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# On Quasi Hemi-Slant Submersions

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**Abstract**
The paper deals with the notion of quasi hemi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds. These submersions are generalization of hemi-slant submersions and semi-slant submersions. In this paper, we also study the geometry of leaves of distributions which are involved in the definition of the submersion. Further, we obtain the conditions for such distributions to be integrable and totally geodesic. Moreover, we also give the characterization theorems for proper quasi hemi-slant submersions and provide some examples of it.

**Keywords:** Hemi-slant submersions, Lorentzian para Sasakian manifolds, Quasi hemi-slant submersions, Slant submersions.
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## 1. Introduction

In differential geometry the theory of Riemannian submersions was firstly defined and studied by O'Neill [1] and Gray [2], in 1966 and 1967, respectively. In 1976, Watson [3] studied almost complex type of Riemannian submersions and introduced almost Hermitian submersions between almost Hermitian manifolds. Latar on, Chinea [4] extended the idea of almost Hermitian submersion to different sub-classes of almost contact manifolds. There are so many important and interesting results about Riemannian and almost Hermitian submersion which are studied in ( [5]- [7]). As a natural generalization of holomorphic submersions and totally real submersions, B. Sahin introduced the notion of slant submersions [8], semi-invariant submersions [9] and hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds in 2011, 2013 and 2015 respectively. There are many research articles on Riemannian submersions between Riemannian manifolds equipped with different structures have been published by several geometers ( [10]- [27]).

Magid and Falcitelli et. al. stablished the theory of Lorentzian submersions in [28] and [29], respectively. In 1989, Matsumoto [30] introduced the notion of Lorentzian para Sasakian manifolds. Later, Mihai and Rosca studied the same notion independently in [31]. Recently, Gunduzalp and Sahin studied paracontact and Lorentzian almost paracontact structures in [32] and [33]. Kumar et. al. in [34] defined and studied conformal semi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds. As a natural generalization of hemi-slant submersions, semi-slant submersions and bi-slant submersions, Prasad, Shukla and Kumar in [35] introduced the notion of quasi bi-slant submersions from Kaehler manifold onto a Riemannian manifold.

Beside the introduction this paper contains three sections. In the second section, we present some basic informations related to quasi hemi-slant Riemannian submersion needed throughout this paper. In the third section, we obtain some results on quasi hemi-slant Riemannian submersions from Lorentzian para Sasakian manifold onto Riemannian manifold. We also study the geometry of leaves of distribution involved in above submersion. Finally, we obtain certain conditions for such submersions to be totally geodesic. In the last section, we provide some examples for such submersions.

## 2. Preliminaries

In this section, we recall main definitions and properties of Lorentzian para Sasakian manifolds.

An $(2n+1)$-dimensional differentiable manifold $M_1$ which admits a $(1,1)$ tensor field $\phi$, a contravariant vector field $\xi$, a 1-form $\eta$ is called Lorentzian para Sasakian manifold with Lorentzian metric $g_{M_1}$ ( [31], [36]) which satisfy:

$$\begin{aligned}
\phi^2 &= I+\eta\otimes\xi, \quad \phi\circ\xi=0, \quad \eta\circ\phi=0, && (2.1)\\
\eta(\xi) &= -1, \quad g_{M_1}(Z_1,\xi)=\eta(Z_1), && (2.2)\\
g_{M_1}(\phi Z_1,\phi Z_2) &= g_{M_1}(Z_1,Z_2)+\eta(Z_1)\eta(Z_2), \quad g_{M_1}(\phi Z_1,Z_2)=g_{M_1}(Z_1,\phi Z_2), && (2.3)\\
\nabla_{Z_1}\xi &= \phi Z_1, && (2.4)\\
(\nabla_{Z_1}\phi)Z_2 &= g_{M_1}(Z_1,Z_2)\xi+\eta(Z_2)X+2\eta(Z_1)\eta(Z_2)\xi, && (2.5)
\end{aligned}$$

where $\nabla$ represents the operator of covariant differentiation with respect to the Lorentzian metric $g_{M_1}$ and $Z_1,Z_2$ vector fields on $M_1$.

In a Lorentzian para Sasakian manifold, it is clear that

$$rank(\phi)=2n. \tag{2.6}$$

Now, if we put

$$\Phi(Z_1,Z_2)=\Phi(Z_2,Z_1)=g_{M_1}(Z_1,\phi Z_2)=g_{M_1}(\phi Z_1,Z_2) \tag{2.7}$$

then the tensor field $\Phi$ is symmetric $(0,2)$ tensor field, for any vector fields $Z_1$ and $Z_2$ on $M_1$.

**Example 2.1** ( [36]). *Let $R^{2k+1}=\{(x^1,x^2,\dots,x^k,y^1,y^2,\dots,y^k,z): x^i,y^i,z\in R, \quad i=1,2,\dots,k\}$. Consider $R^{2k+1}$ with the following structure:*

$$\phi\left(\sum_{i=1}^{k}\left(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right)+Z\frac{\partial}{\partial z}\right)=-\sum_{i=1}^{k}Y_i\frac{\partial}{\partial x_i}-\sum_{i=1}^{k}X_i\frac{\partial}{\partial y_i}+\sum_{i=1}^{k}Y_iy^i\frac{\partial}{\partial z},$$

$$g_{R^{2k+1}}=-(\eta\otimes\eta)+\frac{1}{4}\sum_{i=1}^{k}\left(dx^i\otimes dx^i+dy^i\otimes dy^i\right), \quad \eta=-\frac{1}{2}\left(dz-\sum_{i=1}^{k}y^idx^i\right), \quad \xi=2\frac{\partial}{\partial z}.$$

*Then, $(R^{2k+1},\phi,\xi,\eta,g_{R^{2k+1}})$ is a Lorentzian para-Sasakian manifold. The vector fields $E_i=2\frac{\partial}{\partial y^i}, E_{k+i}=2\left(\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial z}\right)$ and $\xi$ form a $\phi$-basis for the contact metric structure.*

Let $\Pi:(M_1,g_{M_1})\to(M_2,g_{M_2})$ be Riemannian submersions between Riemannian manifolds [7]. Define O'Neill's tensors $\mathscr{T}$ and $\mathscr{A}$ [1] by

$$\begin{aligned}
\mathscr{A}_EL &= \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}L+\mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}L, && (2.8)\\
\mathscr{T}_EL &= \mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}L+\mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}L, && (2.9)
\end{aligned}$$

for any vector fields $E,L$ on $M_1$, where $\nabla$ is the Levi-Civita connection of $g_{M_1}$. It is easy to see that $\mathscr{T}_E$ and $\mathscr{A}_E$ are skew-symmetric operators on the tangent bundle of $M_1$ reversing the vertical and the horizontal distributions.

