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Dynamics and Stability of Ξ-Hilfer Fractional Fuzzy Differential Equations with Impulses

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Abstract
This paper deals with the existence, uniqueness and Ulam-stability outcomes for Ξ-Hilfer fractional fuzzy differential equations with impulse. Further, by using the techniques of nonlinear functional analysis, we study the Ulam-Hyers-Rassias stability.

Keywords: Existence, Fuzzy impulsive differential equation, Ξ-Hilfer fractional derivative, Uniqueness, Ulam-Hyers-Rassias stability
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1. Introduction

The dynamics of processes that are subject to sudden state changes are often studied using differential equations with impulses as models. There are two commonly used types of impulses: instantaneous and non-instantaneous. The investigation of impulsive differential equation involving classical derivatives one can refer to [1]-[9]

Due to its importance in numerous related domains, including physics, mechanics, chemistry, engineering, etc., fractional calculus has received more and more attention in recent years, one can see [10]-[13] and references therein. In [12], Hilfer investigated applications for an extended fractional operator that has the Riemann-Liouville (RL) and Caputo derivatives as special cases. In this study, we deal with the existence, uniqueness, and stability of ψ-Hilfer fractional derivative based fractional differential equations, which Sousa and Oliveira initiated in [14].

Mathematicians have explored fuzzy fractional integrals and differential equations. One can see that RL, Hadamard, and Katugampola fuzzy fractional integrals are the basis for a lot of research on this area. We recommand the reader to the works [15, 16] and references listed therein for details about the basic concepts of fuzzy analysis and fuzzy differential equations. By employing the Caputo-Katugampola fuzzy fractional derivative, Sajedi et al. evaluated the existence, uniqueness, and several types of Ulam-Hyers stability of solutions of an impulsive coupled system of fractional differentia equations [17]. For more facts on fuzzy fractional differential equations and its stability concepts, see, for example, [18]-[25].

In this paper, motivated by the research going on in this direction, we study the Ξ-Hilfer fractional fuzzy differential equations with impulse of the form:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi} z(t) = p(t,z(t)), \quad t \in (s_i, t_{i+1}], \quad i \in M_0 := M \cup \{0\}, \\ z(t) = g_i(t, z(t_i^+)), \quad t \in (t_i, s_i], \quad i \in M, \\ \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = z_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where $M = \{1,2,\cdots,m\}$, $z \in \mathbb{R}$, $\alpha \in (0,1)$, $\beta \in [0,1]$, $p : [0,T] \times E_d \to E_d$ is continuous, and E_d is the space of fuzzy sets and t_i satisfy $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m < t_{m+1} = T$, $g_i : [t_i, s_i] \times E_d \to E_d$ is continuous for all $i = 1,2,\cdots,m$, which is non-instantaneous impulses. Moreover ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}$ and $\mathfrak{I}_{0^+}^{1-\gamma,\Xi}$ are the Ξ-Hilfer fractional derivative and Ξ-RL fractional integral.

2. Preliminaries

If we take $\mathscr{J} = [0,T]$. Let E_d be a family of fuzzy numbers, that is., $z : \mathbb{R} \to [0,1]$ satisfies normal, convex, upper semicontinuous and compactly supported.
The s-level set of $z \in E_d$ are defined by

$$[z]^s = \begin{cases} \{t \in \mathbb{R} | z(t) \geq s\}, \quad \text{if} \quad s \in (0,1], \\ cl\{t \in \mathbb{R} | z(t) > s\}, \quad \text{if} \quad s = 0. \end{cases}$$

So, the s-level set of $z \in E_d$ are compact intervals of the form $[z]^s = [\underline{z}(s), \overline{z}(s)] \subset \mathbb{R}$.

Definition 2.1. *[15] Two fuzzy sets z_1 and z_2 are defined on E_d and $\lambda \in \mathbb{R}$. According to Zadeh's extension principle, $z_1 + z_2$ and λz_1 are in E_d and defined as*

$$\begin{aligned} [z_1 + z_2]^s =& [z_1]^s + [z_2]^s, \\ [\lambda z_1]^s =& \lambda [z_1]^s, \quad \textit{for all} \quad s \in [0,1], \end{aligned}$$

where $[z_1]^s + [z_2]^s$ represents the usual addition of two intervals of $\mathbb{R}$ and $\lambda [z_1]^s$ represents the usual scalar product between λ and an real interval.

