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ANHOLONOMIC CO-ORDINATES AND ELECTROMAGNETIC CURVES WITH ALTERNATIVE MOVING FRAME VIA MAXWELL EVOLUTION

Hazal CEYHAN,[1] Ebru YANIK[2] and Zehra ÖZDEMİR[3]

[1,2]Department of Mathematics, Ankara University, 06100 Ankara, TÜRKİYE
[3]Department of Mathematics, Amasya University, 05189 Amasya, TÜRKİYE

Abstract. In this study, we examine the Berry's phase equation for E-M curves in the $C-direction$ and $W-direction$ throughout an optic fiber via alternative moving frame in three dimensional space. Moreover, electromagnetic curve's $C-direction$ and $W-direction$ Rytov parallel transportation laws are defined. Finally, we examine the electromagnetic curve with anholonomic co-ordinates for Maxwellian evolution by Maxwell's equation.

1. Introduction

Electromagnetic (E-M) theory and magnetic theory are very significant topics for the scientific world. Mathematically, at first, it started to be researched in terms of topology [20]. Then, it was noted that a geometric perspective could be presented with this approach and Berry made a the publication leading the way in this regard [16]. After that, Ross studied the rotational motion of the polarization state together with the optical fiber geometrically and gave a relationship with the most important branch of geometry, curves [11]. Haldane examined the geometric phase of a light wave in tangential vector space [5]. Dandoloff, Zakrzewski and Frins, Dultz researched parallel transport with Berry's phase and correlated the space curve with the trajectories of the light wave along the optical fiber [4,19]. On the other hand, there has been a very important paper that has led to the study of this subject for geometers recently. In that paper, the relationship between the magnetic field and Killing vector field and the connection with the classical elastic

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[1] hazallceyhan@gmail.com-Corresponding author; 0000-0001-6201-5134;

[2] ebruyanik@ankara.edu.tr; 0000-0003-0768-9931

[3] zehra.ozdemir@amasya.edu.tr; 0000-0001-9750-507X.

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theory and the Hall effect were given [12, 13]. Then, the magnetic flow and field were studied in some geometric structures, thanks to the interest in the geometric phase and the important publications written on this subject [7, 9, 10]. With this, the geometrical phase shift of the angular momentum and their densities were researched analytically so that Frenet-Serret coordinate system and the special curve were associated [18]. Afterward, Özdemir [27] and Ceyhan [6] calculated magnetic and electromagnetic trajectories and they presented some motivated examples of motion of the polarization light wave. Finally, in [17, 21], Körpınar and Gürbüz examined the connection between electromagnetic theory and Maxwell's equations from a geometric perspective.

Recently optical fiber is a very important field that come into prominence in physics and geometry. Polarized light is generally thought of as the transport of an electromagnetic wave and its appearances. When it is assumed to propagate within the optical fiber, it is well-defined, owing to the Maxwell's equations. The set of Maxwell's equations implicitly shows how electromagnetic field vectors propagate and explicitly tell sources of the field. In the optical fiber configuration of uniform, isotropic, nonconducting, free-from charge, magnetic flux, and non-dispersive etc. The evolution of the space curve is a very influential way to understand many physical processes such as vortex filaments, dynamics of Heisenberg spin chain, integrable systems, soliton equation theory, sigma models, relativity, water wave theory, fluid dynamics, field theories, linear and nonlinear optics. We give example publications of the applications mentioned above. Authors researched the relationship between non-linear Schöndinger equation rogue soliton equivalent in the spin system [2]. In [14], authors gave a sufficient conclusion by using Da Rios vortex filament equation and the evolution equation for the torsion is the Viscous Burger's equation. Then in comprehensive paper [26], Banica and Miot investigated evolution, interaction and collisions of vortex filaments. Moreover, Körpinar and et. al. studied Binormal Schöndinger system of Heisenberg ferromagnetic equation and flux surface by using normal direction equations [23–25]. In [22], author studied binormal direction with magnetic flows equations for Berry's phase applications.

In this study, we analyze the geometric phase equation for E-M curves in the $C-direction$ and $W-direction$ directions throughout an optic fiber via an alternative moving frame in three dimensional space. At the same time, we research the electromagnetic curve's via anholonomic co-ordinates for Maxwellian evolution by Maxwell's equation. The first section includes the historical background of the work and a description of what has been done in this paper. In the second section, equations of alternative moving frame studied in this study and anholonomic coordinate calculations to be used in other parts are given. The third section comprises evaluation of directional derivative expressions for the alternative moving frame and the $S-direction$, $C-direction$, $W-direction$ derivatives of Serret-Frenet relations in matrix form. The fourth section we calculate $\overrightarrow{\mathbf{E}}_c$ and $\overrightarrow{\mathbf{E}}_w$ electric field, magnetic field, electromagnetic matrix form and $\overrightarrow{\mathbf{E}}_c$, $\overrightarrow{\mathbf{E}}_w$ Rytov curves. Finally, in the last section, we give mathematical approach of Maxwell equations for electromagnetic and magnetic waves via alternative moving frame.

2. Fundamental Background

Let $\gamma = \gamma(s)$ be an arbitrary curve in 3D Riemannian manifolds. If $\langle \overrightarrow{\gamma}'(s), \overrightarrow{\gamma}'(s) \rangle = 0$ for any $s \in I$, γ is called an arc-lenght parametrized curve where $\langle , \rangle$ is defined as;

$$\langle \ , \ \rangle = du_1^2 + du_2^2 + du_3^2$$

that (u_1, u_2, u_3) is a coordinate of $\mathbb{E}^3$.

Alternative frame's fields as $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ Frenet frame are given as below;

$$\overrightarrow{N}, \qquad \overrightarrow{C} = \frac{\overrightarrow{N}'}{\|\overrightarrow{N}'\|}, \qquad \overrightarrow{W} = \frac{\tau t + \kappa b}{\sqrt{\kappa^2 + \tau^2}}$$

where $\overrightarrow{N}$ is a unit principal normal vector field and $\overrightarrow{W}$ is a Darboux vector field.