From equations (2.8) and (2.9), we have

$$\begin{aligned}
\nabla_{Y_1}Y_2 &= \mathscr{T}_{Y_1}Y_2+\mathscr{V}\nabla_{Y_1}Y_2, && (2.10)\\
\nabla_{Y_1}Z_1 &= \mathscr{T}_{Y_1}Z_1+\mathscr{H}\nabla_{Y_1}Z_1, && (2.11)\\
\nabla_{Z_1}Y_1 &= \mathscr{A}_{Z_1}Y_1+\mathscr{V}\nabla_{Z_1}Y_1, && (2.12)\\
\nabla_{Z_1}Z_2 &= \mathscr{H}\nabla_{Z_1}Z_2+\mathscr{A}_{Z_1}Z_2 && (2.13)
\end{aligned}$$

for $Y_1,Y_2\in\Gamma(\ker\Pi_*)$ and $Z_1,Z_2\in\Gamma(\ker\Pi_*)^\perp$, where $\mathscr{H}\nabla_{Y_1}Z_1=\mathscr{A}_{Z_1}Y_1$, if $Z_1$ is basic. It is not difficult to observe that $\mathscr{T}$ acts on the fibers as the second fundamental form, while $\mathscr{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

Since $\mathscr{T}_{Z_1}$ is skew-symmetric, we observe that $\Pi$ has totally geodesic fibres if and only if $\mathscr{T} \equiv 0$.

Let $(M_1, \phi, \xi, \eta, g_{M_1})$ be a Lorentzian para Sasakian manifold and $(M_2, g_{M_2})$ be a Riemannian manifold and $\Pi : M_1 \to M_2$ is smooth map. Then the second fundamental form of $\Pi$ is given by

$$(\nabla \Pi_*)(U_1, U_2) = \nabla^{\Pi}_{U_1} \Pi_* U_2 - \Pi_*(\nabla_{U_1} U_2) \quad \text{for } U_1, U_2 \in \Gamma(T_p M_1), \tag{2.14}$$

where we denote conveniently by $\nabla$ the Levi-Civita connections of the matrices $g_{M_1}$ and $g_{M_2}$ and $\nabla^{\Pi}$ is the pullback connection.

We recall that a differentiable map $\Pi$ between two Riemannian manifolds is totally geodesic if

$$(\nabla \Pi_*)(U_1, U_2) = 0 \text{ for all } U_1, U_2 \in \Gamma(TM_1). \tag{2.15}$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Now, we can easily prove the following lemma as in [12].

**Lemma 2.2.** *Let $\Pi$ be a Riemannian submersion from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto Riemannian manifold $(M_2, g_{M_2})$, then we have*

*(i)* $(\nabla \Pi_*)(W_1, W_2) = 0,$

*(ii)* $(\nabla \Pi_*)(Z_1, Z_2) = -\Pi_*(\mathscr{T}_{Z_1} Z_2) = -\Pi_*(\nabla_{Z_1} Z_2),$

*(iii)* $(\nabla \Pi_*)(W_1, Z_1) = -\Pi_*(\nabla_{W_1} Z_1) = -\Pi_*(\mathscr{A}_{W_1} Z_1),$

*where $W_1, W_2$ are horizontal vector fields and $Z_1, Z_2$ are vertical vector fields.*

## 3. Quasi Hemi-Slant Submersions

In this section, quasi hemi-slant submersions $\Pi$ from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold $(M_2, g_{M_2})$ is defined and studied.

**Definition 3.1** ( [37]). *Let $(M_1, \phi, \xi, \eta, g_{M_1})$ be a Lorentzian para Sasakian manifold and $(M_2, g_{M_2})$ a Riemannian manifold. A Riemannian submersion $\Pi : (M_1, \phi, \xi, \eta, g_{M_1}) \to (M_2, g_{M_2})$ is called a quasi hemi-slant submersion if there exist four mutually orthogonal distribution $D, D^{\theta}, D^{\perp}$ and $< \xi >$ such that*

*(i)* $\ker \Pi_* = D \oplus_{orth} D^{\theta} \oplus_{orth} D^{\perp} \oplus_{orth} < \xi >,$

*(ii)* $\phi(D) = D$ *i.e., $D$ is invariant,*

*(iii) for any non-zero vector field $Z_1 \in (D^{\theta})_p$, $p \in M_1$, the angle $\theta$ between $\phi Z_1$ and $(D^{\theta})_p$ is constant and independent of the choice of point $p$ and $Z_1$ in $(D^{\theta})_p$.*

The angle $\theta$ is called slant angle of the submersion, where $D, D^{\theta}$ and $D^{\perp}$ are space like subspaces.

Let $\Pi$ be quasi hemi-slant submersion from an almost contact metric manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold $(M_2, g_{M_2})$. Then, we have

$$TM_1 = \ker \Pi_* \oplus (\ker \Pi_*)^{\perp}. \tag{3.1}$$

Now, for any vector field $V_1 \in \Gamma(\ker \Pi_*)$, we put

$$V_1 = PV_1 + QV_1 + RV_1 - \eta(V_1)\xi, \tag{3.2}$$

where $P, Q$ and $R$ are projection morphisms of $\ker \Pi_*$ onto $D, D^{\theta}$ and $D^{\perp}$, respectively.

For $Y_1 \in \Gamma(\ker \Pi_*)$, we set

$$\phi Y_1 = \psi Y_1 + \omega Y_1, \tag{3.3}$$

where $\psi Y_1 \in \Gamma(\ker \Pi_*)$ and $\omega Y_1 \in \Gamma(\omega D^{\theta} \oplus \omega D^{\perp})$.

