(z)\|Rf(z)\|<\varepsilon$ for $\|z\|>r_0$. Therefore, it follows from (2.18) that $\lim_{\|z\|\to1}\mu(z)\|\nabla f(z)\|=0$, that means $f\in\mathcal{B}^\nabla_{\mu,0}(B_X,Y)$. □

We can now combine the results of Proposition 2.3 and Lemma 2.2 with an argument analogous to the Theorem 2.6 in [1] and obtain the following theorem:

Theorem 2.2. *The spaces $\mathcal{B}^{\nabla}_{\mu}(B_X,Y)$, $\mathcal{B}^{R}_{\mu}(B_X,Y)$ and $\mathcal{B}^{\mathrm{aff}}_{\mu}(B_X,Y)$ coincide. The spaces $\mathcal{B}^{\nabla}_{\mu,0}(B_X,Y)$, $\mathcal{B}^{R}_{\mu,0}(B_X,Y)$ and $\mathcal{B}^{\mathrm{aff}}_{\mu,0}(B_X,Y)$ coincide. Moreover,*

$$\|f\|_{\mathcal{B}^{R}_{\mu}(B_X,Y)} \le \|f\|_{\mathcal{B}^{\nabla}_{\mu}(B_X,Y)} \le 2\sqrt{2}R_{\mu}\|f\|_{\mathcal{B}^{\mathrm{aff}}_{\mu}(B_X,Y)}.$$

Next, we present a Möbius invariant norm for the Bloch-type space $\mathcal{B}(B_X,Y)$. Möbius transformations on a Hilbert space X are the mappings φ_a, $a \in B_X$, defined as follows:

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathcal{B}_X, \tag{2.19}$$

where $s_a = \sqrt{1-\|a\|^2}$, P_a is the orthogonal projection from X onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from X onto $X \ominus [a]$. It is clear that

$$P_a(z) = \frac{\langle z, a\rangle}{\|a\|^2}a,\ (z \in X) \quad \text{and} \quad Q_a(z) = z - \frac{\langle z, a\rangle}{\|a\|^2}a,\ (z \in B_X).$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is obvious that each φ_a is a holomorphic mapping from B_X into X. We will also need the following facts about the pseudohyperbolic distance in B_X. It is given by

$$\varrho_X(x,y) := \|\varphi_{-y}(x)\| \quad \text{for any } x, y \in B_X.$$

For details concerning Möbius transformations and the pseudohyperbolic distance, we refer to the book of K. Zhu [15]. It is well known that, in the case $n \ge 2$, the equality $\|f\circ\varphi\|_{\mathcal{B}^{\nabla}(\mathbb{B}_n,Y)} = \|f\|_{\mathcal{B}^{\nabla}(\mathbb{B}_n,Y)}$ is false. Our goal is to find a semi-norm on $\mathcal{B}(B_X,Y)$ which is invariant under the automorphisms of the ball B_X.

Definition 2.3. *Let X be a complex Hilbert space, Y be a Banach space and $f \in H(B_X,Y)$. Consider the invariant gradient norm*

$$\|\widetilde{\nabla} f(z)\| := \|\nabla(f\circ\varphi_z)(0)\| \quad \textit{for any } z \in B_X.$$

We recall the following result of Blasco and his colleagues in [1]:

Lemma 2.3 (Lemma 3.5, [1])**.** *Let $f \in H(B_X)$. Then*

$$\|\widetilde{\nabla} f(z)\| = \sup_{w\neq 0} \frac{|\langle \nabla f(z), w\rangle|(1-\|z\|^2)}{\sqrt{(1-\|z\|^2)\|w\|^2 + |\langle w, z\rangle|^2}}.$$

We define invariant semi-norm as follows

$$\|f\|_{s\mathcal{B}^{\mathrm{inv}}(B_X,Y)} := \sup_{z\in B_X} \|\widetilde{\nabla} f(z)\| = \sup_{z\in B_X}\ \sup_{u\in Y', \|u\|\le 1} \|\widetilde{\nabla}(u\circ f)(z)\|.$$

We denote

$$\mathcal{B}^{\mathrm{inv}}(B_X,Y) := \{f \in \mathcal{B}(B_X,Y) :\ \|f\|_{s\mathcal{B}^{\mathrm{inv}}(B_X,Y)} < \infty\}.$$