The one-parameter derivative s, which is the arc-lenght parameter of the alternative moving frame's fields is as follows;

$$\begin{aligned} \overrightarrow{N}'(s) &= f(s)\overrightarrow{C}(s) \\ \overrightarrow{C}'(s) &= -f(s)\overrightarrow{N}(s) + g(s)\overrightarrow{W}(s) \\ \overrightarrow{W}'(s) &= -g(s)\overrightarrow{C}(s) \end{aligned}$$

which $f(s)$ and $g(s)$ are curvature of the curve γ (κ and τ are the curvature and the torsion of the curve γ in terms of Frenet's frame, respectively); are defined as;

$$f = \kappa\sqrt{1 + H^2} \quad g = \sigma f$$

where $H = \frac{\tau}{\kappa}$ is harmonic curvature and $\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'$, ([3]).

With $\overrightarrow{A}$ being an arbitrary vector field, the gradient and the curl of this vector field are given respectively, as ([1])

$$grad\overrightarrow{A} = \overrightarrow{N}\overrightarrow{N} \cdot grad\overrightarrow{A} + \overrightarrow{C}\overrightarrow{C} \cdot grad\overrightarrow{A} + \overrightarrow{W}\overrightarrow{W} \cdot grad\overrightarrow{A},$$

$$curl\overrightarrow{A} = \overrightarrow{N} \times \frac{\partial \overrightarrow{A}}{\partial s} + \overrightarrow{C} \times \frac{\partial \overrightarrow{A}}{\partial s} + \overrightarrow{W} \times \frac{\partial \overrightarrow{A}}{\partial s}.$$

On the other hand, the directional derivatives of arbitrary scalar f, along the vector $\overrightarrow{N}$, vector $\overrightarrow{C}$ and Darboux vector $\overrightarrow{W}$, are defined in the above expression as;

$$\frac{\partial f}{\partial s} = \overrightarrow{N} \cdot gradf,$$
$$\frac{\partial f}{\partial c} = \overrightarrow{C} \cdot gradf,$$
$$\frac{\partial f}{\partial w} = \overrightarrow{W} \cdot gradf.$$

Assume that a directional derivative of an arbitrary vector $\overrightarrow{A}$ with respect to direction η where $\eta \in \{\overrightarrow{N}, \overrightarrow{C}, \overrightarrow{W}\}$ and considering the directional derivative $\frac{\partial \overrightarrow{A}}{\partial \eta}$ which calculated as follows; the divergence operator div acting on an arbitrary vector $\overrightarrow{A}$ is written as;

$$div\overrightarrow{A} = \overrightarrow{N} \cdot \frac{\partial \overrightarrow{A}}{\partial \eta} + \overrightarrow{C} \cdot \frac{\partial \overrightarrow{A}}{\partial \eta} + \overrightarrow{W} \cdot \frac{\partial \overrightarrow{A}}{\partial \eta}.$$

The directional derivative of the vector $\overrightarrow{N}$ can be written in a general form as follows;

$$\left.\begin{aligned} \frac{\partial \overrightarrow{N}}{\partial s} &= (\overrightarrow{N} \cdot \frac{\partial \overrightarrow{N}}{\partial s})\overrightarrow{N} + (\overrightarrow{C} \cdot \frac{\partial \overrightarrow{N}}{\partial s})\overrightarrow{C} + (\overrightarrow{W} \cdot \frac{\partial \overrightarrow{N}}{\partial s})\overrightarrow{W}, \\ \frac{\partial \overrightarrow{N}}{\partial c} &= (\overrightarrow{N} \cdot \frac{\partial \overrightarrow{N}}{\partial c})\overrightarrow{N} + (\overrightarrow{C} \cdot \frac{\partial \overrightarrow{N}}{\partial c})\overrightarrow{C} + (\overrightarrow{W}.\frac{\partial \overrightarrow{N}}{\partial c})\overrightarrow{W}, \\ \frac{\partial \overrightarrow{N}}{\partial w} &= (\overrightarrow{N} \cdot \frac{\partial \overrightarrow{N}}{\partial w})\overrightarrow{N} + (\overrightarrow{C} \cdot \frac{\partial \overrightarrow{N}}{\partial w})\overrightarrow{C} + (\overrightarrow{W} \cdot \frac{\partial \overrightarrow{N}}{\partial w})\overrightarrow{W}. \end{aligned}\right\} \tag{1}$$

For the other directional derivatives of the vectors which are vector $\overrightarrow{C}$ and Darboux vector $\overrightarrow{W}$, we can use the same method. Here we give various interrelationships between directional derivatives found by solving the following sets of equations;

$$\left.\begin{aligned} \frac{\partial}{\partial s}(\overrightarrow{N} \cdot \overrightarrow{N}) &= 2\overrightarrow{N}\frac{\partial \overrightarrow{N}}{\partial s} \\ &= 0 \\ \frac{\partial}{\partial s}(\overrightarrow{C} \cdot \overrightarrow{C}) &= 2\overrightarrow{C}\frac{\partial \overrightarrow{C}}{\partial s} \\ &= 0 \\ \frac{\partial}{\partial s}(\overrightarrow{W} \cdot \overrightarrow{W}) &= 2\overrightarrow{W}\frac{\partial \overrightarrow{W}}{\partial s} \\ &= 0 \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}\frac{\partial}{\partial s}(\overrightarrow{N}\cdot\overrightarrow{C}) &= \overrightarrow{N}\cdot\frac{\partial\overrightarrow{C}}{\partial s}+\overrightarrow{C}\cdot\frac{\partial\overrightarrow{N}}{\partial s}\\ &= 0\\ \overrightarrow{N}\cdot\frac{\partial\overrightarrow{C}}{\partial s} &= -\overrightarrow{C}\cdot\frac{\partial\overrightarrow{N}}{\partial s}\\ &= f(s)\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}\frac{\partial}{\partial s}(\overrightarrow{N}\cdot\overrightarrow{W}) &= \overrightarrow{N}\cdot\frac{\partial\overrightarrow{W}}{\partial s}+\overrightarrow{W}\cdot\frac{\partial\overrightarrow{N}}{\partial s}\\ &= 0\\ \overrightarrow{N}\cdot\frac{\partial\overrightarrow{W}}{\partial s} &= -\overrightarrow{W}\cdot\frac{\partial\overrightarrow{N}}{\partial s}\end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned}\frac{\partial}{\partial s}(\overrightarrow{C}\cdot\overrightarrow{W}) &= \overrightarrow{C}\cdot\frac{\partial\overrightarrow{W}}{\partial s}+\overrightarrow{W}\cdot\frac{\partial\overrightarrow{C}}{\partial s}\\ &= 0\\ \overrightarrow{C}\cdot\frac{\partial\overrightarrow{W}}{\partial s} &= -\overrightarrow{W}\cdot\frac{\partial\overrightarrow{C}}{\partial s}\\ &= g(s)\end{aligned}\right\} \tag{5}$$