Using equations (3.2) and (3.3), we have

$$\begin{aligned} \phi V_1 &= \phi(PV_1) + \phi(QV_1) + \phi(RV_1), \\ &= \psi(PV_1) + \omega(PV_1) + \psi(QV_1) + \omega(QV_1) + \psi(RV_1) + \omega(RV_1). \end{aligned}$$

Since $\phi(D) = D$ and $\phi(D^{\perp}) \subset (\ker \Pi_*)^{\perp}$, we get $\omega(PV_1) = 0$ and $\psi(RV_1) = 0$.

Hence above equation reduces to

$$\phi V_1 = \psi(PV_1) + \psi QV_1 + \omega QV_1 + \omega RV_1. \tag{3.4}$$

Thus we have the following decomposition

$$\phi(\ker \Pi_*) = D \oplus \psi D^{\theta} \oplus (\omega D^{\theta} \oplus \omega D^{\perp}), \tag{3.5}$$

where $\oplus$ denotes orthogonal direct sum. Since $\omega D^{\theta} \subseteq (\ker \Pi_*)^{\perp}$, $\omega D^{\perp} \subseteq (\ker \Pi_*)^{\perp}$. So, we can write

$$(\ker \Pi_*)^{\perp} = \omega D^{\theta} \oplus \omega D^{\perp} \oplus \mu,$$

where $\mu$ is orthogonal complement of $(\omega D^{\theta} \oplus \omega D^{\perp})$ in $(\ker \Pi_*)^{\perp}$.

Also for any non-zero vector field $W_1 \in \Gamma(\ker \Pi_*)^{\perp}$, we have

$$\phi W_1 = BW_1 + CW_1, \tag{3.6}$$

where $BW_1 \in \Gamma(\ker \Pi_*)$ and $CW_1 \in \Gamma(\mu)$.
Span$\{\xi\} = \langle \xi \rangle$ defines time like vector field distribution. If $Z_1$ is a space-like vector field and is orthogonal to $\xi$, then

$$g_{M_1}(\phi Z_1, \phi Z_2) = g_{M_1}(Z_1, Z_2) > 0,$$

so $\phi Z_1$ is also space like. Also $\psi Z_1$ is space-like.

For space-like vector fields the Cauchy-Schwartz inequality, $g_{M_1}(Z_1, Z_2) \leq \mid Z_1 \mid\mid Z_2 \mid$ is verified.

Therefore the Wirtinger angle $\theta$ is given by

$$cos\theta = \frac{g_{M_1}(\phi Z_1, \psi Z_2)}{\mid \phi Z_1 \mid\mid \psi Z_2 \mid}.$$

$g_{M_1}\big|_{\ker \mathrm{F}_*}$ is non degenerate metric of index 1 at any point of $M_1$. So $(\ker \Pi_*)_q$ is time like subspace of $T_qM_1$ at any point of $M_1$,

so $(\ker \Pi_*)_q^{\perp}$ is space like subspace of $T_qM_1$ at any point $q \in M_1$.

We will denote a quasi hemi-slant submersion from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold $(M_2, g_{M_2})$ by $\Pi$.

**Lemma 3.2.** *If $\Pi$ be a quasi hemi-slant submersion then we have*

$$\psi^2 U_1 + B\omega U_1 = U_1 + \eta(U_1)\xi, \quad \omega\psi U_1 + C\omega U_1 = 0, \quad \omega BX_1 + C^2 X_1 = X_1, \quad \psi BX_1 + BCX_1 = 0,$$

*for all $U_1 \in \Gamma(\ker \Pi_*)$ and $X_1 \in \Gamma(\ker \Pi_*)^{\perp}$.*

*Proof.* Using equations (2.1), (3.3) and (3.5), we have Lemma 3.2. □

**Lemma 3.3.** *If $\Pi$ be a quasi hemi-slant submersion then we have*

*(i)* $\psi^2 U_1 = (\cos^2\theta) U_1$,

*(ii)* $g_{M_1}(\psi U_1, \psi U_2) = \cos^2\theta\, g_{M_1}(U_1, U_2)$,

*(iii)* $g_{M_1}(\omega U_1, \omega U_2) = \sin^2\theta\, g_{M_1}(U_1, U_2)$,

*for all $U_1, U_2 \in \Gamma(D^{\theta})$.*

*Proof.* (i) Let $\Pi$ be a quasi hemi-slant submersion from a Lorentzian para Sasakian manifold $(M_1, \phi, \xi, \eta, g_{M_1})$ onto a Riemannian manifold $(M_2, g_{M_2})$ with the quasi hemi-slant angle $\theta$.

Then for a non-vanishing vector field $U_1 \in \Gamma(D^{\theta})$, we have

$$\cos\theta = \frac{\mid \psi U_1 \mid}{\mid \phi U_1 \mid}, \tag{3.7}$$

and

$$\cos\theta = \frac{g_{M_1}(U_1, \psi U_1)}{\mid U_1 \mid\mid \psi U_1 \mid}. \tag{3.8}$$

By using equations (2.1), (3.3) and (3.8) we have

$$\cos\theta = \frac{g_{M_1}(\psi U_1, \psi U_1)}{\mid \phi U_1 \mid\mid \psi U_1 \mid},$$

$$\cos\theta = \frac{g_{M_1}(U_1, \psi^2 U_1)}{|\phi U_1||\psi U_1|}. \tag{3.9}$$

From equations (3.8) and (3.9), we get $\psi^2 U_1 = (\cos^2\theta)U_1$, for $U_1 \in \Gamma(D^\theta)$.

(ii) For all $U_1, U_2 \in \Gamma(D^\theta)$, using equations (2.3), (3.3) and Lemma 3.3 (i), we have

$$\begin{aligned} g_{M_1}(\psi U_1, \psi U_2) &= g_{M_1}(\phi U_1 - \omega U_1, \psi U_2) \\ &= g_{M_1}(U_1, \psi^2 U_2) \\ &= \cos^2\theta g_{M_1}(U_1, U_2). \end{aligned}$$

(iii) Using equation (2.3), (3.3) and Lemma 3.3 (i), (ii) we have Lemma 3.3 (iii).

