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Well-posed problems for the Laplace-Beltrami operator on a punctured two-dimensional sphere

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Abstract

An arbitrary point is removed from a three-dimensional Euclidean space on a two-dimensional sphere. The new well-posed solvable boundary value problems for the corresponding Laplace-Beltrami operator on the resulting punctured sphere are presented. To formulate the well-posed problems some properties of Green's function of the Laplace-Beltrami operator on a two-dimensional sphere are previously studied in detail.

Keywords: Laplace-Beltrami operator two-dimensional punctured sphere well-posed problems Green's functions for elliptic equations.
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1. Introduction

In this paper, the choice of a fixed reversible self-adjoint operator A_1 acting in the Hilbert space H_1 is a starting point. A domain of the operator A_1 is denoted by $D(A_1)$. Suppose, there is a finite set of elements $\{\psi_1, \psi_2, \ldots, \psi_k\}$ that do not belong to $D(A_1)$. In principle, some elements of $\{\psi_1, \ldots, \psi_k\}$ may not belong to H_1. Denote the linear span of $\{\psi_1, \ldots, \psi_k\}$ by H_2 and assume that $\dim H_2 = k$. Let A_2 be some linear operator that maps elements of the space H_2 to the elements of another space H_3. The dimension of the space H_3 is assumed to be equal to the dimension of the space H_2. We add to H_1 a finite-dimensional space H_2 whose basis consists of elements not belonging to $D(A_1)$. We assume that the block operator

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$A = A_1 \oplus A_2$ acts from the space H into the space $\tilde{H} = H_1 + H_3$. Let us introduce D the set of all elements h from H representable in the form

$$h = g + \sum_{j=1}^{k} \alpha_j \psi_j \tag{1}$$

for some g from $D(A_1)$ and $(\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$. Note that representation (1) is unique. Therefore, it is convenient to introduce a linear operator J that associates an element $h \in H$ with an element g from $D(A_1)$. Similarly, α_j for $j = 1, ..., k$ are linear functionals of h.
On the set D, we define the operator A_{max} by the formula

$$A_{max}h = A_1 Jh + \sum_{j=1}^{k} \beta_j(Jh) A_2 \psi_j,$$

where $\{\beta_1(\cdot), \beta_2(\cdot), ..., \beta_k(\cdot)\}$ is a fixed set of linear functionals in the space H_1. The operator A_{max} is called the maximal operator, since the operator equation

$$A_{max}h = f, \qquad \forall f \in \tilde{H} \tag{2}$$

has a solution from the set D. If $h_0 \in D$ is a solution to equation (2), then for any $\gamma_1 \in \mathbb{C}, \ldots, \gamma_k \in \mathbb{C}$ expression $h_0 + \gamma_1\psi_1 + \ldots, +\gamma_k\psi_k$ also represents a solution to equation (2). The notion and definition of a maximal operator can be found in [1].

The question naturally arises: on which subset D_0 of the set D does operator equation (2) have a unique solution for any $f \in \tilde{H}$? This article is devoted to a complete description of such subsets of D_0 in the case when A_1 is the Laplace-Beltrami operator on the two-dimensional sphere S^2 from $\mathbb{R}^3$. In this case we are talking about the description of reversible restrictions of a maximal operator. It is often required that the solution of the operator equation (2) additionally possess the following property: for small changes in f, the corresponding solution h changes little. We call such restrictions of the maximal operator invertible restrictions with continuous inverse operators. In this article, in the case of the Laplace-Beltrami operator on a two-dimensional sphere from $\mathbb{R}^3$, restrictions with continuous inverse operators are described.

Let us dwell on some specific features of such tasks. We illustrate them using the simplest case of the Laplace operator defined on the unit ball $\Omega = \{|x| < 1\}$ of the Euclidean space $\mathbb{R}^3$. As it is known, the Dirichlet problem for the Laplace operator consists in the following:

$$\Delta u(x) = f(x), \Omega, u|_{\partial\Omega} = 0. \tag{3}$$

The Dirichlet problem (3) has a unique solution $u(x)$ for any right hand side $f \in L_2(\Omega)$. Moreover, the solution is given by the formula $u(x) = \int_\Omega G(x,t)f(t)dt$, where $G(x,t)$ is corresponding Green's function. As an operator A_1 $(-\Delta)$ in the Hilbert space $H_1 = L_2(\Omega)$. We fix a point x_0 subject to the requirement $|x_0| < 1$. Consider a set of four functions

$$\{\psi_0(x) = G(x, x_0), \psi_j(x) = \frac{\partial G(x,t)}{\partial t_j}|_{t=x^0}, j = 1, 2, 3\}.$$

It is clear that the functions $\psi_j(x) \notin D(A_1), j = 0, 1, 2, 3$, since they have singularities at $x = x_0$. Denote the linear span of $\{\psi_0(x), \psi_1(x), \psi_2(x), \psi_3(x)\}$ by H_2. In the space $H = H_1 + H_2$, consider the block operator $A = A_1 \oplus A_2$, where A_2 is the Laplace operator. As one can see from this example, the space H is wider than the space H_1. In this case $A_2\psi_j \equiv 0$ for $x \neq x_0$ because $A_2 = -\Delta$.

Let us introduce the maximal operator A_{max} on the set D whose elements h are given by relations (1)