It is also easy to check that $\mathcal{B}^{\mathrm{inv}}(B_X,Y)$ is Banach under the norm

$$\|f\|_{\mathcal{B}^{\mathrm{inv}}(B_X,Y)} := \|f(0)\| + \|f\|_{s\mathcal{B}^{\mathrm{inv}}(B_X,Y)}.$$

Now, applying Theorem 3.8 in [1] to the functions $u \circ f$ for every $u \in Y'$, we obtain the following:

Theorem 2.3. *The spaces $\mathcal{B}^{\nabla}(B_X,Y)$, and $\mathcal{B}^{\mathrm{inv}}(B_X,Y)$ coincide. Moreover,*

$$\|f\|_{\mathcal{B}^{\nabla}(B_X,Y)} \le \|f\|_{\mathcal{B}^{\mathrm{inv}}(B_X,Y)} \lesssim \|f\|_{\mathcal{B}^{\nabla}(B_X,Y)}.$$

3. Weak Holomorphic Spaces Associated with Bloch-Type Spaces

Let X, Y be complex Banach spaces and $\mathcal{W} \subset Y'$ be a separating subspace of the dual Y' of Y. Let $\mathcal{E} \subset \mathscr{H}(B_X)$ be a Banach space. We say that the space

$$\mathcal{W}\mathcal{E} := \{f : B_X \to Y : \ f \text{ is locally bounded and } w \circ f \in \mathcal{E}, \ \forall w \in \mathcal{W}\}$$

equipped with the norm

$$\|f\|_{\mathcal{W}\mathcal{E}} := \sup_{w \in \mathcal{W}, \|w\| \le 1} \|w \circ f\|_{\mathcal{E}} \tag{3.1}$$

is the Banach space $\mathcal{W}$-associated with $\mathcal{E}$ of Y-valued functions.

Remark 3.2. *In the case the norm $\|\cdot\|_{\mathcal{E}}$ of Banach space $\mathcal{E}$ is written in the form*

$$\|f\| = |f(0)| + \|f\|_{s\mathcal{E}} \quad \forall f \in \mathcal{E}$$

the space $\mathcal{W}\mathcal{E}$ can be equipped with the norm

$$\|f\|_{\mathcal{W}\mathcal{E}^+} := \sup_{w \in \mathcal{W}, \|w\| \le 1} |w \circ f(0)| + \sup_{w \in \mathcal{W}, \|w\| \le 1} \|w \circ f\|_{s\mathcal{E}} \quad \forall f \in \mathcal{E}. \tag{3.2}$$

However, it is easy to check that $\mathcal{W}\mathcal{E} = \mathcal{W}\mathcal{E}^+$ and

$$\|\cdot\|_{\mathcal{W}\mathcal{E}} \asymp \|\cdot\|_{\mathcal{W}\mathcal{E}^+}$$

on $\mathcal{W}\mathcal{E}$ where

$$\mathcal{W}\mathcal{E}^+ := \Big\{ f : B_X \to Y : \ f \text{ is locally bounded and } \sup_{w \in \mathcal{W}, \|w\| \le 1} \|w \circ f\|_{s\mathcal{E}} < \infty \Big\}.$$

Therefore, by $\mathcal{W}\mathcal{E}$ we always mean that is $(\mathcal{W}\mathcal{E}, \|\cdot\|_{\mathcal{W}\mathcal{E}})$.

Suppose now that $\mathcal{E} \subset \mathscr{H}(B_X)$ is a Banach space such that

(e1) $\mathcal{E}$ contains the constant functions,

(e2) the closed unit ball $B_{\mathcal{E}}$ is compact in the compact open topology τ_{co} of B_X.

It is easy to check that the properties (e1), (e2) are satisfied by a large number of well-known function spaces, such as classical Hardy, Bergman, BMOA, and Bloch spaces.