□

**Lemma 3.4.** *If $\Pi$ be a quasi hemi-slant submersion then we have*

$$\mathscr{V}\nabla_{Y_1}\psi Y_2 + \mathscr{T}_{Y_1}\omega Y_2 - g_{M_1}(Y_1,Y_2)\xi - 2\eta(Y_1)\eta(Y_2)\xi - \eta(Y_2)Y_1 = \psi\mathscr{V}\nabla_{Y_1}Y_2 + B\mathscr{T}_{Y_1}Y_2, \tag{3.10}$$
$$\mathscr{T}_{Y_1}\psi Y_2 + \mathscr{H}\nabla_{Y_1}\omega Y_2 = \omega\mathscr{V}\nabla_{Y_1}Y_2 + C\mathscr{T}_{Y_1}Y_2, \tag{3.11}$$
$$\mathscr{V}\nabla_{U_1}BU_2 + \mathscr{A}_{U_1}CU_2 - g_{M_1}(CU_1,U_2)\xi = \psi\mathscr{A}_{U_1}U_2 + B\mathscr{H}\nabla_{U_1}U_2, \tag{3.12}$$
$$\mathscr{A}_{U_1}BU_2 + \mathscr{H}\nabla_{U_1}CU_2 = \omega\mathscr{A}_{U_1}U_2 + C\mathscr{H}\nabla_{U_1}U_2, \tag{3.13}$$
$$\mathscr{V}\nabla_{Y_1}BU_1 + \mathscr{T}_{Y_1}CU_1 = \psi\mathscr{T}_{Y_1}U_1 + B\mathscr{H}\nabla_{Y_1}U_1, \tag{3.14}$$
$$\mathscr{T}_{Y_1}BU_1 + \mathscr{H}\nabla_{Y_1}CU_1 = \omega\mathscr{T}_{Y_1}U_1 + C\mathscr{H}\nabla_{Y_1}U_1, \tag{3.15}$$
$$\mathscr{V}\nabla_{U_1}\psi Y_1 + \mathscr{A}_{U_1}\omega Y_1 = B\mathscr{A}_{U_1}Y_1 + \psi\mathscr{V}\nabla_{U_1}Y_1, \tag{3.16}$$
$$\mathscr{A}_{U_1}\psi Y_1 + \mathscr{H}\nabla_{U_1}\omega Y_1 - \eta(Y_1)U_1 = C\mathscr{A}_{U_1}Y_1 + \omega\mathscr{V}\nabla_{U_1}Y_1, \tag{3.17}$$

*for any $Y_1, Y_2 \in \Gamma(\ker\Pi_*)$ and $U_1, U_2 \in \Gamma(\ker\Pi_*)^\perp$.*

*Proof.* Using equations (2.5), (2.10)-(2.13), (3.3) and (3.5), we get equations (3.10)-(3.17). □

Now, we define

$$(\nabla_{Y_1}\psi)Y_2 = \mathscr{V}\nabla_{Y_1}\psi Y_2 - \psi\mathscr{V}\nabla_{Y_1}Y_2, \tag{3.18}$$
$$(\nabla_{Y_1}\omega)Y_2 = \mathscr{H}\nabla_{Y_1}\omega Y_2 - \omega\mathscr{V}\nabla_{Y_1}Y_2, \tag{3.19}$$
$$(\nabla_{X_1}C)X_2 = \mathscr{H}\nabla_{X_1}CX_2 - C\mathscr{H}\nabla_{X_1}X_2, \tag{3.20}$$
$$(\nabla_{X_1}B)X_2 = \mathscr{V}\nabla_{X_1}BX_2 - B\mathscr{H}\nabla_{X_1}X_2 \tag{3.21}$$

for any $Y_1, Y_2 \in \Gamma(\ker\Pi_*)$ and $X_1, X_2 \in \Gamma(\ker\Pi_*)^\perp$.

**Lemma 3.5.** *If $\Pi$ be a quasi hemi-slant submersion then we have*

$$(\nabla_{Y_1}\phi)Y_2 = B\mathscr{T}_{Y_1}Y_2 - \mathscr{T}_{Y_1}\omega Y_2 + g_{M_1}(Y_1,Y_2)\xi + 2\eta(Y_1)\eta(Y_2)\xi + \eta(Y_2)Y_1,$$
$$(\nabla_{Y_1}\omega)Y_2 = C\mathscr{T}_{Y_1}Y_2 - \mathscr{T}_{Y_1}\psi Y_2,$$
$$(\nabla_{U_1}C)U_2 = \omega\mathscr{A}_{U_1}U_2 - \mathscr{A}_{U_1}BU_2,$$
$$(\nabla_{U_1}B)U_2 = \psi\mathscr{A}_{U_1}U_2 - \mathscr{A}_{U_1}CU_2 + g_{M_1}(U_1,U_2)\xi,$$

*for any vectors $Y_1, Y_2 \in \Gamma(\ker\Pi_*)$ and $U_1, U_2 \in \Gamma(\ker\Pi_*)^\perp$.*

*Proof.* Using equations (3.10), (3.11), (3.12), (3.13) and (3.18)-(3.21), we get all equations of Lemma 3.5. □

If the tensors $\phi$ and $\omega$ are parallel with respect to the linear connection $\nabla$ on $M_1$ respectively, then

$$B\mathscr{T}_{Y_1}Y_2 = \mathscr{T}_{Y_1}\omega Y_2 - g_{M_1}(Y_1,Y_2)\xi - 2\eta(Y_1)\eta(Y_2)\xi - \eta(Y_2)Y_1, C\mathscr{T}_{Y_1}Y_2 = \mathscr{T}_{Y_1}\psi Y_2$$

for any $Y_1,Y_2 \in \Gamma(TM_1)$.