$$A_{max}h = A_1 Jh.$$

It is required to give a description of reversible restrictions of the maximal operator. In [2], to solve such problems, methods of the theory of extensions of linear operators are used. In contrast to [2], in this paper we use the methods of restriction of linear operators [3]-[5]. The paper [6] gives a correct definition of the formal Laplace operator with a delta-like perturbation. To determine the region, the limiting potentials of the simple and double layers are used. It is proved that the boundary densities of the potentials and the double layer can be uniquely restored from a set of spectrum of some reference operators. In this case, one of the origin operators coincides with the original Laplace operator with a delta-like perturbation. In [7], a correct definition of the Laplace operator with delta-like potentials is given. A well-resolvable point perturbation is investigated and the resolvent formulas are described. Some properties of the resolvent are studied. The article [8] studies the localization of the discrete spectrum of some perturbations of a two-dimensional harmonic oscillator. The convergence of the expansion of the source function in terms of eigenfunctions of a two-dimensional harmonic oscillator is also studied there, and a representation of Green's function of a two-dimensional harmonic oscillator is obtained. In [9], an internal closed (without boundary) smooth manifold of lower dimension is cut out from a multidimensional ball. In this region, the reversible constrictions of the Laplace operator are well defined. In particular, the correct nonsmooth Bitsadze-Samarskii problem for the Laplace equation is defined. The article [10] shows a correct definition of elliptic operators with second-order variable coefficients with point interactions and gives formulas for their resolvents. In [11], the changes in the finite part of the spectrum of the Laplace operator under delta-like perturbations were studied. The article [12] studies differential operators on arbitrary geometric graphs without loops. For the introduced maximal operator, an analogue of Lagrange's formula is proved. An algorithm for constructing adjoint boundary forms for an arbitrary set of boundary conditions is presented. A complete description of all self-adjoint constraints of the maximal operator is also given. In [13], a class of well-posed problems for the Poisson equation in a punctured domain was considered in a Hilbert space. The properties of Green's function are investigated. The article [14] gives a complete description of correctly solvable boundary value problems for the Laplace operator in a punctured circle. Also formulas for the resolvents of well-posed problems for the Laplace operator in a punctured circle are given. The paper [15] describes the resolvents of well-posed problems for perturbations of finite rank of a polyharmonic operator in a punctured ball.

In contrast to the indicated works [2]-[15], in this paper the original Laplace-Beltrami operator is defined on a Riemannian manifold without boundary. The following paragraphs describe well-posed problems for the Laplace-Beltrami operator on a punctured two-dimensional sphere S^2 of $\mathbb{R}^3$.

2. Known facts about the Laplace-Beltrami operator on a two-dimensional sphere

In the function space $L_2(S^2)$, consider the Laplace-Beltrami operator B_0 on the two-dimensional sphere, defined as the operator that transfers any function $v(x) \in W_2^2(S^2)$ into the function $(I - \hat{\Delta}'_{S^2})v(x) \in L_2(S^2)$. The expression for the Laplace-Beltrami operator $-\hat{\Delta}'_{S^2}$ in spherical coordinates is given by the formula

$$\widehat{\Delta'_{S^2} v}(\theta,\varphi) = \hat{\Delta}'_{\theta,\varphi}\hat{v}(\theta,\varphi),$$

where $\hat{v}(\theta,\varphi) = v(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, and (θ,φ) are angles of spherical coordinates. Here $\hat{\Delta}'_{\theta,\varphi}$ is the second order differential operator defined by the formula

$$\hat{\Delta}'_{\theta,\varphi}\hat{v} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\hat{v}}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2\hat{v}}{\partial\varphi^2}.$$

Here we use the definitions of spaces of smooth functions on a smooth manifold and a differential operator on such a manifold [16]. Operator expression $\hat{\Delta}_{\theta,\varphi}$ of the Laplace-Beltrami operator $\hat{\Delta}'_{S^2}$ in the spherical system depends on the angles (θ,φ). Moreover, the expression $\hat{\Delta}'_{\theta,\varphi}$ has singularities for $\theta = 0$ and $\theta = \pi$. In fact, $\hat{\Delta}'_{S^2}$ at all points of the sphere S^2 has exactly the same structure and therefore has no singularities.

Note [16] that the operator B_0 is symmetric and non-negative in the space $L_2(S^2)$.

It is known [17] that the eigenvalues of the operator B_0 can only be numbers $\lambda_l = 1 + l(l+1)$, where $l \geq 0$ are integers. The eigenfunctions of the operator B_0 corresponding to the operator are given by the formula:

$$\hat{Y}_{lm}(\theta,\varphi) = \begin{cases} P_{lm}(\cos\theta)\cos m\varphi, & m = 0,\dots,l \\ P_{l|m|}(\cos\theta)\sin|m|\varphi, & m = -1,-2,\dots,-l. \end{cases}$$

where $P_{lm}(\cos\theta)$ represents the associated Legendre functions.

Note that the eigenfunctions of the operator B_0 do have the property

$$\hat{Y}_{lm}(\theta,\varphi) = \hat{Y}_{lm}(\pi-\theta,\pi+\varphi). \tag{4}$$

To verify the required equality, it suffices to check that the parity of the associated Legendre functions $P_{lm}(t)$ coincide with the parity of $l + |m|$.

It is known [17] that the system of eigenfunctions of the operator B_0 is an orthogonal basis in the function space $L_2(S^2)$.

Note that the following equalities hold:

$$(Y_{lm}(\theta,\varphi)Y_{lm}(\theta,\varphi))_{S^2} = \begin{cases} \frac{4\pi}{2l+1}, & m = 0, \\ \frac{(l+|m|)!}{(l-|m|)!}\frac{2\pi}{2l+1}, & 1 \leq |m| \leq l. \end{cases}$$

3. Green's function of the Laplace-Beltrami operator on a two-dimensional sphere

In this paragraph, we introduce the function $\hat{\varepsilon}(\theta,\varphi,\alpha,\beta)$ by the formula

$$\hat{\varepsilon}(\theta,\varphi,\alpha,\beta) = \sum_{l=0}^{\infty}\frac{1}{1+l(l+1)}\sum_{m=-l}^{l}\frac{\hat{Y}_{lm}(\theta,\varphi)\hat{Y}_{lm}(\alpha,\beta)}{\frac{2\pi}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}} \tag{5}$$

at

$$0 \leq \theta,\alpha \leq \pi,$$
$$0 \leq \varphi,\beta \leq 2\pi,$$

where

$$\hat{Y}_{lm}(\theta,\varphi) = \begin{cases} P_{lm}(\cos\theta)\cos m\varphi, m = 0,\dots,l, \\ P_l(\cos\theta)\sin|m|\varphi, m = -1,-2,\dots,-l, \end{cases}$$

$$P_{l|m|}(t) = (1-t^2)^{\frac{|m|}{2}}\frac{d^{|m|}}{dt^{|m|}}P_l(t),$$

$$P_l(t) = \frac{1}{2^l l!}\frac{d^l}{dt^l}(t^2-1)^l.$$

The specified representations $\hat{Y}_{lm}(\theta,\varphi), P_l(t), P_{l|m|}(t)$ are given in [16].