Proposition 3.4. *Let X, Y be complex Banach spaces and $\mathcal{W} \subset Y'$ be a separating subspace. Let $\mathcal{E} \subset \mathscr{H}(B_X)$ a Banach space satisfying (e1)-(e2) and $\mathcal{W}\mathcal{E}$ be the Banach space $\mathcal{W}$-associated with $\mathcal{E}$. Then, the following assertions hold:*

(we1) *$f \mapsto f \otimes y$ defines a bounded linear operator $P_y : \mathcal{E} \to \mathcal{W}\mathcal{E}$ for any $y \in Y$, where $(f \otimes y)(z) = f(z)y$ for $z \in B_X$,*

(we2) *$g \mapsto w \circ g$ defines a bounded linear operator $Q_w : \mathcal{W}\mathcal{E} \to \mathcal{E}$ for any $w \in \mathcal{W}$,*

(we3) *For all $z \in B_X$ the point evaluations $\widetilde{\delta}_z : \mathcal{W}\mathcal{E} \to (Y, \sigma(Y, W))$, where $\widetilde{\delta}_z(g) = g(z)$, are continuous.*

In the case the hypothesis "separating" of $\mathcal{W}$ is replaced by a stronger one that $\mathcal{W}$ is "almost norming", we obtain the assertion (we3′) below instead of (we3):

(we3′) *For all $z \in B_X$ the point evaluations $\widetilde{\delta}_z : \mathcal{W}\mathcal{E} \to Y$ are bounded.*

Here, the subspace $\mathcal{W}$ of Y' is called *almost norming* if

$$q_{\mathcal{W}}(x) := \sup_{w \in \mathcal{W}, \|w\| \le 1} |w(x)|$$

defines an equivalent norm on Y.

Proof. (i) Fix $y \in Y$. In fact, for every $f \in \mathcal{E}$ we have $w \circ (f \otimes y) = w(y)f$. Then

$$\begin{aligned}\|P_y(f)\|_{\mathcal{W}\mathcal{E}} = \sup_{\|w\|\le 1} \|w \circ (f \otimes y)\|_{\mathcal{E}} &= \sup_{\|w\|\le 1} \|w(y)f\|_{\mathcal{E}} \\ &\le \|w\| \cdot \|y\| \cdot \|f\|_{\mathcal{E}} \\ &\le \|y\| \cdot \|f\|_{\mathcal{E}}.\end{aligned}$$

Thus (we1) holds.

(ii) Fix $w \in \mathcal{W}$, for every $g \in \mathcal{W}\mathcal{E}$ we have

$$\begin{aligned}\|Q_w(g)\|_{\mathcal{E}} = \|w \circ g\|_{\mathcal{E}} &= \|w\| \Big\| \frac{w}{\|w\|} \circ g \Big\|_{\mathcal{E}} \\ &\le \|w\| \sup_{\|u\|\le 1} \|u \circ g\|_{\mathcal{E}} \\ &= \|w\| \cdot \|g\|_{\mathcal{W}\mathcal{E}}.\end{aligned}$$

Thus (we2) is true.

(iii) Fix $z \in B_X$. Note first that since $\mathcal{E}$ satisfies (e1) and (e2), then the evaluation maps $\delta_z \in \mathcal{E}'$ for $z \in B_X$ where $\delta_z(f) = f(z)$ for $f \in \mathcal{E}$. It is obvious that $w(\widetilde{\delta}_z(g)) = \delta_z(w \circ g)$ for every $g \in \mathcal{W}\mathcal{E}$ and for every $w \in \mathcal{W}$. Let V be a $\sigma(Y, \mathcal{W})$-neighbourhood of 0 in Y. Without loss of genarality we may assume $V = \{y \in Y : |w(y)| < 1\}$ for some $w \in \mathcal{W}$. Then $\widetilde{\delta}_z(\|\delta_z\|^{-1}\|w\|^{-1}B_{\mathcal{W}\mathcal{E}}) \subset V$, where $B_{\mathcal{W}\mathcal{E}}$ is the unit ball of $\mathcal{W}\mathcal{E}$. Indeed, for every $g \in B_{\mathcal{W}\mathcal{E}}$ we have

$$\begin{aligned}|w(\widetilde{\delta}_z(\|\delta_z\|^{-1}\|w\|^{-1}g))| &= \|\delta_z\|^{-1}|\delta_z(\|w\|^{-1}w \circ g)| \\ &\le \|\delta_z\|^{-1}\|\delta_z\| \cdot \big\| \|w\|^{-1} w \circ g \big\|_{\mathcal{E}} \\ &\le \sup_{u \in \mathcal{W}, \|u\|\le 1} \|u \circ g\|_{\mathcal{E}} \\ &= \|g\|_{\mathcal{W}\mathcal{E}} < 1.\end{aligned}$$

Thus, (we3) holds.