**Theorem 3.6.** *Let $\Pi$ be a quasi hemi-slant submersion. Then, the invariant distribution $D$ is integrable if and only if*

$$g_{M_1}(\mathscr{T}_{X_1}\phi X_2 - \mathscr{T}_{X_2}\phi X_1, \omega QY_1 + \omega RY_1) = g_{M_1}(\mathscr{V}\nabla_{X_2}\phi X_1 - \mathscr{V}\nabla_{X_1}\phi X_2, \psi QY_1),$$

*for $X_1,X_2 \in \Gamma(D)$ and $Y_1 \in \Gamma(D^\theta \oplus D^\perp)$.*

*Proof.* For $X_1,X_2 \in \Gamma(D)$, and $Y_1 \in \Gamma(D^\theta \oplus D^\perp)$, using equations (2.3), (2.5), (2.10), (3.2) and (3.3), we have

$$\begin{aligned} g_{M_1}([X_1,X_2],Y_1) &= g_{M_1}(\nabla_{X_1}\phi X_2, \phi Y_1) - g_{M_1}(\nabla_{X_2}\phi X_1, \phi Y_1), \\ &= g_{M_1}(\mathscr{T}_{X_1}\phi X_2 - \mathscr{T}_{X_2}\phi X_1, \omega QY_1 + \omega RY_1) + g_{M_1}(\mathscr{V}\nabla_{X_1}\phi X_2 - \mathscr{V}\nabla_{X_2}\phi X_1, \psi QY_1), \end{aligned}$$

which completes the proof. □

**Theorem 3.7.** *Let $\Pi$ be a quasi hemi-slant submersion. Then, the slant distribution $D^\theta$ is integrable if and only if*

$$g_{M_1}(\mathscr{H}\nabla_{Z_2}\omega Z_1 - \mathscr{H}\nabla_{Z_1}\omega Z_2, \phi RX_1) = g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2 - \mathscr{T}_{Z_2}\omega Z_1, \phi PX_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2 - \mathscr{T}_{Z_2}\omega\psi Z_1, X_1)$$

*for all $Z_1,Z_2 \in \Gamma(D^\theta)$ and $X_1 \in \Gamma(D \oplus D^\perp)$.*

*Proof.* For all $Z_1,Z_2 \in \Gamma(D^\theta)$ and $X_1 \in \Gamma(D \oplus D^\perp)$, we have

$$g_{M_1}([Z_1,Z_2],X_1) = g_{M_1}(\nabla_{Z_1}Z_2,X_1) - g_{M_1}(\nabla_{Z_2}Z_1,X_1).$$

Using equations (2.3), (2.5), (3.2), (3.3) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}([Z_1,Z_2],X_1) =& g_{M_1}(\phi\nabla_{Z_1}Z_2, \phi X_1) - g_{M_1}(\phi\nabla_{Z_2}Z_1, \phi X_1) \\ =& g_{M_1}(\nabla_{Z_1}\phi Z_2, \phi X_1) - g_{M_1}(\nabla_{Z_2}\phi Z_1, \phi X_1) \\ =& g_{M_1}(\nabla_{Z_1}\psi Z_2, \phi X_1) + g_{M_1}(\nabla_{Z_1}\omega Z_2, \phi X_1) - g_{M_1}(\nabla_{Z_2}\psi Z_1, \phi X_1) - g_{M_1}(\nabla_{Z_1}\omega Z_2, \phi X_1) \\ =& \cos^2\theta g_{M_1}(\nabla_{Z_1}Z_2,X_1) - \cos^2\theta g_{M_1}(\nabla_{Z_2}Z_1,X_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2 - \mathscr{T}_{Z_2}\omega\psi Z_1, X_1) \\ &+ g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2 + \mathscr{T}_{Z_1}\omega Z_2, \phi PX_1 + \phi RX_1) - g_{M_1}(\mathscr{H}\nabla_{Z_2}\omega Z_1 + \mathscr{T}_{Z_2}\omega Z_1, \phi PX_1 + \phi RX_1). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2\theta g_{M_1}([Z_1,Z_2],X_1) =& g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2 - \mathscr{T}_{Z_2}\omega Z_1, \phi PX_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2 - \mathscr{H}\nabla_{Z_2}\omega Z_1, \phi RX_1) \\ &+ g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2 - \mathscr{T}_{Z_2}\omega\psi Z_1, X_1), \end{aligned}$$

which completes the proof. □

**Theorem 3.8.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the anti-invariant distribution $D^\perp$ is always integrable.*

*Proof.* The proof of the above theorem is exactly the same as that one for hemi-slant submersions, see Theorems 3.13 of [38]. So we omit it. □

**Proposition 3.9.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the vertical distribution* $(\ker \Pi_*)$ *does not defines a totally geodesic foliation on $M_1$.*

*Proof.* Let $Z_1 \in \Gamma(\ker \Pi_*)$ and $Z_2 \in \Gamma(\ker \Pi_*)^\perp$, using equation (2.4), we have

$$g_{M_1}(\nabla_{Z_1}\xi, Z_2) = g_{M_1}(\phi Z_1, Z_2),$$

since $g_{M_1}(\phi Z_1,Z_2) \neq 0$, so $g_{M_1}(\nabla_{Z_1}\xi,Z_2) \neq 0$. Hence, $(\ker \Pi_*)$ does not defines a totally geodesic foliation on $M_1$. □

**Theorem 3.10.** *Let $\Pi$ be a proper quasi hemi-slant submersion. Then the distribution $(\ker\Pi_*)- < \xi >$ defines a totally geodesic foliation on $M_1$ if and only if*

$$g_{M_1}(\mathscr{T}_{Z_1}PZ_2 + \cos^2\theta\,\mathscr{T}_{Z_1}QZ_2, V_1) = -g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi QZ_2, V_1) - g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, BV_1) - g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, CV_1)$$

*for all $Z_1, Z_2 \in \Gamma(\ker\Pi_*)- < \xi >$ and $V_1 \in \Gamma(\ker\Pi_*)^\perp$.*

*Proof.* For all $Z_1, Z_2 \in \Gamma(\ker\Pi_*)- < \xi >$ and $V_1 \in \Gamma(\ker\Pi_*)^\perp$, using equations (2.3), (2.5) and (3.2), we have

$$g_{M_1}(\nabla_{Z_1}Z_2, V_1) = g_{M_1}(\nabla_{Z_1}\phi PZ_2, \phi V_1) + g_{M_1}(\nabla_{Z_1}\phi QZ_2, \phi V_1) + g_{M_1}(\nabla_{Z_1}\phi RZ_2, \phi V_1).$$

Now, using equations (2.10), (2.11), (3.3), (3.5) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}(\nabla_{Z_1}Z_2, V_1) =& g_{M_1}(\mathscr{T}_{Z_1}PZ_2, V_1) + \cos^2\theta\, g_{M_1}(\mathscr{T}_{Z_1}QZ_2, V_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi QZ_2, V_1) \\ &+ g_{M_1}(\nabla_{Z_1}(\omega PZ_2 + \omega QZ_2 + \omega RZ_2), \phi V_1). \end{aligned}$$