The following lemma gives a compact representation of the function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ above.

Lemma 3.1. *Function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ for $0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$ has the representation*

$$\hat{\varepsilon}(\theta,\varphi;\alpha,\beta) = \frac{1}{2\pi^2}\int_0^{\omega}\frac{1}{\sqrt{2\cos t - \cos\omega}}(\cos\frac{t}{2}\ln\frac{1}{2(1-\cos t)} - \kappa(t))dt,$$

where $\cos\omega = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\varphi-\beta), \kappa(t) \in \mathbb{C}^2[0,2\pi]$.

Proof. The monograph [16] gives the addition formula for a spherical function:

$$P_l(\cos\omega) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\frac{\hat{Y}_{lm}(\theta,\varphi)\hat{Y}_{lm}(\alpha,\beta)}{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}},$$

where

$$\cos\omega = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\varphi-\beta).$$

Therefore, from relation (5) we can write the following representation for the function

$$\hat{\varepsilon}(\theta,\varphi;\alpha,\beta) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{2l+1}{1+l(l+1)}P_l(\cos\omega). \tag{6}$$

The monograph [17] gives the Meyer integral representation for the Legendre polynomial:

$$P_l(\cos\omega) = \frac{2}{\pi}\int_0^{\omega}\frac{\cos(l+\frac{1}{2})t}{\sqrt{2\cos t - 2\cos\omega}}dt.$$

Therefore, we can transform the function as follows

$$\hat{\varepsilon}(\theta,\varphi;\alpha,\beta) = \frac{1}{2\pi^2}\int_0^{\omega}\frac{1}{\sqrt{2\cos t-2\cos\omega}}(\sum_{l=0}^{\infty}\frac{2l+1}{1+l(l+1)}\cos(l+\frac{1}{2})t)dt =$$

$$= \frac{1}{2\pi^2}\int_0^{\omega}\frac{1}{\sqrt{2\cos t-2\cos\omega}}[\sum_{l=1}^{\infty}\cos(l+\frac{1}{2})t + \cos\frac{1}{2} +$$

$$+\sum_{l=1}^{\infty}[\frac{2l+1}{1+l(l+1)} - \frac{2l+1}{l(l+1)}]\cos(l+\frac{1}{2})t]dt.$$

Since in the reference book [19] the sums of the series are calculated:

$$\sum_{l=1}^{\infty}\frac{\cos lt}{l} = \frac{1}{2}\ln\frac{1}{2(1-\cos t)}, \quad 0 < t < 2\pi,$$

then we write the sum of the series in the form:

$$\sum_{l=2}^{\infty}\frac{2l+1}{l(l+1)}\cos(l+\frac{1}{2})t = \cos\frac{t}{2}\ln\frac{1}{2(1-\cos t)} - \cos(\frac{t}{2}),$$

This implies the following representation

$$\hat{\varepsilon}(\theta,\varphi;\alpha,\beta) = \frac{1}{2\pi^2}\int_0^{\omega}\frac{1}{\sqrt{2\cos t-2\cos\omega}}(\cos\frac{t}{2}\ln\frac{1}{2(1-\cos t)} - \kappa(t))dt,$$

where $\kappa(t) = \sum_{l=1}^{\infty}\frac{(-2l-1)}{l(l+1)(l^2+l+1)}\cos(l+\frac{1}{2})t, \kappa(t)\in\mathbb{C}^2[0,2\pi]$.

Lemma 3.1 is completely proved. □

Further the following assertion will be useful.

Lemma 3.2. *For any function $f(\theta,\varphi)\in L_2(S^2)$ the following formula is valid:*

$$(I-\hat{\Delta}'_{\theta,\varphi})(\int_0^{\pi}\sin\alpha d\alpha\int_0^{2\pi}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)f(\alpha,\beta)d\beta) = f(\theta,\varphi),$$

where the Laplace-Beltrami operator is defined by the formula

$$\hat{\Delta}'_{\theta,\varphi} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\theta^2}{\partial\varphi^2}.$$

Proof. Denote by

$$\hat{u}(\theta,\varphi)=\int_0^{\pi}\sin\alpha d\alpha\int_0^{2\pi}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)f(\alpha,\beta)d\beta. \tag{7}$$

Taking into account the representation (5) for the function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$, we rewrite equality (7):

$$\hat{u}(\theta,\varphi)=\sum_{l=0}^{\infty}\frac{1}{1+l(l+1)}\sum_{m=-l}^{l}\frac{\hat{Y}_{lm}(\theta,\varphi)}{\sqrt{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}}\int_0^{\pi}\sin\alpha d\alpha\int_0^{2\pi}\frac{\hat{Y}_{lm}(\alpha,\beta)}{\sqrt{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}}f(\alpha,\beta)d\beta.$$

Since the function $\frac{\hat{Y}_{lm}(\theta,\varphi)}{\sqrt{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}}$ represents the normalized eigenfunction of the Laplace-Beltrami operator $(-\hat{\Delta}'_{\theta,\varphi})$ corresponding to the eigenvalue $l(l+1)$, then the following equality holds:

$$(I-\hat{\Delta}'_{\theta,\varphi})\hat{Y}_{lm}(\theta,\varphi)+\hat{Y}_{lm}(\theta,\varphi)=(1+l(l+1))\hat{Y}_{lm}(\theta,\varphi).$$

Therefore, the expression $\hat{\Delta}'_{\theta,\varphi}\hat{u}(\theta,\varphi)$ has the form

$$(I-\hat{\Delta}'_{\theta,\varphi})\hat{u}(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\hat{Y}_{lm}(\theta,\varphi)}{\sqrt{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}}\int_0^{\pi}\sin\alpha d\alpha\int_0^{2\pi}\frac{\hat{Y}_{lm}(\alpha,\beta)}{\sqrt{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}}f(\alpha,\beta)d\beta.$$

The right side of the last equality is $f(\theta,\varphi)$, since $\{\frac{1}{4\pi},\frac{\hat{Y}_{lm}(\theta,\varphi)}{\sqrt{\frac{2\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}},l\geq 1,l\leq m\leq l\}$ represents an orthonormal basis of the space $L_2(S^2)$. Lemma 3.2 is completely proved. □

Lemma 3.1 and Lemma 3.2 imply that the function

$$\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)=\frac{1}{2\pi^2}\int_0^{\omega}\frac{1}{\sqrt{2\cos t-2\cos\omega}}(\cos\frac{t}{2}\ln\frac{1}{2(1-\cos t)}-\kappa(t))dt$$

represents Green's function of the inhomogeneous Laplace-Beltrami equation on a two-dimensional sphere $-\hat{\Delta}'_{\theta,\varphi}u(\theta,\varphi)=f(\theta,\varphi)$ in the function space $L_2(S^2)$.