In the case where $\mathcal{W}$ is almost norming, since $q_{\mathcal{W}}$ defines an equivalent norm, there exists $C > 0$ such that

$$\begin{aligned}\|\widetilde{\delta}_z(g)\| = \|g(z)\| &\le C q_{\mathcal{W}}(g(z)) \\ &= C \sup_{w \in \mathcal{W}, \|w\|\le 1} |w(g(z))| \\ &\le C \sup_{w \in \mathcal{W}, \|w\|\le 1} \|w \circ g\| \\ &= C\|g\|_{\mathcal{W}\mathcal{E}} \quad \forall g \in \mathcal{W}\mathcal{E}.\end{aligned}$$

The assertion (we3′) is proved. □

Now let $\mathcal{W} \subset Y'$ be a separating subspace of the dual Y'. Applying Proposition 2.3, Theorems 2.2 and 2.3 to functions $w \circ f$ for each $f \in H(B_X, Y)$ and $w \in \mathcal{W}$, we obtain the equivalence of the norms in $\mathcal{W}$-associated Bloch-type spaces:

$$\|\cdot\|_{\mathcal{W}\mathcal{B}^R_\mu(B_X)} \cong \|\cdot\|_{\mathcal{W}\mathcal{B}^\nabla_\mu(B_X)} \cong \|\cdot\|_{\mathcal{W}\mathcal{B}^{\mathrm{aff}}_\mu(B_X)},$$

$$\|\cdot\|_{\mathcal{W}\mathcal{B}^R(B_X)} \cong \|\cdot\|_{\mathcal{W}\mathcal{B}^\nabla(B_X)} \cong \|\cdot\|_{\mathcal{W}\mathcal{B}^{\mathrm{aff}}(B_X)} \cong \|\cdot\|_{\mathcal{W}\mathcal{B}^{\mathrm{inv}}(B_X)}.$$

Hence, for the sake of simplicity, from now on we write $\mathcal{B}_\mu$ instead of $\mathcal{B}^R_\mu$. Recall that, the space $\mathcal{W}\mathcal{B}_\mu(B_X)$ equipped with the norm in the form either (3.1) or (3.2). It is clear that for every separating subspace $\mathcal{W}$ of Y' we have

$$\mathcal{B}^\lozenge_\mu(B_X, Y) \subset \mathcal{W}\mathcal{B}^\lozenge_\mu(B_X), \quad \mathcal{B}^\lozenge_{\mu,0}(B_X, Y) \subset \mathcal{W}\mathcal{B}^\lozenge_{\mu,0}(B_X).$$

The main result of this section is the following:

Theorem 3.4. *Let $\mathcal{W} \subset Y'$ be a separating subspace. Let μ be a normal weight on B_X. Then $\mathcal{W}\mathcal{B}_\mu(B_X)$ and $\mathcal{W}\mathcal{B}_{\mu,0}(B_X)$ satisfy (we1)-(we3).*

We need the following lemma whose proof parallels that of Lemma 13 in [11] and will be omitted.

Lemma 3.4. *Let μ be a normal weight on B_X. Then there exists $C_\mu > 0$ such that*

$$C_\mu \leq \frac{\mu(r)}{\mu(r^2)} \leq 1 \quad \forall r \in [0,1).$$

Proof of Theorem 3.4. By Proposition 3.4, it suffices to show that $\mathcal{B}_\mu(B_X)$, $\mathcal{B}_{\mu,0}(B_X)$ satisfy (e1) and (e2). It is obvious that $\mathcal{B}_\mu(B_X)$, $\mathcal{B}_{\mu,0}(B_X)$ satisfy (e1). Because $\mathcal{B}_{\mu,0}(B_X)$ is the subspace of $\mathcal{B}_\mu(B_X)$, it suffices to check (e2) for the space $\mathcal{B}_\mu(B_X)$. In order to prove (e2) holds for $\mathcal{B}_\mu(B_X)$, we will show that the closed unit ball U of $\mathcal{B}_\mu(B_X)$ is pointwise bounded and equicontinuous.