Definition 2.2. *[16] The distance $D_0[z_1, z_2]$ between two fuzzy numbers is defined by*

$$D_0[z_1, z_2] = \sup_{0 \leq s \leq 1} H([z_1]^s, [z_2]^s) \quad \textit{for all} \quad z_1, z_2 \in E_d, \tag{2.1}$$

where $H([z_1]^s, [z_2]^s) = max\{|\underline{z_1}(s) - \underline{z_2}(s)|, |\overline{z_1}(s) - \overline{z_2}(s)|\}$ is the Hausdroff distance between $[z_1]^s$ and $[z_2]^s$.

Definition 2.3. *[16] Let $z_1, z_2 \in E_d$. There exists $z_3 \in E_d$ such that $z_1 = z_2 + z_3$, that is., $z_3 = z_1 \ominus z_2$, where z_3 is Hukuhara difference of z_1 and z_2.*
The generalized Hukuhara difference of two fuzzy numbers $z_1, z_2 \in E_d$ [gH-difference] is defined as

$$z_1 \ominus_{gH} z_2 = z_3 \Leftrightarrow z_1 = z_2 + z_3, \quad or \quad z_2 = z_1 + (-1) z_3, \tag{2.2}$$

where $z_1 \ominus_{gH} z_2$ is called as gH-difference of z_1 and z_2 in E_d.

Definition 2.4. *[15] Let $z : [a,b] \to E_d$ be a fuzzy function, then for each $s \in [0,1]$, the function $t \mapsto d([z(t)]^s)$ is nondecreasing (nonincreasing) on $[a,b]$. In addition, z is called d-monotone on $[a,b]$, if z is d-increasing or d-decreasing on $[a,b]$.*

Definition 2.5. *[15] Let $z : (a,b] \to E_d$ and $t \in [a,b]$. If z is a fuzzy function of gH-differentiable with respect to t then there exists an element $z'_{gH}(t) \in E_d$ such that*

$$z'_{gH}(t) = \lim_{h \to 0} \frac{z(t+h) \ominus_{gH} z(t)}{h}. \tag{2.3}$$

Definition 2.6. *Let $z : \mathscr{J} \to E_d$ be a continuous fuzzy mapping. The fuzzy Ξ-type RL fractional integral of z is defined by*

$$\left({}^{RL}\mathfrak{I}_{0^+}^{\alpha,\Xi} z \right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t) - \Xi(\tau))^{\alpha-1} z(\tau) d\tau, \quad \textit{for all} \quad t \in \mathscr{J}. \tag{2.4}$$

Definition 2.7. *Let $z: \mathscr{J} \to E_d$ be a continuous fuzzy mapping. The fuzzy Ξ-type RL fractional derivative of order $n-1<\alpha<n$ for fuzzy-valued function z is defined by*

$$\left({}^{RL}\mathfrak{D}_{0^+}^{\alpha,\Xi}z\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right)^n\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{n-\alpha-1}z(\tau)d\tau, \quad \text{for all} \quad t\in\mathscr{J}. \tag{2.5}$$

Definition 2.8. *The fuzzy Ξ-Hilfer fractional derivative of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$ is defined by*

$${}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}z(t)=\mathfrak{I}_{0^+}^{\alpha(1-\beta),\Xi}\left(\frac{1}{\Xi'(t)}\frac{d}{dt}\right)\mathfrak{I}_{0^+}^{(1-\alpha)(1-\beta),\Xi}z(t). \tag{2.6}$$

for a fuzzy function $z: \mathscr{J}\to E_d$ so that the expression on the right side exists.

Lemma 2.9. *Let $\alpha\in(0,1)$, $\beta\in[0,1]$ and $z\in\mathscr{AC}(\mathscr{J},E_d)$ be a d-monotone fuzzy function, then*

$$\left(\mathfrak{I}_{0^+}^{\alpha,\Xi}{}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}z\right)(t)=z(t)\ominus_{gH}\frac{\left(\mathfrak{I}_{0^+}^{1-\gamma,\Xi}z\right)(0)}{\Gamma(\gamma)}(\Xi(t)-\Xi(0))^{\gamma-1}, \quad t\in\mathscr{J}. \tag{2.7}$$

$$\left({}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}\mathfrak{I}_{0^+}^{\alpha,\Xi}z\right)(t)=z(t), \quad t\in\mathscr{J}. \tag{2.8}$$

Theorem 2.10. *[3] Let (S,D) be a generalized complete metric space. Suppose that the operator $T: S\to S$ is strictly contractive with Lipschitz constant $L<1$. If there exists a non-negative integer k such that $D[T^{k+1},T^k]<\infty$ for some $z\in S$, then the following are true:*

(i) The sequence $\{T^k z\}_{k\geq 1}$ converges to a fixed point z^* of T;

(ii) z^* is the unique fixed point of $T\in S^*$; where $S^*=\{v\in S|D[T^k z,v]<\infty\}$.