Let us now find out the smoothness properties of Green's function.

In the monograph [17], an asymptotic representation at $l\to\infty$ is given for the Legendre polynomial

$$P_l(\cos\omega)=\sqrt{\frac{2}{\pi l\sin\omega}}\sin[(l+\frac{1}{2})\omega+\frac{\pi}{4}]+r_l(\omega), \tag{8}$$

here $0<\delta\leq\omega\leq\pi-\delta$. Note that $r_l(\omega)=\frac{q_l(\omega)}{l}$ and $|q_l(\omega)|<c$ uniformly over the entire closed interval $[\delta,\pi-\delta]$. Using the indicated asymptotic representation (8), we find the smoothness properties of Green's functions $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$.

For further purposes, it is convenient to introduce the following three functions:

$$K_1(\omega)=\sum_{l=0}^{\infty}\frac{2l+1}{(1+l(l+1))\sqrt{l}}\sin[(l+\frac{1}{2})\omega+\frac{\pi}{4}],$$

$$K_2(\omega)=\frac{1}{4\pi}\sum_{l=1}^{\infty}\frac{2l+1}{(1+l(l+1))l}q_l(\omega).$$

Lemma 3.3. *Let δ be an arbitrary positive number. Then Green's function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ has the following representation for $\delta<\omega<\pi-\delta$*

$$\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)=\frac{1}{4\pi}\sqrt{\frac{2}{\pi\sin\omega}}K_1(\omega)+K_2(\omega). \tag{9}$$

Here $\cos\omega=\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\varphi-\beta)$.

Proof. In the course of the proof of Lemma 3.1, for Green's function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ representation (6) is obtained. We substitute the asymptotic formula (8) into representation (6). As a result, we obtain representation (9). Note that the series $K_1(\omega), K_2(\omega)$ converge absolutely and uniformly over the entire closed interval $[\delta,\pi-\delta]$. Lemma 3.3 is completely proved. □

Remark 3.4. *The functions $K_1(\omega), K_2(\omega)$ are continuously differentiable functions on the interval $(0,\pi)$.*

Remark 3.5. *According to Lemma 3.3, Green's function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ is given by formula (9) for $\omega\in(0,\pi)$ and, taking into account Remark 3.4, represents a continuously differentiable function of ω on the interval $(0,\pi)$.*

Remark 3.6. *Representation (9) implies that Green's function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ has singularities at $\sin\omega=0$. At the same time, $\sin\omega=0$ for $\omega=0$ and $\omega=\pi$. Thus, Green's function $\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)$ has two singularities at the points $(\theta,\varphi)=(\alpha,\beta)$ and $(\theta,\varphi)=(\pi-\alpha,\pi+\beta)$. In fact, from the relation (4) implies that*

$$\hat{\varepsilon}(\theta,\varphi,\alpha,\beta)=\hat{\varepsilon}(\pi-\theta,\pi+\varphi,\alpha,\beta)=\hat{\varepsilon}(\theta,\varphi,\pi-\alpha,\pi+\beta). \tag{10}$$

4. Correct definition of the maximal operator

We choose an arbitrary point (θ_0,φ_0) on the two-dimensional sphere S^2. Denote by $S_0^2=S^2\backslash\{(\theta_0,\varphi_0)\}$. Consider, on the two-dimensional sphere S^2, a neighborhood of the point (θ_0,φ_0) defined as follows

$$\Pi_1^0(\delta)=\{(\theta,\varphi)\in S^2:|\theta-\theta_0|<\delta,|\varphi-\varphi_0|<\delta\},$$

where $\delta>0$. The neighborhood of the point $(\pi-\theta_0,\pi+\varphi_0)$ is denoted by

$$\Pi_2^0(\delta)=\{(\theta,\varphi)\in S^2:|\theta-\pi+\theta_0|<\delta,|\varphi-\pi-\varphi_0|<\delta\}.$$

It is also convenient to denote by

$$\Pi^0(\delta)=\Pi_1^0(\delta)\cup\Pi_2^0(\delta).$$

We introduce the class of functions

$$W_{2,loc}^2(S_0^2)=\bigcup_{\delta>0}W_2^2(S^2\backslash\Pi^0(\delta)).$$

For further purposes, it is convenient for us to introduce the following linear functionals by the formulas:

$$U_0(\hat{h})=\sin\theta_0\lim_{\delta\to0}[\int_{\varphi_0-\delta}^{\varphi_0+\delta}(\frac{\partial\hat{h}(\theta_0-\delta,\beta)}{\partial\alpha}-\frac{\partial\hat{h}(\theta_0+\delta,\beta)}{\partial\alpha})d\beta+$$
$$+\int_{\theta_0-\delta}^{\theta_0+\delta}(\frac{\partial\hat{h}(\alpha,\varphi_0-\delta)}{\partial\beta}-\frac{\partial\hat{h}(\alpha,\varphi_0+\delta)}{\partial\beta})\frac{d\alpha}{\sin\alpha}]+$$
$$+\sin\theta_0\lim_{\delta\to0}[\int_{\pi+\varphi_0-\delta}^{\pi+\varphi_0+\delta}(\frac{\partial\hat{h}(\pi-\theta_0-\delta,\beta)}{\partial\alpha}-\frac{\partial\hat{h}(\pi-\theta_0+\delta,\beta)}{\partial\alpha})d\beta+$$
$$+\int_{\pi-\theta_0-\delta}^{\pi-\theta_0+\delta}(\frac{\partial\hat{h}(\alpha,\pi+\varphi_0-\delta)}{\partial\beta}-\frac{\partial\hat{h}(\alpha,\pi+\varphi_0+\delta)}{\partial\beta})\frac{d\alpha}{\sin\alpha}],$$