The equations obtained above for the $\frac{\partial}{\partial s}$ are also written in the same way for the directional derivatives $\frac{\partial}{\partial c}$ and $\frac{\partial}{\partial w}$.

And also in [15], defined four additional terms of the directional derivatives. These are as follows

$$\left.\begin{aligned}\overrightarrow{C}\cdot\frac{\partial\overrightarrow{N}}{\partial c} &= -\overrightarrow{N}\cdot\frac{\partial\overrightarrow{C}}{\partial c}\\ &= \Psi_{cs}\\ \overrightarrow{W}\cdot\frac{\partial\overrightarrow{N}}{\partial w} &= -\overrightarrow{N}\cdot\frac{\partial\overrightarrow{W}}{\partial w}\\ &= \Psi_{ws}\\ \overrightarrow{W}\cdot\frac{\partial\overrightarrow{N}}{\partial c} &= -\overrightarrow{N}\cdot\frac{\partial\overrightarrow{W}}{\partial c}\\ &= \frac{1}{2}(\Phi_s+\Lambda_s)\\ \overrightarrow{C}\cdot\frac{\partial\overrightarrow{N}}{\partial w} &= -\overrightarrow{N}\cdot\frac{\partial\overrightarrow{C}}{\partial w}\\ &= \frac{1}{2}(\Phi_s-\Lambda_s)\end{aligned}\right\} \tag{6}$$

The symbol Ψ_{cs} represents the normal deformation of the vector tube in the direction of the $\overrightarrow{C}$, Ψ_{ws} represents the normal deformation of the vector tube in the direction of the $\overrightarrow{W}$, Φ_s is called the abnormality parameter of the vector $s-line$, Λ_s is the shear deformation in the normal plane (i.e. plane containing the $\overrightarrow{N}$ and $\overrightarrow{W}$ vectors).
Note that the abnormality parameter of the vectors $\overrightarrow{N}, \overrightarrow{C}, \overrightarrow{W}$ are denoted by the symbols Λ_s, Λ_c and Λ_w and defined as

$$\left.\begin{aligned} \Lambda_c &= -g(s) - \frac{1}{2}(\Phi_s - \Lambda_s) \\ &= -g(s) + \overrightarrow{N} \cdot \frac{\delta \overrightarrow{C}}{\delta w}, \\ \Lambda_w &= -g(s) + \frac{1}{2}(\Phi_s + \Lambda_s) \\ &= -g(s) - \overrightarrow{N} \cdot \frac{\delta \overrightarrow{W}}{\delta c}, \\ \Lambda_s &= -\overrightarrow{W} \cdot \frac{\delta \overrightarrow{N}}{\delta c} + \overrightarrow{C} \cdot \frac{\delta \overrightarrow{N}}{\delta w}, \end{aligned}\right\} \tag{7}$$

respectively. The three abnormalities can be written as follows upon examining the above expression as follows

$$\left.\begin{aligned} curl\overrightarrow{N} \cdot \overrightarrow{N} &= \Lambda_s \\ curl\overrightarrow{C} \cdot \overrightarrow{C} &= \Lambda_c \\ curl\overrightarrow{W} \cdot \overrightarrow{W} &= \Lambda_w \end{aligned}\right\} \tag{8}$$

The abnormality parameter of the vector $s - line$ is obtained by setting the torsion τ in (7) equal to each other, such that,

$$\Phi_s = \Lambda_w - \Lambda_c$$

3. Evaluation of Directional Derivative Expressions for the Alternative Moving Frame

Here we consider that $\gamma(s, c, w)$ be a curve that exists in the 3D space. Via (1)-(8) we can write the other geometric equations in terms of anholonomic coordinates respectively;