(i) First, we prove that U is pointwise bounded. It suffices to prove that

$$|f(z)| \leq \max\Big\{1, I_\mu(z)\Big\}\|f\|_{\mathcal{B}_\mu(B_X)} \quad \forall f \in \mathcal{B}_\mu(B_X), \forall z \in B_X. \tag{3.3}$$

Fix $f \in \mathcal{B}_\mu(B_X)$ and put $g(z) = f(z) - f(0)$ for every $z \in B_X$. Note that $g(0) = 0$ and $\|g\|_{\mathcal{B}_\mu(B_X)} = \|f\|_{s\mathcal{B}_\mu(B_X)}$. As in Lemma 2.2 by Cauchy-Schwarz inequality, we have

$$|g(z)| \leq \int_0^1 \frac{\|f\|_{s\mathcal{B}_\mu(\mathcal{B}_X)}\|z\|}{\mu(tz)} dt = \|f\|_{s\mathcal{B}_\mu(B_X)} I_\mu(z) = \|g\|_{\mathcal{B}_\mu(B_X)} I_\mu(z).$$

Consequently,

$$\begin{aligned} |f(z)| &\leq |f(0)| + |g(z)| \leq |f(0)| + \|g\|_{\mathcal{B}_\mu(B_X)} I_\mu(z) \\ &\leq \max\Big\{1, I_\mu(z)\Big\}\big(|f(0)| + \|f\|_{s\mathcal{B}_\mu(B_X)}\big) \\ &= \max\Big\{1, I_\mu(z)\Big\}\|f\|_{\mathcal{B}_\mu(B_X)}. \end{aligned}$$

(ii) Next, we show that U is equicontinuous. For each $f \in U$, by Proposition 2.1 we can find $m \geq 2$ such that

$$\|f\|_{s\mathcal{B}_\mu(B_X)} = \sup_{y \in OS_m} \|f_y\|_{s\mathcal{B}_\mu(\mathbb{B}_m)}.$$

Fix $e_{[m]} = (e_1, \ldots, e_m) \in OS_m$. Then, for every $z = (z_k)_{k\in\Gamma}, w = (w_k)_{k\in\Gamma} \in B_X$, we consider $z_{[m]} := (z_1, \ldots, z_m)$, $w_{[m]} := (w_1, \ldots, w_m)$. By Theorem 3.6 in [15] and Lemma 3.4, we have

$$\begin{aligned} &|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \\ \leq &\beta(z_{[m]}, w_{[m]}) \sup_{x_{[m]} \in \mathbb{B}_m} \|\widetilde{\nabla} f_{e_{[m]}}(x_{[m]})\| \\ \leq &\beta(z_{[m]}, w_{[m]}) \sup_{x_{[m]} \in \mathbb{B}_m} \sup_{y \in \mathbb{B}_m \setminus \{0\}} \frac{|\langle \nabla f_{e_{[m]}}(x_{[m]}), y\rangle|(1 - \|x_{[m]}\|^2)}{\sqrt{(1 - \|x_{[m]}\|^2)\|y\|^2 + |\langle y, x_{[m]}\rangle|^2}} \\ \leq &\beta(z_{[m]}, w_{[m]}) C_\mu^{-1} \sup_{x_{[m]} \in \mathbb{B}_m} \frac{\mu^{[m]}(\|x_{[m]}\|)|\nabla f_{e_{[m]}}(x_{[m]})\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \\ \leq &\beta(z_{[m]}, w_{[m]}) C_\mu^{-1} \|f_{e_{[m]}}\|_{\mathcal{B}_\mu(\mathbb{B}_m)} \frac{\sqrt{1 - \|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)}, \end{aligned}$$

where β is the Bergman metric on $\mathbb{B}_m$ given by

$$\beta(s,t) = \frac{1}{2}\log\frac{1+|(\varphi_m)_s(t)|}{1-|(\varphi_m)_s(t)|}$$

with $(\varphi_m)_s$ is the involutive automorphism of $\mathbb{B}_m$ that interchanges 0 and s. If $\|x_{[m]}\|^2 \leq \delta$ it is clear that

$$\frac{\sqrt{1-\|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \leq \frac{1}{m_{\mu,\delta}} < \infty,$$

where $m_{\mu,\delta} = \min_{t\in[0,\delta]}\mu(t) > 0$; if $\|x_{[m]}\|^2 > \delta$ and $b \geq 1/2$, by (2.1) we have

$$\frac{\sqrt{1-\|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} \leq \frac{(1-\|x_{[m]}\|^2)^b}{\mu^{[m]}(\|x_{[m]}\|^2)} < S_\mu < \infty;$$

if $\|x_{[m]}\|^2 > \delta$ and $b < 1/2$, we get

$$\frac{\sqrt{1-\|x_{[m]}\|^2}}{\mu^{[m]}(\|x_{[m]}\|^2)} = \frac{(1-\|x_{[m]}\|^2)^b}{\mu^{[m]}(\|x_{[m]}\|^2)}(1-\|x_{[m]}\|^2)^{1/2-b} \leq S_\mu(1-\delta)^{1/2-b} < \infty.$$