$$U_1(\hat{h})=\sin\theta_0\lim_{\delta\to0}[\int_{\varphi_0-\delta}^{\varphi_0+\delta}(\hat{h}(\theta_0+\delta,\beta)-\hat{h}(\theta_0-\delta,\beta))d\beta]+$$
$$+\sin\theta_0\lim_{\delta\to0}[\int_{\pi+\varphi_0-\delta}^{\pi+\varphi_0+\delta}(\hat{h}(\pi-\theta_0+\delta,\beta)-\hat{h}(\pi-\theta_0-\delta,\beta))d\beta],$$

$$U_2(\hat{h}) = \lim_{\delta\to 0}[\int_{\theta_0-\delta}^{\theta_0+\delta}(\hat{h}(\alpha,\varphi_0+\delta)-\hat{h}(\alpha,\varphi_0-\delta))\frac{d\alpha}{\sin\alpha}]+$$
$$+\lim_{\delta\to 0}[\int_{\pi-\theta_0-\delta}^{\pi-\theta_0+\delta}(\hat{h}(\alpha,\pi+\varphi_0+\delta)-\hat{h}(\alpha,\pi+\varphi_0-\delta))\frac{d\alpha}{\sin\alpha}].$$

We also need the following function class:

$$W_{2,U}^2(S_0^2) = \{\hat{h}(\theta,\varphi)\in W_{2,loc}^2(S_0^2):$$
$$\exists U_0(\hat{h}), U_1(\hat{h}), U_2(\hat{h}) \quad \text{finite values}\}.$$

Take an arbitrary function $\hat{h}(\theta,\varphi)$ from the class $W_{2,U}^2(S_0^2)$. Denote by

$$\hat{g}(\theta,\varphi) = \lim_{\delta\to 0}\int_{S^2\setminus\Pi^0(\delta)}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)(-\hat{\Delta}'_{\alpha\beta}\hat{h}(\alpha,\beta))\sin\alpha d\alpha.$$

An important property of the function $\hat{g}(\theta,\varphi)$ is indicated in the following statement.

Lemma 4.1. *Let $\hat{h}(\alpha,\beta)$ be an arbitrary element from the class $W_{2,U}^2(S_0^2)$. Then the function $\hat{g}(\theta,\varphi)$ is represented as the following:*

$$\hat{g}(\theta,\varphi) = \hat{h}(\theta,\varphi)+\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)U_0(\hat{h})+$$
$$+\frac{\partial\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)}{\partial\alpha}U_1(\hat{h})+\frac{\partial\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)}{\partial\beta}U_2(\hat{h}). \tag{11}$$

Remark 4.2. *Representation Lemma 4.1 implies that the function $\hat{g}(\theta,\varphi)$ is continuous at the points (θ_0,φ_0), $(\pi-\theta_0,\pi+\varphi_0)$ and is uniquely determined by the function $\hat{h}(\theta,\varphi)$. At the same time, the function $\hat{h}(\theta,\varphi)$ can have singularities at the points (θ_0,φ_0) and $(\pi-\theta_0,\pi+\varphi_0)$. Therefore, the function $\hat{g}(\theta,\varphi)$ can be considered as a regularization of the functions $\hat{h}(\theta,\varphi)$.*

Proof. We transform the function $\hat{g}(\theta,\varphi)$ using its definition and the definition of the Laplace-Beltrami operator. We calculate the following limit separately:

$$\hat{T}(\theta,\varphi,\theta_0,\varphi_0) = \lim_{\delta\to 0}\int_{\varphi_0-\delta}^{\varphi_0+\delta}[(\int_0^{\theta_0-\delta}+\int_{\theta_0+\delta}^{\pi})\frac{\partial}{\partial\alpha}(\sin\alpha\frac{\partial\hat{h}(\alpha,\beta)}{\partial\alpha})\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)d\alpha]d\beta+$$
$$+(\int_0^{\theta_0-\delta}+\int_{\theta_0+\delta}^{\pi})[\int_{\varphi_0-\delta}^{\varphi_0+\delta}\frac{1}{\sin\alpha}\frac{\partial^2\hat{h}(\alpha,\beta)}{\partial\beta^2}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)d\beta]d\alpha+$$
$$+(\int_0^{\varphi_0-\delta}+\int_{\varphi_0+\delta}^{2\pi})(\int_0^{\pi}\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial\hat{h}(\alpha,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)d\alpha)d\beta+$$
$$+\int_0^{\pi}[(\int_0^{\varphi_0-\delta}+\int_{\varphi_0+\delta}^{2\pi})\frac{1}{\sin\alpha}\frac{\partial^2\hat{h}(\alpha,\beta)}{\partial\beta^2}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)d\beta]d\alpha=$$
$$=\int_{\varphi_0-\delta}^{\varphi_0+\delta}[\sin\alpha\frac{\partial\hat{h}(\alpha,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)-\hat{h}(\alpha,\beta)\sin\alpha\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)}{\partial\alpha}]_{\alpha=\theta_0+\delta}^{\alpha=\theta_0-\delta}d\beta+$$
$$+(\int_0^{\theta_0-\delta}+\int_{\theta_0+\delta}^{\pi})\frac{1}{\sin\alpha}[\frac{\partial\hat{h}(\alpha,\beta)}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)-\hat{h}(\alpha,\beta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)}{\partial\beta}]_{\beta=\varphi_0-\delta}^{\beta=\varphi_0+\delta}d\alpha+$$
$$+(\int_0^{\varphi_0-\delta}+\int_{\varphi_0+\delta}^{2\pi})[\sin\alpha\frac{\partial\hat{h}(\alpha,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)d\alpha)-$$

$$-\hat{h}(\alpha,\beta)\sin(\alpha)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)}{\partial\alpha}]_{\alpha=0}^{\alpha=\pi}d\beta+$$