Now, since $\omega PZ_2 + \omega QZ_2 + \omega RZ_2 = \omega Z_2$ and $\omega PZ_2 = 0$, we have

$$\begin{aligned} g_{M_1}(\nabla_{Z_1}Z_2, V_1) =& g_{M_1}(\mathscr{T}_{Z_1}PZ_2 + \cos^2\theta\,\mathscr{T}_{Z_1}QZ_2, V_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi QZ_2, V_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, BV_1) \\ &+ g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, CV_1), \end{aligned}$$

which completes the proof. □

**Theorem 3.11.** *Let $\Pi$ be a quasi hemi-slant submersion. Then, the horizontal distribution $(\ker\Pi_*)^\perp$ does not defines a totally geodesic foliation on $M_1$.*

*Proof.* Let $X_1, X_2 \in \Gamma(\ker\Pi_*)^\perp$, using equation ( 2.4), we have

$$g_{M_1}(\nabla_{X_1}X_2, \xi) = -g_{M_1}(X_2, \nabla_{X_1}\xi) = -g_{M_1}(X_2, \phi X_1),$$

since $g_{M_1}(X_2, \phi X_1) \neq 0$, so $g_{M_1}(\nabla_{X_1}X_2, \xi) \neq 0$. Hence, $(\ker\Pi_*)^\perp$ does not defines a totally geodesic foliation on $M_1$. □

**Proposition 3.12.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the distribution $D$ does not defines a totally geodesic foliation on $M_1$.*

*Proof.* For all $Y_1, Y_2 \in \Gamma(D)$, using equation (2.4), we have

$$g_{M_1}(\nabla_{Y_1}Y_2, \xi) = -g_{M_1}(Y_2, \phi Y_1),$$

since $g_{M_1}(Y_2, \phi Y_1) \neq 0$, so $g_{M_1}(\nabla_{Y_1}Y_2, \xi) \neq 0$. Hence $D$ does not defines a totally geodesic foliation on $M_1$. □

**Theorem 3.13.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the distribution $D\oplus < \xi >$ defines a totally geodesic foliation if and only if*

$$g_{M_1}(\mathscr{T}_{X_1}\phi PX_2, \omega QY_1 + \phi RY_1) = -g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2, \psi QY_1),$$

$$g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2, BY_2) = -g_{M_1}(\mathscr{T}_{X_1}\phi PX_2, CY_2),$$

*for all $X_1, X_2 \in \Gamma(D\oplus < \xi >), Y_1 = QY_1 + RY_1 \in \Gamma(D^\theta \oplus D^\perp)$ and $Y_2 \in \Gamma(\ker\Pi_*)^\perp$.*

*Proof.* For all $X_1, X_2 \in \Gamma(D\oplus < \xi >), Y_1 = QY_1 + RY_1 \in \Gamma(D^\theta \oplus D^\perp)$ and $Y_2 \in \Gamma(\ker\Pi_*)^\perp$, using equations (2.3), (2.5), (2.10), (3.2) and (3.3), we have

$$\begin{aligned} g_{M_1}(\nabla_{X_1}X_2, Y_1) =& g_{M_1}(\nabla_{X_1}\phi X_2, \phi Y_1) \\ =& g_{M_1}(\nabla_{X_1}\phi PX_2, \phi QY_1 + \phi RY_1) \\ =& g_{M_1}(\mathscr{T}_{X_1}\phi PX_2, \omega QY_1 + \phi RY_1) + g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2, \psi QY_1). \end{aligned}$$

Now, again using equations (2.3), (2.5), (2.10), (3.2) and (3.5), we have

$$\begin{aligned} g_{M_1}(\nabla_{X_1}X_2, Y_2) =& g_{M_1}(\nabla_{X_1}\phi X_2, \phi Y_2) \\ =& g_{M_1}(\nabla_{X_1}\phi PX_2, BY_2 + CY_2) \\ =& g_{M_1}(\mathscr{V}\nabla_{X_1}\phi PX_2, BY_2) + g_{M_1}(\mathscr{T}_{X_1}\phi PX_2, CY_2), \end{aligned}$$

which completes the proof. □

**Proposition 3.14.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the distribution $D^\theta$ does not defines a totally geodesic foliation on $M_1$.*

*Proof.* For all $Z_1, Z_2 \in \Gamma(D^\theta)$, using equation (2.4), we have

$$g_{M_1}(\nabla_{Z_1} Z_2, \xi) = -g_{M_1}(Z_2, \phi Z_1),$$

since $g_{M_1}(Z_2, \phi Z_1) \neq 0$, so $g_{M_1}(\nabla_{Z_1} Z_2, \xi) \neq 0$. Hence $D^\theta$ does not defines a totally geodesic foliation on $M_1$. □

**Theorem 3.15.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the distribution $D^\theta \oplus <\xi>$ defines a totally geodesic foliation on $M_1$ if and only if*

$$g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2, X_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, \phi P X_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, \phi R X_1) = \eta(Z_2) g_{M_1}(Z_1, \phi P X_1),$$
$$g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, C X_2) + g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, B X_2) = \eta(Z_2) g_{M_1}(Z_1, B X_2),$$

*for all $Z_1, Z_2 \in \Gamma(D^\theta \oplus <\xi>), X_1 \in \Gamma(D \oplus D^\perp)$ and $X_2 \in \Gamma(\ker \Pi_*)^\perp$.*

*Proof.* For all $Z_1, Z_2 \in \Gamma(D^\theta \oplus <\xi>)$, $X_1 \in \Gamma(D \oplus D^\perp)$ and $X_2 \in \Gamma(\ker \Pi_*)^\perp$, using equations (2.3), (2.5), (2.11), (3.2), (3.3) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}(\nabla_{Z_1} Z_2, X_1) =& g_{M_1}(\nabla_{Z_1} \phi Z_2, \phi X_1) - \eta(Z_2) g_{M_1}(Z_1, \phi X_1) \\ =& g_{M_1}(\nabla_{Z_1} \psi Z_2, \phi X_1) + g_{M_1}(\nabla_{Z_1} \omega Z_2, \phi X_1) - \eta(Z_2) g_{M_1}(Z_1, \phi P X_1) \\ =& \cos^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2, X_1) + g_{M_1}(\mathscr{T}_Z \omega Z_2, \phi P X_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, \phi R X_1) \\ & - \eta(Z_2) g_{M_1}(Z_1, \phi P X_1). \end{aligned}$$