Consequently,

$$|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \leq \beta(z_{[m]}, w_{[m]})\widehat{S}_\mu\|f_{e_{[m]}}\|_{\mathcal{B}_\mu(\mathbb{B}_m)},$$

where

$$\widehat{S}_\mu := C_\mu^{-1}\max\left\{m_{\mu,\delta}^{-1}, S_\mu(1-\delta)^{1/2-b}\right\}.$$

Since $\beta(s,t)$ is the infimum of the set consisting of all $\ell(\gamma)$ where γ is a piecewise smooth curve in $\mathbb{B}_m$ from s to t (see [15, p. 25]) we have

$$|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \leq \|z_{[m]} - w_{[m]}\|\widehat{S}_\mu\|f_{e_{[m]}}\|_{\mathcal{B}_\mu(\mathbb{B}_m)} \leq \widehat{S}_\mu\|z-w\|.$$

Consequently,

$$|f(z) - f(w)| = \lim_{m\to\infty}|f_{e_{[m]}}(z_{[m]}) - f_{e_{[m]}}(w_{[m]})| \leq \widehat{S}_\mu\|z-w\|.$$

This yields that U is equicontinuous. □

Remark 3.3. *In fact, the estimate (3.3) can be written as follows*

$$|f(z)| \leq |f(0)| + I_\mu(z)\|f\|_{s\mathcal{B}_\mu}.$$

Finally, we discuss the linearization theorem for spaces $\mathcal{WB}_\mu(B_X)$ which will be usefull in investigation some related problems, especially the theory of operators between these spaces. In fact, this theorem holds for spaces $\mathcal{WE}$ where $\mathcal{E} \subset \mathscr{H}(B_X)$ is a Banach space satisfying (e1)-(e2). In this paper, we will state and prove this theorem for the general case.

Theorem 3.5 (Linearization). *Let X, Y be complex Banach spaces and $\mathcal{W} \subset Y'$ be a separating subspace. Let $\mathcal{E} \subset \mathscr{H}(B_X)$ be a Banach space satisfying (e1)-(e2). Then there exist a Banach space $^*\mathcal{E}$ and a mapping $\delta_X \in \mathscr{H}(B_X, {}^*\mathcal{E})$ with the following universal property: A function $f \in \mathcal{WE}$ if and only if there is a unique mapping $T_f \in L({}^*\mathcal{E}, Y)$ such that $T_f \circ \delta_X = f$. This property characterize $^*\mathcal{E}$ uniquely up to an isometric isomorphism.*

Moreover, the mapping

$$\Phi : f \in \mathcal{WE} \mapsto T_f \in L({}^*\mathcal{E}, Y)$$

is a topological isomorphism.

We will prove this theorem by Mujica's method [8, Theorem 2.1], which is based on the Dixmier–Ng theorem, with a little improvement.

Proof. Let us denote by ${}^*\mathcal{E}$ the closed subspace of all linear functionals $u \in \mathcal{E}'$ such that $u\big|_{B_\mathcal{E}}$ is τ_{co}-continuous. As in the proof of Mujica [8, Theorem 2.1], we get the evaluation mapping $\delta_X : B_X \to {}^*\mathcal{E}$ given by $\delta_X(x) = \delta_x$ with $\delta_x(g) := g(x)$ for all $g \in \mathcal{E}$, is holomorphic and

$$\text{span}\{\delta_x : x \in B_X\} \text{ is a dense subspace of } {}^*\mathcal{E}. \tag{3.4}$$

Now, we show that ${}^*\mathcal{E}$ and δ_X have required universal property. First, given a locally bounded function $f : B_X \to Y$. Assume that there exists $T_f \in L({}^*\mathcal{E}, Y)$ such that $T_f \circ \delta_X = f$. Since T_f is continuous and δ_X is holomorphic, it follows that $u \circ f \in \mathscr{H}(B_X)$ for every $u \in \mathcal{W}$. Since $\mathcal{W}$ is separating, according to [10, Lemma 4.2] we have $f \in \mathscr{H}_{LB}(B_X, Y)$. Next, it follows from $({}^*\mathcal{E})' = \mathcal{E}$ (see [9]) that $u \circ f \in \mathcal{E}$ for each $u \in \mathcal{W}$, and then $f \in \mathcal{W}\mathcal{E}$. Now, we will prove the converse of the statement. Fix $f \in \mathcal{W}\mathcal{E}$.