$$+\int_0^{\pi}[\frac{\partial\hat{h}(\alpha,\beta)}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\alpha,\beta)-\hat{h}(\alpha,\beta)\frac{\partial\varepsilon(\theta,\hat{\varphi};\alpha,\beta)}{\partial\beta}]_{\beta=\varphi_0+\delta}^{\beta=\varphi_0-\delta}\frac{d\alpha}{\sin\alpha}+$$

$$+\int_0^{2\pi}\int_0^{\pi}\hat{h}(\theta,\varphi)\Delta_S\varepsilon(\theta,\varphi;\alpha,\beta)d\alpha d\beta=$$

$$=\lim_{\delta\to 0}\sin\theta_0\int_{\varphi_0-\delta}^{\varphi_0+\delta}(\frac{\partial\hat{h}(\theta_0-\delta,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\theta_0-\delta,\beta)-\frac{\partial\hat{h}(\theta_0+\delta,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\theta_0+\delta,\beta))d\beta-$$

$$-\lim_{\delta\to 0}\sin\theta_0\int_{\varphi_0-\delta}^{\varphi_0+\delta}[\hat{h}(\theta_0-\delta,\beta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\theta_0-\delta,\beta)}{\partial\alpha}-\hat{h}(\theta_0+\delta,\beta)\frac{\partial\hat{\varepsilon}(\theta,\varphi,\theta_0+\delta,\beta)}{\partial\alpha}]d\beta+$$

$$+\lim_{\delta\to 0}\int_{\theta_0-\delta}^{\theta_0+\delta}(\frac{\partial\hat{h}(\alpha,\varphi_0-\delta)}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\alpha,\varphi_0-\delta)-\frac{\partial\hat{h}(\alpha,\varphi_0+\delta)}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\alpha,\varphi_0+\delta))\frac{d\alpha}{\sin\alpha}-$$

$$-\lim_{\delta\to 0}\int_{\theta_0-\delta}^{\theta_0+\delta}(\hat{h}(\alpha,\varphi_0-\delta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\varphi_0-\delta)}{\partial\beta}-\hat{h}(\alpha,\varphi_0+\delta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\varphi_0+\delta)}{\partial\beta})\frac{d\alpha}{sin\alpha}+$$

$$+\int_0^{2\pi}\int_0^{\pi}\hat{h}(\theta,\varphi)\Delta_S\varepsilon(\theta,\varphi;\alpha,\beta)d\alpha d\beta.$$

In the same way, the value is calculated
$\hat{T}(\theta,\varphi,\pi-\theta_0,\pi+\varphi_0)=$

$$=\lim_{\delta\to 0}\sin\theta_0\int_{\pi+\varphi_0-\delta}^{\pi+\varphi_0+\delta}(\frac{\partial\hat{h}(\pi-\theta_0-\delta,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\pi-\theta_0-\delta,\beta)-$$

$$-\frac{\partial\hat{h}(\pi-\theta_0+\delta,\beta)}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\pi-\theta_0+\delta,\beta))d\beta-$$

$$-\lim_{\delta\to 0}\sin\theta_0\int_{\pi+\varphi_0-\delta}^{\pi+\varphi_0+\delta}[\hat{h}(\pi-\theta_0-\delta,\beta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\pi-\theta_0-\delta,\beta)}{\partial\alpha}-$$

$$-\hat{h}(\pi-\theta_0+\delta,\beta)\frac{\partial\hat{\varepsilon}(\theta,\varphi,\pi-\theta_0+\delta,\beta)}{\partial\alpha}]d\beta+$$

$$+\lim_{\delta\to 0}\int_{\pi-\theta_0-\delta}^{\pi-\theta_0+\delta}(\frac{\partial\hat{h}(\alpha,\pi+\varphi_0-\delta)}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\alpha,\pi+\varphi_0-\delta)-$$

$$-\frac{\partial\hat{h}(\alpha,\pi+\varphi_0+\delta)}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\alpha,\pi+\varphi_0+\delta))\frac{d\alpha}{\sin\alpha}-$$

$$-\lim_{\delta\to 0}\int_{\pi-\theta_0-\delta}^{\pi-\theta_0+\delta}(\hat{h}(\alpha,\pi+\varphi_0-\delta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\pi+\varphi_0-\delta)}{\partial\beta}-$$

$$-\hat{h}(\alpha,\pi+\varphi_0+\delta)\frac{\partial\hat{\varepsilon}(\theta,\varphi;\alpha,\pi+\varphi_0+\delta)}{\partial\beta})\frac{d\alpha}{sin\alpha}.$$

Next, we use the fact that $\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)$, $\frac{\partial}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)$, $\frac{\partial}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)$ are continuous functions for $(\theta,\varphi)\neq(\theta_0,\varphi_0)$ and $(\theta,\varphi)\neq(\pi-\theta_0,\pi+\varphi_0)$, since the expressions (10) are correct. As a result, we obtain representation Lemma 4.1. Thus Lemma 4.1 is completely proved. □

Denote by

$$\hat{\psi}_0(\theta,\varphi) = \hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0), \quad \hat{\psi}_1(\theta,\varphi) = \frac{\partial}{\partial\alpha}\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0),$$
$$\hat{\psi}_2(\theta,\varphi) = \frac{\partial}{\partial\beta}\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)$$

at $(\theta,\varphi) \in S_0^2$.

Lemma 4.1 implies the following corollary.

Corollary 4.3. *Equalities hold for $(\theta,\varphi) \in S_0^2$:*

$$U_j(\hat{\psi}_k) = \delta_{jk}, j,k = 0,1,2$$

where δ_{jk} is the Kronecker symbol.

In fact, from the definition of Green's functions $\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0)$ follows the equality

$$-\hat{\Delta}'_{\theta,\varphi}\hat{\varepsilon}(\theta,\varphi;\theta_0,\varphi_0) = 0,$$

under $(\theta,\varphi) \neq (\theta_0,\varphi_0)$ and $(\theta,\varphi) \neq (\pi-\theta_0,\pi+\varphi_0)$. If in Lemma 4.1 $\hat{h}(\theta,\varphi)$ are chosen equal to $\hat{\psi}_j(\theta,\varphi)$ for $j = \overline{0,2}$, then Corollary 4.3 follows from Lemma 4.1.