$$
\begin{aligned}
div\overrightarrow{N} &= \overrightarrow{N}\cdot\frac{\partial\overrightarrow{N}}{\partial s}+\overrightarrow{C}\cdot\frac{\partial\overrightarrow{N}}{\partial c}+\overrightarrow{W}\cdot\frac{\partial\overrightarrow{N}}{\partial w}\\
&= \Psi_{cs}+\Psi_{ws},\\
div\overrightarrow{C} &= \overrightarrow{N}\cdot\frac{\partial\overrightarrow{C}}{\partial s}+\overrightarrow{C}\cdot\frac{\partial\overrightarrow{C}}{\partial c}+\overrightarrow{W}\cdot\frac{\partial\overrightarrow{C}}{\partial w}\\
&= -f(s)+\overrightarrow{W}\cdot\frac{\partial\overrightarrow{C}}{\partial w},\\
div\overrightarrow{W} &= \overrightarrow{N}\cdot\frac{\partial\overrightarrow{W}}{\partial s}+\overrightarrow{C}\cdot\frac{\partial\overrightarrow{W}}{\partial c}+\overrightarrow{W}\cdot\frac{\partial\overrightarrow{W}}{\partial w}\\
&= \overrightarrow{C}\frac{\partial\overrightarrow{W}}{\partial c}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
curl\overrightarrow{N} &= \overrightarrow{N}\times\frac{\partial\overrightarrow{N}}{\partial s}+\overrightarrow{C}\times\frac{\partial\overrightarrow{N}}{\partial c}+\overrightarrow{W}\times\frac{\partial\overrightarrow{N}}{\partial w}\\
&= \Lambda_s+f(s)\overrightarrow{W},\\
curl\overrightarrow{C} &= \overrightarrow{N}\times\frac{\partial\overrightarrow{C}}{\partial s}+\overrightarrow{C}\times\frac{\partial\overrightarrow{C}}{\partial c}+\overrightarrow{W}\times\frac{\partial\overrightarrow{C}}{\partial w}\\
&= (-div\overrightarrow{W})\overrightarrow{N}+(-g(s)-\frac{1}{2}(\Phi_s-\Lambda_s))\overrightarrow{C}+\Psi_{cs}\cdot\overrightarrow{W},\\
curl\overrightarrow{W} &= \overrightarrow{N}\times\frac{\partial\overrightarrow{W}}{\partial s}+\overrightarrow{C}\times\frac{\partial\overrightarrow{W}}{\partial c}+\overrightarrow{W}\times\frac{\partial\overrightarrow{W}}{\partial w}\\
&= (div\overrightarrow{C}+f(s))\overrightarrow{N}+(-\Psi_{ws})\overrightarrow{C}+(-g(s)+\frac{1}{2}(\Phi_s+\Lambda_s))\cdot\overrightarrow{W}.
\end{aligned}
$$

Thanks to the above equations, the Serret-Frenet formulas for each direction of the frame together with the anholonomic coordinates of the $\{N, C, W\}$ frame are obtained as the following matrix forms;

$$
\begin{aligned}
\frac{\partial}{\partial s}\begin{pmatrix}\overrightarrow{N}\\ \overrightarrow{C}\\ \overrightarrow{W}\end{pmatrix} &= \begin{pmatrix}0 & f(s) & 0\\ -f(s) & 0 & g(s)\\ 0 & -g(s) & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{N}\\ \overrightarrow{C}\\ \overrightarrow{W}\end{pmatrix},\\
\frac{\partial}{\partial c}\begin{pmatrix}\overrightarrow{N}\\ \overrightarrow{C}\\ \overrightarrow{W}\end{pmatrix} &= \begin{pmatrix}0 & \Psi_{cs} & \Lambda_w+g(s)\\ -\Psi_{cs} & 0 & -div\overrightarrow{W}\\ -(\Lambda_w+g(s)) & div\overrightarrow{W} & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{N}\\ \overrightarrow{C}\\ \overrightarrow{W}\end{pmatrix},\\
\frac{\partial}{\partial w}\begin{pmatrix}\overrightarrow{N}\\ \overrightarrow{C}\\ \overrightarrow{W}\end{pmatrix} &= \begin{pmatrix}0 & -(\Lambda_c+g(s)) & \Psi_{ws}\\ \Lambda_c+g(s) & 0 & f(s)+div\overrightarrow{C}\\ -\Psi_{ws} & -(f(s)+div\overrightarrow{C}) & 0\end{pmatrix}\begin{pmatrix}\overrightarrow{N}\\ \overrightarrow{C}\\ \overrightarrow{W}\end{pmatrix}.
\end{aligned} \tag{9}
$$

4. Relationship between anholonomic coordinates and Electromagnetic curves

Berry's (geometric) phase in the directions throughout $C-direction$ and $W-direction$ arises with the dissemination of an E-M wave along with the optical fiber for the alternative moving frame of curve γ. Optical fiber can be defined as a curve $\gamma(s,c,w)$ via alternative moving frame in three dimensional space. The E-M wave dissemination is in the direction of $\overrightarrow{N}=(s,c,w)$ the polarization of the E-M wave is mentioned by the direction of the electric field vector $\overrightarrow{\mathbf{E}}=(s,c,w)$ and magnetic field is described as $\overrightarrow{V}=(s,c,w)$. Here basically the electric field will be shown perpendicular to the direction of W will be examined.

Case 1 : The derivation of the $\overrightarrow{\mathbf{E}}$ between any two points in the $C-direction$ for the alternative moving frame $\{N,C,W\}$ of the curve $\gamma(s,c,w)$ can be defined as

$$\frac{\partial}{\partial c}\overrightarrow{\mathbf{E}}(s,c,w)=\lambda_1\overrightarrow{N}+\lambda_2\overrightarrow{C}+\lambda_3\overrightarrow{W} \tag{10}$$

where $\lambda_i(s,c,w),\ i=1,2,3$ are sufficiently smooth arbitrary functions along the γ. The electric field is right angle to $\overrightarrow{N}$ and if we consider that because of the absorption, there is no mechanism loss in the optical fiber, we can write the following equations;

$$\langle\overrightarrow{N},\overrightarrow{\mathbf{E}}\rangle=0,\qquad\langle\overrightarrow{\mathbf{E}},\overrightarrow{\mathbf{E}}\rangle=c. \tag{11}$$

Taking the derivative of (11) and using the Eqs. (9)-(11), we get

$$\langle\frac{\partial\overrightarrow{N}}{\partial c},\overrightarrow{\mathbf{E}}\rangle=-\lambda_1.$$

Using Eqs. (10) and (11) we can calculate,

$$\lambda_1=-(\Psi_{cs}\mathbf{E}^C+(\Lambda_w+g(s))\mathbf{E}^W) \tag{12}$$

where $\mathbf{E}^C$ and $\mathbf{E}^W$ are smooth components of the $\overrightarrow{C}$ and $\overrightarrow{W}$. If we taking derivative of the second one in (11), we can get

$$\langle\frac{\partial\overrightarrow{\mathbf{E}}}{\partial c},\overrightarrow{\mathbf{E}}\rangle=0.$$