(iii) If $v\in S^*$, then $D[v,z^*]\leq\frac{1}{1-L}D[Tv,v]$.

3. Existence Theory

In this section, we consider $\mathscr{PC}(\mathscr{J},E_d)$ the family of piecewise continuous fuzzy function, we say that $v(t)$ is continuous on $\mathscr{J}_i$, $\quad i=0,1,\cdots,m$, where $\mathscr{J}_i=(t_i,t_{i+1}]$ and $t_0=0$, $t_{m+1}=T$.

We introduce the following hypotheses:

$(H1)$ There exists function $m^*,n^*\in C(\mathscr{J},\mathbb{R}^+)$ such that

$$D_0[p(t,u(t)),\widehat{0}]\leq m^*(t)D_0[u(t),\widehat{0}]+n^*(t),$$

where $M^*=\sup_{t\in\mathscr{J}}m^*(t)$ and $N^*=\sup_{t\in\mathscr{J}}n^*(t)$.

$(H2)$ $p\in C([s_i,t_{i+1}],E_d)$ and there exists a positive constants L_p such that

$$D_0[p(t,u_1),p(t,u_2)]\leq L_p D_0[u_1,u_2], \quad t\in\mathscr{J}.$$

$(H3)$ $g_i\in C([t_i,s_i],E_d)$ and there exists a positive constants L_{g_i}

$$D_0[g_i(t,u_1),g_i(t,u_2)]\leq L_{g_i}D_0[u_1,u_2].$$

$(H4)$ There exists function $q\in C(\mathscr{J},\mathbb{R}^+)$ such that

$$D_0[g_i(t,u(t_i^+)),\widehat{0}]\leq q(t)D_0[u(t),\widehat{0}].$$

$(H5)$ Let $\varphi\in C(\mathscr{J},\mathbb{R}^+)$ be a non-decreasing function, then there exists $C_\varphi>0$ such that

$$\frac{1}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}\varphi(\tau)d\tau<C_\varphi\varphi(t) \quad \text{for each} \quad t\in\mathscr{J}.$$

Lemma 3.1. *Let $p \in C(\mathscr{J}, E_d)$ be a continuous fuzzy function. Then, a d-monotone fuzzy function $z \in \mathscr{PC}(\mathscr{J}, E_d)$ is a solution of the following problem*

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi} z(t) = p(t,z(t)), \quad t \in \mathscr{J}, \\ \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = z_0. \end{cases}$$

if and only if $z \in \mathscr{PC}(\mathscr{J}, E_d)$ satisfies the integral equation provided as follows:

$$z(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} z_0 = \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau, \quad t \in \mathscr{J}.$$

Lemma 3.2. *Let $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\gamma = \alpha + \beta(1-\alpha)$. Suppose that $p : \mathscr{J} \times E_d \to E_d$ be a continuous fuzzy function and $g_i : [t_i, s_i] \times E_d \to E_d$ is a continuous for every $i \in M$. Then a d-monotone continuous function z is a solution of the following integral equation:*

$$\begin{cases} z(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} z_0 = \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau, \quad t \in (s_i, t_{i+1}], \\ z(t) = g_i(t, z(t_i^+)), \quad t \in (t_i, s_i], \quad k \in M, \\ z(t) \ominus_{gH} z(s_i) = \frac{1}{\Gamma(\alpha)} \int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau, \quad t \in (s_i, t_{i+1}], \\ where \quad z(s_i) = g_i(s_i, z(t_i^+)) \end{cases} \tag{3.1}$$

if and only if z is a d-monotone solution of the fuzzy impulsive of Ξ-Hilfer fractional problem is

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi} z(t) = p(t,z(t)), \quad t \in (s_i, t_{i+1}], \quad i \in M_0 := M \cup \{0\}, \\ z(t) = g_i(t, z(t_i^+)), \quad t \in (t_i, s_i], \quad i \in M, \\ \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = z_0. \end{cases} \tag{3.2}$$

Proof. Suppose that z satisfies the problem (1.1), that is, z is a solution of Eqn.(1.1).
Let $t \in (0, t_1]$, then