The space $W^2_{2,U}(S_0^2)$ introduced above is wider than the domain $D(B_0)$ of the operator B_0. Let us define the maximal operator B_{max} on the set of functions $W^2_{2,U}(S_0^2)$ by the formula

$$B_{max}\hat{h}(\theta,\varphi) = B_0(\hat{h}(\theta,\varphi) + \sum_{j=0}^{2}\hat{\psi}_j(\theta,\varphi)U_j(\hat{h}))$$

for $(\theta,\varphi) \neq (\theta_0,\varphi_0)$ and $(\theta,\varphi) \neq (\pi-\theta_0,\pi+\varphi_0)$. The operator B_{max} is called the maximal operator, since the nonhomogeneous operator equation $B_{max}\hat{h}(\theta,\varphi) = \hat{F}(\theta,\varphi), (\theta,\varphi) \neq (\theta_0,\varphi_0), (\theta,\varphi) \neq (\pi-\theta_0,\pi+\varphi_0)$ has a solution for any right side of $L_2(S^2)$. However, an inhomogeneous operator equation can have several solutions.

In the next subsection, the domain of definition of the maximal operator is narrowed so that the indicated inhomogeneous equation on the narrowed domain has a unique solution. Such operators are called invertible restrictions of the maximal operator.

5. Reversible restrictions of a maximal operator

To describe the reversible restrictions of the maximal operator B_{max}, we follow the methods of Otelbaev's papers [3]-[5]. According to Otelbaev [3]-[5], for an arbitrary function $h(x)$ from the space $W^2_{2,U}(S_0^2)$ we consider the solution of the operator equation $B_0 g(x) = B_{max}h(x)$. Using Green's function $\varepsilon(x,t)$ from Lemma 3.1 and Lemma 3.2, this solution can be written as:

$$g(x) = \lim_{\delta\to 0}\int_{S^2\setminus\Pi^0(\delta)} \varepsilon(x,t)B_{max}h(t)dt.$$

According to Lemma 4.1, the solution $g(x)$ in the spherical coordinate system takes the form

$$\hat{g}(\theta,\varphi) = \hat{h}(\theta,\varphi) + \hat{\psi}_0(\theta,\varphi)U_0(\hat{h}) + \hat{\psi}_1(\theta,\varphi)U_1(\hat{h}) + \hat{\psi}_2(\theta,\varphi)U_2(\hat{h}).$$

The last equality implies a complete description of the domain of definition of the maximal operator $D(B_{max}) = W^2_{2,U}(S_0^2)$.

Lemma 5.1. *Any element $h(x)$ from the space $W^2_{2,U}(S^2_0)$ has the following representation*

$$h(x) = g(x) - \sum_{j=0}^{2} \psi_j(x)\gamma_j,$$

where a smooth function $g(x) \in W^2_2(S^2)$, $\gamma_0, \gamma_1, \gamma_2$ are some complex numbers.

Thus, an arbitrary element from $D(B_{max})$ is given by the smooth part $g(x)$ and three numbers. Moreover, the numbers $\gamma_0, \gamma_1, \gamma_2$ and the function $g(x)$ are independent of each other. When the domain of definition of the reversible restriction is chosen, then the domain of definition of the maximal operator is narrowed. The domain of definition of $D(B_{max})$ is narrowed due to the fact that the numbers $\gamma_0, \gamma_1, \gamma_2$ and the smooth function $g(x)$ depend on the function $F(x)$ which runs through all $L_2(S^2)$.

Now we formulate a statement about the invertible restriction of the maximal operator B_{max}.

Theorem 5.2. *Let $\gamma_0, \gamma_1, \gamma_2$ be linear functionals defined on over the entire space $L_2(S^2)$. Then the restriction of the operator B_{max} on the set*

$$D_{\gamma_0,\gamma_1,\gamma_2} = \{h(x) = B_0^{-1}F(x) - \gamma_0(F)\psi_0(x) - \gamma_1(F)\psi_1(x) - \gamma_2(F)\psi_2(x), \forall F \in L_2(S^2)\}$$

represents an invertible operator on the entire space $L_2(S^2)$. If $\gamma_0, \gamma_1, \gamma_2$ are linear bounded functionals on $L_2(S^2)$, then the corresponding restriction of the maximal operator to $D_{\gamma_0,\gamma_1,\gamma_2}$ has an inverse operator, which is limited in the following sense:

$$\{\gamma_0(F), \gamma_1(F), \gamma_2(F), B_0^{-1}F(x)\} \to 0, F \xrightarrow[L_2(S^2)]{} 0.$$

Let $B_\gamma = B_{\gamma_0,\gamma_1,\gamma_2}$ denote the reversible restriction from Theorem 5.2. It is clear that the following boundary value problems

$$\begin{gathered} B_{max}h(x) = F(x), x \in S^2_0, \\ U_0(h) = \gamma_0(B_{max}h), U_1(h) = \gamma_1(B_{max}h), U_2(h) = \gamma_2(B_{max}h), \end{gathered} \tag{12}$$

correspond to the restriction B_γ. The solution to this problem is unique in $W^2_{2,U}(S^2_0)$ and has the representation $h = (B_\gamma)^{-1}F$, where

$$(B_\gamma)^{-1}F = B_0^{-1}F(x) - \gamma_0(F)\psi_0(x) - \gamma_1(F)\psi_1(x) - \gamma_2(F)\psi_2(x).$$

One useful observation is given in the following statement.