Therefore, using (9), (11) and (12), we obtain

$$\overrightarrow{\mathbf{E}}_c=-(\Psi_{cs}\mathbf{E}^C+(\Lambda_w+g(s))\mathbf{E}^W)\overrightarrow{N}+\lambda(\overrightarrow{\mathbf{E}}\times\overrightarrow{N}) \tag{13}$$

that λ is a constant term.
The last equation allows us to find the rotation of the electric field in the $C-direction$ around the $\overrightarrow{n}$. Moreover, we can assume that $\lambda=0$, with that we can

finalize which $\overrightarrow{\mathbf{E}}$ is a Rytov parallel transport in the $C-direction$ by the conditions given above

$$\overrightarrow{\mathbf{E}}_c = -(\overrightarrow{\mathbf{E}} \cdot N_c)N. \tag{14}$$

Furthermore, the Fermi-Walker transportation law is given as

$$\overrightarrow{B}_c^{FW} = \overrightarrow{B}_c \pm (\overrightarrow{B} \cdot \overrightarrow{N}_c)\overrightarrow{N} + (\overrightarrow{B} \cdot \overrightarrow{N})\overrightarrow{N}_c. \tag{15}$$

Generally, we can write

$$\overrightarrow{\mathbf{E}} = \mathbf{E}^C \overrightarrow{C} + \mathbf{E}^W \overrightarrow{W}. \tag{16}$$

Deriving (16) and combining with (9) we can write,

$$\frac{\partial}{\partial c}\overrightarrow{\mathbf{E}} = -(\Psi_{cs}\mathbf{E}^C + (\Lambda_w + g(s))\mathbf{E}^W)\overrightarrow{N} + (\mathbf{E}_c^C + div\overrightarrow{W} \cdot \mathbf{E}^W)\overrightarrow{C} + (\mathbf{E}_c^W - div\overrightarrow{W} \cdot \mathbf{E}^C)\overrightarrow{W}. \tag{17}$$

If the electric field is assumed to be Rytov parallel transported in $C - direction$, then comparing (14) and (17) satisfies that;

$$\begin{pmatrix} \mathbf{E}_c^C \\ \mathbf{E}_c^W \end{pmatrix} = \begin{pmatrix} 0 & -div\overrightarrow{W} \\ div\overrightarrow{W} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}^C \\ \mathbf{E}^W \end{pmatrix}. \tag{18}$$

Therefore, we can accomplish that (18) describes the motion of the polarization plane in the $C - direction$ along the optical fiber thus a Berry's phase $\rho = (s, c, w)$ in the c direction is defined by;

$$\frac{\partial}{\partial c}\rho = div\overrightarrow{W}.$$

Using the information provided it is found the magnetic field vector in relation to the ingredient of the electric field as ;

$$\overrightarrow{V} = \mathbf{E}^C \overrightarrow{W} - \mathbf{E}^W \overrightarrow{C} \tag{19}$$

that provides the following conditions;

$$\overrightarrow{V} \perp \overrightarrow{\mathbf{E}} \qquad \overrightarrow{V} \perp \overrightarrow{N} \tag{20}$$

where

$$V^C = -\mathbf{E}^W \qquad V^W = \mathbf{E}^C.$$

Using (20) and (9), deriving (19), we get

$$\frac{\partial \overrightarrow{V}}{\partial c} = (\mathbf{E}^W \Psi_{cs} - \mathbf{E}^C(\Lambda_w + g(s))\overrightarrow{N} + (\mathbf{E}^C div\overrightarrow{W} - \mathbf{E}_c^W)\overrightarrow{C} + (\mathbf{E}_c^C + \mathbf{E}^W div\overrightarrow{W})\overrightarrow{W} \tag{21}$$

which satisfies

$$\langle \frac{\partial \overrightarrow{V}}{\partial c}, \overrightarrow{\mathbf{E}} \rangle + \langle \frac{\partial \overrightarrow{\mathbf{E}}}{\partial c}, \overrightarrow{V} \rangle = 0$$

and

$$\langle \frac{\partial \overrightarrow{V}}{\partial c}, \overrightarrow{N} \rangle + \langle \frac{\partial \overrightarrow{N}}{\partial c}, \overrightarrow{V} \rangle = 0.$$

Within the results obtained, we can make the following inference, magnetic field and electric field have alike Berry's phase in the same conditions as follows

$$\overrightarrow{V}_c = -(\overrightarrow{V} \cdot \overrightarrow{N}_c)\overrightarrow{N}. \tag{22}$$

We show that if $\overrightarrow{\mathbf{E}}$ is the Rytov parallel transported the $C-direction$ if and only if it is Fermi-Walker parallel transported in the $C-direction$ throughout optical fiber via alternative moving frame of the curve γ.

The Lorentz force is the force acting on a charged particle moving in electromagnetic field in three dimensional space. At that time, the electromagnetic field in the $C-direction$ along with the curve γ via alternative moving frame with concerning anholonomic coordinates help of Lorentz equation $\phi(\overrightarrow{\mathbf{E}}) = \overrightarrow{X} \times \overrightarrow{\mathbf{E}}$ where $\overrightarrow{X}$ is a Killing magnetic field in three dimensional space and (9) is given as follows;

$$\langle \phi_c(\overrightarrow{\mathbf{E}}), \overrightarrow{N} \rangle = -\langle \phi(\overrightarrow{N}), \overrightarrow{\mathbf{E}}_C \rangle = \langle \frac{\partial \overrightarrow{\mathbf{E}}}{\partial c}, \overrightarrow{N} \rangle = -\Psi_{cs}\mathbf{E}^C - (\Lambda_w + g(s)\mathbf{E}^W.$$