Now, we have

$$\sin^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_1) = g_{M_1}(\mathscr{T}_{Z_1}\omega\psi Z_2, X_1) + g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, \phi P X_1) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, \phi R X_1) - \eta(Z_2) g_{M_1}(Z_1, \phi P X_1)$$

Next, from equations (2.3), (2.5), (2.11), (3.2), (3.3), (3.5) and Lemma 3.3, we have

$$\begin{aligned} g_{M_1}(\nabla_{Z_1} Z_2, X_2) =& g_{M_1}(\nabla_{Z_1} \phi Z_2, \phi X_2) - \eta(Z_2) g_{M_1}(Z_1, \phi X_2), \\ =& g_{M_1}(\nabla_{Z_1} \psi Z_2, \phi X_2) + g_{M_1}(\nabla_{Z_1} \omega Z_2, \phi X_2) - \eta(Z_2) g_{M_1}(Z_1, \phi X_2), \\ =& \cos^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, C X_2) + g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, B X_2) \\ & - \eta(Z_2) g_{M_1}(Z_1, B X_2). \end{aligned}$$

Now, we have

$$\sin^2\theta_1 g_{M_1}(\nabla_{Z_1} Z_2, X_2) = g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega\psi Z_2, X_2) + g_{M_1}(\mathscr{H}\nabla_{Z_1}\omega Z_2, C X_2) + g_{M_1}(\mathscr{T}_{Z_1}\omega Z_2, B X_2) - \eta(Z_2) g_{M_1}(Z_1, B X_2),$$

which completes the proof. □

**Theorem 3.16.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the distribution $D^\perp$ defines a totally geodesic foliation on $M_1$ if and only if*

$$g_{M_1}(\mathscr{T}_{X_1} X_2, \omega\psi Q Y_1) = -g_{M_1}(\mathscr{H}\nabla_{X_1}\omega R X_2, \omega Y_1),$$
$$g_{M_1}(\mathscr{T}_{X_1}\omega R X_2, B Y_2) = g_{M_1}(\nabla_{\omega R X_2} \phi C Y_2, \omega R X_1),$$

*for all $X_1, X_2 \in \Gamma(D^\perp), Y_1 \in \Gamma(D \oplus D^\theta)$, and $Y_2 \in \Gamma(\ker \pi_*)^\perp$.*

*Proof.* For all $X_1, X_2 \in \Gamma(D^\perp), Y_1 \in \Gamma(D \oplus D^\theta)$, and $Y_2 \in \Gamma(\ker \pi_*)^\perp$. Using equation (2.4), we have

$$g_{M_1}(\nabla_{X_1} X_2, \xi) = 0.$$

Next, using equations (2.3), (2.5), (3.2), (3.3) and Lemma 3.3, we have

$$g_{M_1}(\nabla_{X_1} X_2, Y_1) = g_{M_1}(\phi\nabla_{X_1} X_2, \phi P Y_1 + \psi Q Y_1) + g_{M_1}(\nabla_{X_1} \phi X_2, \omega Q Y_1),$$
$$g_{M_1}(\nabla_{X_1} X_2, P Y_1 + Q Y_1) = g_{M_1}(\nabla_{X_1} X_2, P Y_1) + \cos^2\theta g_{M_1}(\nabla_{X_1} X_2, Q Y_1) + g_{M_1}(\nabla_{X_1} X_2, \omega\psi Q Y_1) + g_{M_1}(\nabla_{X_1} \phi X_2, \omega Q Y_1).$$

Now, using equations (2.10) and (2.11), we have

$$\sin^2\theta g_{M_1}(\nabla_{X_1}X_2, QY_1) = g_{M_1}(\mathscr{T}_{X_1}X_2, \omega\psi QY_1) + g_{M_1}(\mathscr{H}\nabla_{X_1}\omega RX_2, \omega Y_1).$$

Next, using equations (2.3), (2.5), (2.11), (2.13), (3.3) and (3.5), we have

$$\begin{aligned} g_{M_1}(\nabla_{X_1}X_2, Y_2) =& g_{M_1}(\nabla_{X_1}\omega RX_2, BY_2) + g_{M_1}(\nabla_{X_1}\omega RX_2, CY_2),\\ =& g_{M_1}(\mathscr{T}_{X_1}\omega RX_2, BY_2) - g_{M_1}(\mathscr{H}\nabla_{\omega RX_2}\phi CY_2, \omega RX_1), \end{aligned}$$

which is complete proof. □

Using Proposition 3.9 and Theorem 3.11, one can give the following theorem:

**Theorem 3.17.** *Let $\Pi$ be a quasi hemi-slant submersion. Then the map $\Pi$ is not a totally geodesic map.*

## 4. Examples

**Example 4.1.** *Consider the Euclidean space $R^{11}$ with coordinates $(x_1,...,x_5,,y_1.....,y_5,z)$ and base field $\{E_i,\ E_{5+i},\xi\}$ where $E_i = 2\frac{\partial}{\partial y^i}$, $E_{5+i} = 2\left(\frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}\right)$, $i = 1,\dots,5$ and contravariant vector field $\xi = 2\frac{\partial}{\partial z}$. Define Lorentzian almost para contact structure on $R^{11}$ as follows:*

$$\phi\left(\sum_{i=1}^{5}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right) = -\sum_{i=1}^{5}Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^{5}X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^{5}Y_i y^i\frac{\partial}{\partial z},$$

$$\xi = 2\frac{\partial}{\partial z},\quad \eta = -\frac{1}{2}\left(dz - \sum_{i=1}^{5}y^i dx^i\right),\quad g_{R^{11}} = -(\eta\otimes\eta) + \frac{1}{4}\left(\sum_{i=1}^{5}dx^i\otimes dx^i + \sum_{i=1}^{5}dy^i\otimes dy^i\right).$$