Theorem 5.3. *The resolvent of the restriction B_γ has the following representation*

$$\begin{gathered} (B_\gamma - \lambda I)^{-1}F = (B_0 - \lambda I)^{-1}F(x) - \\ -\gamma_0(B_0(B_0 - \lambda I)^{-1}F)B_\gamma(B_\gamma - \lambda I)^{-1}\psi_0(x) - \\ -\gamma_1(B_0(B_0 - \lambda I)^{-1}F)B_\gamma(B_\gamma - \lambda I)^{-1}\psi_1(x) - \\ -\gamma_2(B_0(B_0 - \lambda I)^{-1}F)B_\gamma(B_\gamma - \lambda I)^{-1}\psi_2(x). \end{gathered}$$

Theorem 5.3 is proved in the same way as Theorem 2.3 was proved in [15]. Now we calculate the resolution of the restriction B_γ. To do this, we introduce the notations

$$D(\lambda) = \begin{vmatrix} 1 + \lambda\gamma_0(B_0(B_0 - \lambda I)^{-1}\psi_0) & \lambda\gamma_1(B_0(B_0 - \lambda I)^{-1}\psi_0) \\ \lambda\gamma_0(B_0(B_0 - \lambda I)^{-1}\psi_1) & 1 + \lambda\gamma_1(B_0(B_0 - \lambda I)^{-1}\psi_1) \\ \lambda\gamma_0(B_0(B_0 - \lambda I)^{-1}\psi_2) & \lambda\gamma_1(B_0(B_0 - \lambda I)^{-1}\psi_2) \end{vmatrix},$$

$$\begin{vmatrix} \lambda\gamma_2(B_0(B_0-\lambda I)^{-1}\psi_0) \\ \lambda\gamma_2(B_0(B_0-\lambda I)^{-1}\psi_1) \\ 1+\lambda\gamma_2(B_0(B_0-\lambda I)^{-1}\psi_2) \end{vmatrix}.$$

$$H(x;\lambda;F) = \begin{vmatrix} \gamma_0(B_0(B_0-\lambda I)^{-1}F) & \gamma_1(B_0(B_0-\lambda I)^{-1}F) \\ 1+\lambda\gamma_0(B_0(B_0-\lambda I)^{-1}\psi_0) & \lambda\gamma_1(B_0(B_0-\lambda I)^{-1}\psi_0) \\ \lambda\gamma_0(B_0(B_0-\lambda I)^{-1}\psi_1) & 1+\lambda\gamma_1(B_0(B_0-\lambda I)^{-1}\psi_1) \\ \lambda\gamma_0(B_0(B_0-\lambda I)^{-1}\psi_2) & \lambda\gamma_1(B_0(B_0-\lambda I)^{-1}\psi_2) \end{vmatrix}$$

$$\begin{vmatrix} \gamma_2(B_0(B_0-\lambda I)^{-1}F) & 0 \\ \lambda\gamma_2(B_0(B_0-\lambda I)^{-1}\psi_0) & B_0(B_0-\lambda I)^{-1}\psi_0 \\ \lambda\gamma_2(B_0(B_0-\lambda I)^{-1}\psi_1) & B_0(B_0-\lambda I)^{-1}\psi_1 \\ 1+\lambda\gamma_2(B_0(B_0-\lambda I)^{-1}\psi_2) & B_0(B_0-\lambda I)^{-1}\psi_2 \end{vmatrix}.$$

Theorem 5.4. *Let λ be chosen so that*

(1) $1 + l(l+1) - \lambda \neq 0, \quad l \geq 0$

(2) $D(\lambda) \neq 0$.

Then there is a resolution $(B_\gamma - \lambda I)^{-1}$, which is defined by the formula:

$$(B_\gamma - \lambda I)^{-1}F = (B_0 - \lambda I)^{-1}F(x) - \frac{H(x;\lambda;F)}{D(\lambda)}. \tag{13}$$

Theorem 5.4 is proved in the same way as Theorem 5.3 was proved in [12]. Representation (13) implies that the resolvent is a meromorphic function of λ.

In conclusion of this subsection, we give an example for calculating the resolvent. Let $\gamma_1 = \gamma_2 \equiv 0, \gamma_0(F) = \int_0^{2\pi}\int_0^{\pi} \xi(\theta,\varphi)F(\theta,\varphi)\sin\theta d\theta d\varphi$ at $\xi(\theta,\varphi)$ arbitrary twice continuously differentiable function on S^2. Denote by $\eta(\theta,\varphi) = (I - \hat{\Delta}'_{\theta,\varphi})\xi(\theta,\varphi)$. In this case, the boundary value problem (12) takes the form

$$\begin{aligned} (I - \hat{\Delta}'_{\theta,\varphi})\hat{h}(\theta,\varphi) = F(\theta,\varphi), \\ (\theta,\varphi) \neq (\theta_0,\varphi_0), \quad (\theta,\varphi) \neq (\pi-\theta_0, \pi+\varphi_0), \\ U_0(\hat{h}) = \int_0^{2\pi}\int_0^{\pi} \eta(\theta,\varphi)\hat{g}(\theta,\varphi)\sin\theta d\theta d\varphi, U_1(\hat{h}) = 0, U_2(\hat{h}) = 0, \end{aligned} \tag{14}$$

where $\hat{g}(\theta,\varphi) = (B_0^{-1}F)(\theta,\varphi)$ is a regularization of the element $\hat{h}(\theta,\varphi)$ of $W^2_{2,U}(S_0^2)$. The resolvent of the operator $B_{\alpha_0 00}$ corresponding to the boundary value problem (14) takes the form

$$(B_{\gamma_0 00} - \lambda I)^{-1}F(x) = (B_0 - \lambda I)^{-1}F(x) -$$

$$-\frac{\gamma_0(B_0(B_0-\lambda I)^{-1}F)\cdot(\psi_0(x) + \lambda(B_0-\lambda I)^{-1}\psi_0(x))}{1 + \lambda\gamma_0(\psi_0) + \lambda^2\gamma_0((B_0-\lambda I)^{-1}\psi_0(x))}.$$

For $\lambda = 0$ we obtain a formula for solving the boundary value problem (14)

$$h(x) = B_0^{-1}F(x) - \gamma_0(F)\psi_0(x), x \in S_0^2.$$

6. Conclusion

In this paper, delta-like perturbations of the Laplace-Beltrami operator on a Riemannian manifold without boundary are studied. In this article, the two-dimensional sphere of $\mathbb{R}^3$ as an Riemannian manifold is considered. Similar constructions can be done for the Laplace-Beltrami operator on smooth manifolds without boundary. Another possibilities are connected with the description of correct restrictions of linear differential operators on smooth manifolds with boundary. Finally, the problem of describing well-posed problems for elliptic differential operators of derivative orders on manifolds with boundary consisting of components of different dimensions is of interest.

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