When necessary arrangements are made, we can write;

$$\begin{aligned} \phi_c(\overrightarrow{N}) &= \Psi_{cs}\mathbf{E}^C + (\Lambda_w + g(s))\mathbf{E}^W + a_1\mathbf{E}^N \\ \phi_c(\overrightarrow{C}) &= -\lambda\mathbf{E}^W + a_2\mathbf{E}^N \\ \phi_c(\overrightarrow{W}) &= \lambda\mathbf{E}^C + a_3\mathbf{E}^N. \end{aligned} \tag{23}$$

Taking (23) and (9) into account, Lorentz force in the $C-direction$ throughout the optical fiber that is determined curve γ for the alternative moving frame implies the following matrix form;

$$\begin{pmatrix} \phi_c(\overrightarrow{N}) \\ \phi_c(\overrightarrow{C}) \\ \phi_c(\overrightarrow{W}) \end{pmatrix} = \begin{pmatrix} 0 & \Psi_{cs} & (\Lambda_w + g(s) \\ -\Psi_{cs} & 0 & -\lambda \\ -(\Lambda_w + g(s) & \lambda & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}^N \\ \mathbf{E}^C \\ \mathbf{E}^W \end{pmatrix}$$

Case 2 : The derivation of the electric field vector $\overrightarrow{\mathbf{E}}$ between any two points in the $W-direction$ for the alternative moving frame $\{N, C, W\}$ of the curve $\gamma(s, c, w)$ can be defined as

$$\frac{\partial}{\partial w}\overrightarrow{\mathbf{E}}(s, c, w) = \lambda_1\overrightarrow{N} + \lambda_2\overrightarrow{C} + \lambda_3\overrightarrow{W} \tag{24}$$

where $\lambda_i(s, c, w),\ i = 1, 2, 3$ are sufficiently smooth arbitrary functions along the γ. The electric field is right angle to $\overrightarrow{N}$ and if we consider that because of the absorption, there is no mechanism loss in the optical fiber, we can write the following equations;

$$\langle \overrightarrow{N}, \overrightarrow{\mathbf{E}} \rangle = 0, \qquad \langle \overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{E}} \rangle = c. \tag{25}$$

Taking derivative of (25) and using the Eqs. (9)-(25), we get

$$\langle \frac{\partial \overrightarrow{N}}{\partial w}, \overrightarrow{\mathbf{E}} \rangle = \lambda_1$$

Using Eqs. (24) and (25) we can calculate,

$$\lambda_1 = ((\Lambda_c + g(s))\mathbf{E}^C - \Psi_{ws}\mathbf{E}^W)\overrightarrow{N}. \tag{26}$$

If we taking derivative of the second one in (25), we can get

$$\langle \frac{\partial \overrightarrow{\mathbf{E}}}{\partial w}, \overrightarrow{\mathbf{E}} \rangle = 0.$$

After that we collect (9), (25) and (25) we obtain

$$\overrightarrow{\mathbf{E}}_w = ((\Lambda_c + g(s))\mathbf{E}^C - \Psi_{ws}\mathbf{E}^W)\overrightarrow{N} + \lambda(\overrightarrow{\mathbf{E}} \times \overrightarrow{N}) \tag{27}$$

that λ is a constant.
Considering the last equation we get the rotation of the $\overrightarrow{\mathbf{E}}$ in the $W - direction$ around the $\overrightarrow{N}$. Furthermore, we assume that $\lambda = 0$, in this manner we finalize that $\overrightarrow{\mathbf{E}}$ is a parallel transport in the $W - direction$ with the above terms

$$\overrightarrow{\mathbf{E}}_w = -(\overrightarrow{\mathbf{E}}, \overrightarrow{N}_w)\overrightarrow{N}. \tag{28}$$

Additionally, this motion can be defined through the Fermi-Walker transportation law in three dimensional space is as follows;

$$\overrightarrow{B}_w^{FW} = \overrightarrow{B}_w \pm (\overrightarrow{B} \cdot \overrightarrow{N}_w)\overrightarrow{N} + (\overrightarrow{B} \cdot \overrightarrow{N})\overrightarrow{N}_w. \tag{29}$$

Generally, we get

$$\overrightarrow{\mathbf{E}} = \mathbf{E}^C\overrightarrow{C} + \mathbf{E}^W\overrightarrow{W} \tag{30}$$

where $\mathbf{E}^C$ and $\mathbf{E}^W$ are smooth components of the $\overrightarrow{C}$ and $\overrightarrow{W}$. Deriving (30) and combining with (9) we can write,

$$\begin{aligned} \frac{\partial}{\partial w}\overrightarrow{\mathbf{E}} &= ((\Lambda_c + g(s))\mathbf{E}^C - \Psi_{ws}\mathbf{E}^W)\overrightarrow{N} + (\mathbf{E}_w^C - (div\overrightarrow{C} + f(s))\mathbf{E}^W)\overrightarrow{C} \\ &+ (\mathbf{E}_w^W + (f(s) + div\overrightarrow{C})\mathbf{E}^C)\overrightarrow{W}. \end{aligned} \tag{31}$$

If the electric field is presumed to be Rytov parallel transported in the direction W, then comparing (28) and (31) implies that

$$\begin{pmatrix} \mathbf{E}_w^C \\ \mathbf{E}_w^W \end{pmatrix} = \begin{pmatrix} 0 & div\overrightarrow{C} + f(s) \\ -(div\overrightarrow{C} + f(s)) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}^C \\ \mathbf{E}^W \end{pmatrix}. \tag{32}$$