*Then $(R^{11},\phi,\xi,\eta,g_{R^{11}})$ is Lorentzian para Sasakian manifold. Let the Riemannian metric tensor field $g_{R^4}$ is defined by*

$$g_{R^4} = \frac{1}{4}\sum_{i=1}^{4}(dv_i\otimes dv_i).$$

*on $R^4$, where $\{v_1,v_2,v_3,v_4\}$ is local coordinate system on $R^4$.*

*Let $\Pi : R^{11}\to R^4$ be a map defined by*

$$\Pi(x_1,...,x_5,y_1....,y_5,z) = (x_2, \sin\alpha x_3 - \cos\alpha x_4, y_1, y_4).$$

*which is quasi hemi-slant submersion map such that*

$$\begin{aligned} X_1 &= 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right),\quad X_2 = 2\cos\alpha\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) + 2\sin\alpha\left(\frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z}\right),\quad X_3 = 2\left(\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial z}\right),\\ X_4 &= 2\frac{\partial}{\partial y_2},\quad X_5 = 2\frac{\partial}{\partial y_3},\quad X_6 = 2\frac{\partial}{\partial y_5},\quad X_7 = \xi = 2\frac{\partial}{\partial z}, \end{aligned}$$

$$(\ker\Pi_*) = (D\oplus D^\theta\oplus D^\perp\oplus <\xi>),$$

$$D = \left\langle X_3 = 2\left(\frac{\partial}{\partial x_5} + y_5\frac{\partial}{\partial z}\right), X_6 = 2\frac{\partial}{\partial y_5}\right\rangle,$$

$$D^\theta = \left\langle X_2 = 2\cos\alpha\left(\frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z}\right) + 2\sin\alpha\left(\frac{\partial}{\partial x_4} + y_1\frac{\partial}{\partial z}\right), X_5 = 2\frac{\partial}{\partial y_3}\right\rangle,$$

$$D^\perp = \left\langle X_1 = 2\left(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z}\right), X_4 = 2\frac{\partial}{\partial y_2}\right\rangle,\quad \langle\xi\rangle = \left\langle X_7 = 2\frac{\partial}{\partial z}\right\rangle,$$

$$(\ker\Pi_*)^\perp = \left\langle V_1 = 2\left(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z}\right), V_2 = 2\sin\alpha\left(\frac{\partial}{\partial x_3} + y_2\frac{\partial}{\partial z}\right) - 2\cos\alpha\left(\frac{\partial}{\partial x_4} + y_1\frac{\partial}{\partial z}\right), V_3 = 2\frac{\partial}{\partial y_1}, V_4 = 2\frac{\partial}{\partial y_4}\right\rangle,$$

*with quasi hemi-slant angle $\alpha$. Also by direct computations, we obtain*

$$\Pi_*V_1 = 2\frac{\partial}{\partial v_1},\quad \Pi_*V_2 = 2\frac{\partial}{\partial v_2},\quad \Pi_*V_3 = 2\frac{\partial}{\partial v_3},\quad \Pi_*V_4 = 2\frac{\partial}{\partial v_4}.$$

**Example 4.2.** *Consider $R^{11}$ and $R^4$ has same structure as in Example 4.1. Let $\Pi : R^{11} \to R^4$ be a map defined by*

$$\Pi(x_1,\ldots,x_5,y_1,\ldots,y_5,z) = \left(\frac{\sqrt{3}x_1+x_2}{2}, x_4, y_1, y_3\right).$$

*which is quasi hemi-slant submersion map such that*

$$X_1 = 2\left(\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial z}\right)-2\sqrt{3}\left(\frac{\partial}{\partial x_2}+y_2\frac{\partial}{\partial z}\right), \quad X_2 = 2\left(\frac{\partial}{\partial x_3}+y_3\frac{\partial}{\partial z}\right), \quad X_3 = 2\left(\frac{\partial}{\partial x_5}+y_5\frac{\partial}{\partial z}\right),$$

$$X_4 = 2\frac{\partial}{\partial y_2}, \quad X_5 = 2\frac{\partial}{\partial y_4}, \quad X_6 = 2\frac{\partial}{\partial y_5}, \quad X_7 = 2\frac{\partial}{\partial z},$$

$$(\ker\Pi_*) = (D\oplus D^{\theta}\oplus D^{\perp}\oplus <\xi>),$$

$$D = \left\langle X_3 = 2\left(\frac{\partial}{\partial x_5}+y_5\frac{\partial}{\partial z}\right), X_6 = 2\frac{\partial}{\partial y_5}\right\rangle,$$

$$D^{\theta} = \left\langle X_1 = 2\left(\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial z}\right)-2\sqrt{3}\left(\frac{\partial}{\partial x_2}+y_1\frac{\partial}{\partial z}\right), X_4 = 2\frac{\partial}{\partial y_2}\right\rangle,$$

$$D^{\perp} = \left\langle X_5 = 2\left(\frac{\partial}{\partial x_3}+y_3\frac{\partial}{\partial z}\right), X_2 = 2\frac{\partial}{\partial y_4}\right\rangle, \langle\xi\rangle = < X_7 = 2\frac{\partial}{\partial z}>,$$

$$(\ker\Pi_*)^{\perp} = \left\langle V_1 = 2\sqrt{3}\left(\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial z}\right)+2\left(\frac{\partial}{\partial x_2}+y_2\frac{\partial}{\partial z}\right), V_2 = 2\left(\frac{\partial}{\partial x_4}+y_4\frac{\partial}{\partial z}\right), V_3 = 2\frac{\partial}{\partial y_1}, V_4 = 2\frac{\partial}{\partial y_3}\right\rangle,$$

*with quasi hemi-slant angle $\theta = \frac{\pi}{6}$. Also by direct computations, we obtain*

$$\Pi_*V_1 = 2\frac{\partial}{\partial v_1}, \quad \Pi_*V_2 = 2\frac{\partial}{\partial v_2}, \quad \Pi_*V_3 = 2\frac{\partial}{\partial v_3}, \quad \Pi_*V_4 = 2\frac{\partial}{\partial v_4}.$$

## 5. Conclusion

In this paper, integrability conditions and conditions for defining a totally geodesic foliation by certain distributions were found. Then, by applying the notion of quasi hemi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds.

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