(i) *The case of* $Y = \mathbb{C}$*:* We define $T_f := Jf$, where $J : \mathcal{E} \to ({}^*\mathcal{E})'$ is the evaluation mapping given by $(Jf)(u) = u(f)$ for all $u \in {}^*\mathcal{E}$, which is a topological isomorphism by the Ng Theorem [9, Theorem 1]. Since $(Jg) \circ \delta_X(x) = \delta_x(g) = g(x)$ for all $g \in \mathcal{E}$, $x \in B_X$, it implies that $T_f \circ \delta_X = f$. From (3.4) we obtain the uniqueness of T_f.

(ii) *The case of* Y *is Banach:* We define $T_f : {}^*\mathcal{E} \to \mathcal{W}'$ by

$$(T_f u)(\varphi) = T_{\varphi\circ f}(u) = u(\varphi \circ f) \quad \forall u \in {}^*\mathcal{E}, \forall \varphi \in \mathcal{W}, \tag{3.5}$$

i.e. $T_{\varphi\circ f}$ is defined as in the case (i).

It is easy to check that $T_f \in L({}^*\mathcal{E}, \mathcal{W}')$ and $\|T_f\| = \|f\|_{\mathcal{W}\mathcal{E}}$, hence, Φ is a isometric isomorphism. Furthermore,

$$(T_f\delta_x)(\varphi) = (\varphi \circ f)(x)$$

for every $x \in B_X$ and $\varphi \in \mathcal{W}$ and, therefore, since $\mathcal{W}$ is separating we get $T_f\delta_x = f(x) \in Y$ for every $x \in B_X$. Then, by (3.4) $T_f \in L({}^*\mathcal{E}, Y)$. The uniqueness of T_f follows also from the fact (3.4) that $\delta_X(B_X)$ generates a dense subspace of ${}^*\mathcal{E}$.

Finally, the uniqueness of ${}^*\mathcal{E}$ up to an isometric isomorphism follows from the universal property, together with the isometry $\|T_f\| = \|f\|_{\mathcal{W}\mathcal{E}}$. This completes the proof. □

Our results suggest the following questions.

Problems.

(1) Let $\mathcal{E}_i$, $i = 1, 2$, be spaces of holomorphic functions on the unit ball B_X of a Banach space X and $\mathcal{W}\mathcal{E}_i$ be the Banach spaces $\mathcal{W}$-associated with $\mathcal{E}_i$ of Y-valued functions. Let ψ be a holomorphic on B_X and φ a holomorphic self-map of B_X. Consider the extended Cesàro composition operators $C_{\psi,\varphi} : \mathcal{E}_1 \to \mathcal{E}_2$, $\widetilde{C}_{\psi,\varphi} : \mathcal{W}\mathcal{E}_1 \to \mathcal{W}\mathcal{E}_2$, and the weighted composition operators $W_{\psi,\varphi} : \mathcal{E}_1 \to \mathcal{E}_2$, $\widetilde{W}_{\psi,\varphi} : \mathcal{W}\mathcal{E}_1 \to \mathcal{W}\mathcal{E}_2$.

Is there any relationship between the boundedness as well as the (weak) compactness of $C_{\psi,\varphi}$, $W_{\psi,\varphi}$ and of $\widetilde{C}_{\psi,\varphi}$, $\widetilde{W}_{\psi,\varphi}$? How does separating subspace $\mathcal{W} \subset Y'$ affect that relationship?

(2) In the case where $\mathcal{E}_1 = \mathcal{B}_\nu(B_X)$, $\mathcal{E}_2 = \mathcal{B}_\mu(B_X)$ with ν and μ are normal weights on the unit ball B_X of a infinite dimensional Hilbert space X, is it possible to characterize the boundedness as well as the (weak) compactness of $\widetilde{C}_{\psi,\varphi}$, $\widetilde{W}_{\psi,\varphi}$ via the estimates for the restrictions of ψ and φ to a m-dimensional subspace of X for some $m \geq 2$?

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