Therefore, (32) describes the rotation of the polarization plane in the $W - direction$ along the optical fiber thus a Berry's phase $\rho = (s, c, w)$ in the $W - direction$ described by;

$$\frac{\partial}{\partial w}\rho = div\overrightarrow{C} + f(s).$$

We can indicate the magnetic field vector in relation to the ingredient of the electric field as;

$$\overrightarrow{V} = \mathbf{E}^C \overrightarrow{W} - \mathbf{E}^W \overrightarrow{C} \tag{33}$$

that ensures the following conditions;

$$\overrightarrow{V} \perp \overrightarrow{\mathbf{E}} \qquad \overrightarrow{V} \perp \overrightarrow{N} \tag{34}$$

where

$$V^C = \mathbf{E}^W \qquad V^W = \mathbf{E}^C.$$

Using (9), (34) and deriving (33), we can get;

$$\begin{aligned}\frac{\partial \overrightarrow{V}}{\partial w} &= (-\mathbf{E}^C \Psi_{ws} - \mathbf{E}^W (\Lambda_C + g(s))\overrightarrow{N} + (-\mathbf{E}^C (f(s) + div \overrightarrow{C}) - \mathbf{E}^W_w)\overrightarrow{C} \\ &\quad + (\mathbf{E}^C_w + \mathbf{E}^W (f(s) + div \overrightarrow{C}))\overrightarrow{W}\end{aligned} \tag{35}$$

which satisfies

$$\langle \frac{\partial \overrightarrow{V}}{\partial w}, \overrightarrow{\mathbf{E}} \rangle + \langle \frac{\partial \overrightarrow{\mathbf{E}}}{\partial w}, \overrightarrow{V} \rangle = 0$$

and

$$\langle \frac{\partial \overrightarrow{V}}{\partial w}, \overrightarrow{N} \rangle + \langle \frac{\partial \overrightarrow{N}}{\partial w}, \overrightarrow{V} \rangle = 0.$$

Consequently, we can say that magnetic field and electric field have Berry's phase in the same conditions as follows;

$$\overrightarrow{V}_w = -(\overrightarrow{V} \cdot \overrightarrow{N}_w)\overrightarrow{N}$$

if $\overrightarrow{\mathbf{E}}$ is the Rytov parallel transported the $W - direction$ if and only if it is Fermi-Walker parallel transported in the $W - direction$ along with optical fiber via alternative moving frame of the curve γ.
The electromagnetic field in the $W - direction$ along with the curve γ via alternative moving frame with respect to anholonomic coordinates help of Lorentz equation and (9) is given as follows;

$$\langle \phi_w(\overrightarrow{\mathbf{E}}), \overrightarrow{N} \rangle = -\langle \phi(\overrightarrow{N}), \overrightarrow{\mathbf{E}}_W \rangle = \langle \frac{\partial \overrightarrow{\mathbf{E}}}{\partial w}, \overrightarrow{N} \rangle = (\Lambda_c + g(s)\mathbf{E}^C - \Psi_{ws}\mathbf{E}^W. \tag{36}$$

When necessary arrangements are made, we can write;

$$\begin{aligned}\phi_w(\overrightarrow{N}) &= \Psi_{ws}\mathbf{E}^W - (\Lambda_c + g(s)\mathbf{E}^C + a_1\mathbf{E}^N \\ \phi_w(\overrightarrow{C}) &= -\lambda\mathbf{E}^W + a_2\mathbf{E}^N \\ \phi_w(\overrightarrow{W}) &= \lambda\mathbf{E}^C + a_3\mathbf{E}^N.\end{aligned} \tag{37}$$

Taking (37) and (9) into account, the Lorentz force in the direction W along with the optical fiber that is determined curve γ for the alternative moving frame implies the following matrix form;

$$\begin{pmatrix} \phi_w(\overrightarrow{N}) \\ \phi_w(\overrightarrow{C}) \\ \phi_w(\overrightarrow{W}) \end{pmatrix} = \begin{pmatrix} 0 & -(\Lambda_c + g(s)) & \Psi_{ws} \\ (\Lambda_c + g(s)) & 0 & -\lambda \\ -\Psi_{ws} & \lambda & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}^N \\ \mathbf{E}^C \\ \mathbf{E}^W \end{pmatrix} \tag{38}$$

5. Mathematical Approach of Maxwell equations for Electromagnetic and Magnetic waves via Alternative Moving frame

Maxwell's equations consist of four main equations that are very important for understanding electromagnetic theory. Maxwell's equations, together with the Lorentz force law, are a set of partial differential equations that form the basis for the fields of classical electrodynamics and optics.These equations describe how magnetic and electric fields are exchanged and produced by each other, by charges and currents. Maxwell equations are given by,

$$\nabla \cdot \overrightarrow{\mathbf{E}} = 0 \tag{39}$$

$$\nabla \cdot \overrightarrow{V} = 0 \tag{40}$$

$$\nabla \times \overrightarrow{V} = \epsilon v \frac{\partial \mathbf{E}}{\partial u} \tag{41}$$

$$\nabla \times \overrightarrow{\mathbf{E}} = -\frac{\partial V}{\partial u} \tag{42}$$

where ϵ and v have the same values at all points and (s, c, w) and u space, time variables. If we assume that the electric field is perpendicular to the tangent direction and (17), (31) and (39), we can obtain;

$$\begin{aligned} \nabla \cdot \overrightarrow{\mathbf{E}} &= (\overrightarrow{N} \cdot \frac{\partial}{\partial s} + \overrightarrow{C} \frac{\partial}{\partial c} + \overrightarrow{W} \frac{\partial}{\partial w}) \cdot \overrightarrow{\mathbf{E}} \\ &= \overrightarrow{N} \cdot \frac{\partial \overrightarrow{\mathbf{E}}}{\partial s} + \overrightarrow{C} \cdot \frac{\partial \overrightarrow{\mathbf{E}}}{\partial c} + \overrightarrow{W} \cdot \frac{\partial \overrightarrow{\mathbf{E}}}{\partial w} \\ &= 0 \end{aligned}$$

that satisfies;

$$\mathbf{E}_c^C - \mathbf{E}_w^W = -\mathbf{E}^C div \overrightarrow{C} + \mathbf{E}^W div \overrightarrow{W}. \tag{43}$$

In the same way, we are aware that $\mathbf{E}$ is right angle to the tangent directional and using (17), (31) and (40), we can compute that;

$$\begin{aligned}\nabla \cdot \overrightarrow{V} &= (\overrightarrow{N} \cdot \frac{\partial}{\partial s} + \overrightarrow{C}\frac{\partial}{\partial c} + \overrightarrow{W}\frac{\partial}{\partial w}) \cdot \overrightarrow{V} \\ &= \overrightarrow{N} \cdot \frac{\partial \overrightarrow{V}}{\partial s} + \overrightarrow{C} \cdot \frac{\partial \overrightarrow{V}}{\partial c} - \overrightarrow{W} \cdot \frac{\partial \overrightarrow{V}}{\partial w} \\ &= 0\end{aligned}$$

which implies that,

$$\mathbf{E}_c^W - \mathbf{E}_w^C = \mathbf{E}^C div\overrightarrow{W} - \mathbf{E}^W div\overrightarrow{C}. \tag{44}$$

If we think comprehensively (43) and (44), then it is calculated that Laplacian-like equations through $C - lines$ and $W - lines$ of the electromagnetic waves are as follows;

$$\begin{aligned}\frac{\partial^2}{\partial c^2}\mathbf{E}^W - \frac{\partial^2}{\partial w^2}\mathbf{E}^W &= \mathbf{E}^C((div\overrightarrow{W})_c + (div\overrightarrow{C})_w) + \mathbf{E}^W((div\overrightarrow{W})_w + (div\overrightarrow{C})_c) \\ &\quad + div\overrightarrow{W}(\mathbf{E}_c^C + \mathbf{E}_w^W) + div\overrightarrow{C}(\mathbf{E}_c^W + \mathbf{E}_w^C) \\ \frac{\partial^2}{\partial c^2}\mathbf{E}^C - \frac{\partial^2}{\partial w^2}\mathbf{E}^C &= \mathbf{E}^C((div\overrightarrow{W})_w - (div\overrightarrow{C})_c) + \mathbf{E}^W((div\overrightarrow{W})_w - (div\overrightarrow{C})_c) \\ &\quad + div\overrightarrow{W}(\mathbf{E}_w^C - \mathbf{E}_c^W) + div\overrightarrow{C}(\mathbf{E}_w^W - \mathbf{E}_c^C).\end{aligned}$$

If we consider that the electric field is right angle to the tangential direction and (17), (31) and (41), we get;

$$\begin{aligned}\nabla \times \overrightarrow{V} = \epsilon v \frac{\partial \overrightarrow{\mathbf{E}}}{\partial u} &= (\overrightarrow{N} \cdot \frac{\partial}{\partial s} + \overrightarrow{C}\frac{\partial}{\partial c} + \overrightarrow{W}\frac{\partial}{\partial w}) \times \overrightarrow{V} \\ &= (\overrightarrow{N} \times \frac{\partial}{\partial s}\overrightarrow{V} + \overrightarrow{C} \times \frac{\partial}{\partial c}\overrightarrow{V} + \overrightarrow{W} \times \frac{\partial}{\partial w}\overrightarrow{V})\end{aligned}$$

which satisfies that;

$$\begin{aligned}\epsilon v \frac{\partial \overrightarrow{\mathbf{E}}}{\partial u} &= -(\mathbf{E}_c^C + \mathbf{E}^W div\overrightarrow{W} + \mathbf{E}_w^W + \mathbf{E}^C(f(s) + div\overrightarrow{C})\overrightarrow{N} \\ &\quad + (\mathbf{E}_s^C + \Lambda_c \mathbf{E}^W + \Psi_{ws} E^W)\overrightarrow{C} + (-\mathbf{E}_s^W - \Lambda_w \mathbf{E}^C + \Psi_{cs}\mathbf{E}^W)\overrightarrow{W}.\end{aligned}$$

In the same sense, we attention to the $\overrightarrow{\mathbf{E}}$ is right angle to the tangent directional and (17), (31) and (42), we can write that;

$$\begin{aligned}-\frac{\partial}{\partial u}\overrightarrow{V} = \nabla \times \overrightarrow{\mathbf{E}} &= (\overrightarrow{N}\frac{\partial}{\partial s} + \overrightarrow{C}\frac{\partial}{\partial c} + \overrightarrow{W}\frac{\partial}{\partial w}) \times \overrightarrow{\mathbf{E}} \\ &= \overrightarrow{N} \times \frac{\partial}{\partial s}\overrightarrow{\mathbf{E}} + \overrightarrow{C} \times \frac{\partial}{\partial c}\overrightarrow{\mathbf{E}} + \overrightarrow{W} \times \frac{\partial}{\partial w}\overrightarrow{\mathbf{E}}\end{aligned}$$

which implies that,

$$-\frac{\partial}{\partial u}\overrightarrow{V} = (-\mathbf{E}_c^W + \mathbf{E}^C div\overrightarrow{W} + E_w^C - \mathbf{E}^W(f(s) + div\overrightarrow{C}))\overrightarrow{N}$$
$$+ (-\mathbf{E}_s^C - \Lambda_c\mathbf{E}^C + \Psi_{ws}\mathbf{E}^W)\overrightarrow{C} + (-\mathbf{E}_s^W - \Psi_{cs}\mathbf{E}^C - \Lambda_w\mathbf{E}^W)\overrightarrow{W}.$$

6. Conclusion

In this study, we found the movement of polarized light along the optical fiber by calculating the equations of the electric field and magnetic field in cases where the frame of the space is at a right angle with respect to the alternative frame's vector fields. Thus, we had the opportunity to examine the motion of light in the field of geometry. In this way, the relationship of the motion of light in space with special curves, which is an important subject of geometry, can be investigated. At the same time, we investigated the geometric phase issue and Maxwell's equations together. We have obtained two important cases. These situations gave us the chance to examine the motion of light in the $C-direction$ and in the direction of the Darboux vector. We also give their connections with Fermi-Walker parallel transportation laws via alternative moving frame. For further research, we aim to study Maxwellian evolution equations relationship between spherical coordinates to better understand the solutions of the equations.

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