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi} z(t) = p(t,z(t)), \quad t \in (s_i, t_{i+1}], \\ \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = z_0, \end{cases}$$

is equivalent to the equation

$$z(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} z_0 = \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau. \tag{3.3}$$

Now, it follows from Eqn.(3.2) of second equation that when $t \in (t_1, s_1]$, $z(t) = g_i(t, z(t_i^+))$. If $t \in (s_1, t_2]$ then

$$\begin{aligned} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi} z(t) =& p(t,z(t)), \quad t \in (s_1, t_2] \\ z(s_1) =& g_1(s_1, z(t_1^+)). \end{aligned} \tag{3.4}$$

Applying an operator $\mathfrak{I}_{0^+}^{1-\gamma,\Xi}$ over $(0, t_2]$ on both sides of Eqn.(3.4) , we get

$$z(t) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau, \tag{3.5}$$

which yields

$$z(s_1) \ominus_{gH} \frac{(\Xi(s_1)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = \frac{1}{\Gamma(\alpha)} \int_0^{s_1} \Xi'(\tau)(\Xi(s_1)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau.$$

From the second equation of problem (3.4), we get

$$\begin{cases} g_1(s_1, z(t_1^+)) \ominus_{gH} \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) = \frac{1}{\Gamma(\alpha)} \int_0^{s_1} \Xi'(\tau)(\Xi(s_1)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau \\ \mathfrak{I}_{0^+}^{1-\gamma,\Xi} z(0) \\ = \big(g_1(s_1, z(t_1^+)) \ominus_{gH} \frac{1}{\Gamma(\alpha)} \int_0^{s_1} \Xi'(\tau)(\Xi(s_1)-\Xi(\tau))^{\alpha-1} p(\tau,z(\tau))d\tau\big)\Gamma(\gamma)(\Xi(t)-\Xi(0))^{1-\gamma}. \end{cases} \tag{3.6}$$

Substituting Eqn.(3.6) in Eqn.(3.5), we obtain

$$\begin{cases} z(t)\ominus_{gH}\left(\frac{\Xi(t)-\Xi(0)}{\Xi(s_1)-\Xi(0)}\right)^{\gamma-1}\left(g_1(s_1,u(t_1^+))\ominus_{gH}\frac{1}{\Gamma(\alpha)}\int_0^{s_1}\Xi'(\tau)(\Xi(s_1)-\Xi(\tau))^{\alpha-1}p(\tau,z(\tau))d\tau\right) \\ =\frac{1}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(s_1)-\Xi(\tau))^{\alpha-1}p(\tau,z(\tau))d\tau, \quad t\in(s_1,t_2]. \end{cases}$$

Now, it follows from Eqn.(3.2) of second equation that when $t\in(t_2,s_2]$ with $z(s_2)=g_2(s_2,u(t_2^+))$.
Repeating the same process for $t\in(s_i,t_{i+1}]$, we obtain

$$z(t)\ominus_{gH}z(s_i)=\frac{1}{\Gamma(\alpha)}\int_{s_i}^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,z(\tau))d\tau, \quad t\in(s_i,t_{i+1}],$$

where, $z(s_i)=g_i(s_i,z(t_i^+))$.
Conversely, suppose that z satisfies the integral Eqn.(3.1). If $t\in(0,t_1]$, then $\mathfrak{I}_{0^+}^{1-\gamma,\Xi}z(0)=z_0$ and applying ${}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}$ fact that, we obtain

$$\begin{aligned} {}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}\left(z(t)\ominus_{gH}\mathfrak{I}_{0^+}^{1-\gamma,\Xi}z(0)\right) &={}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}\left(\frac{1}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,z(\tau))d\tau\right), \\ &={}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}\left(\mathfrak{I}_{0^+}^{\alpha,\Xi}p(t,z(t))\right). \\ {}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}\left(\mathfrak{I}_{0^+}^{\alpha,\Xi H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}z(t)\right) &={}^H\mathfrak{I}_{0^+}^{\alpha,\Xi H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}p(t,z(t)) \\ {}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}z(t) &=p(t,z(t)). \end{aligned}$$

And, next we can easily prove that $z(t)=g_i(t,z(t_i^+)), \quad t\in(t_i,s_i]$. □

Theorem 3.3. *Assume that $(H1)-(H3)$ hold. Then, the problem (1.1) has at least one solution.*

Proof. Define a operator $T:\mathscr{PC}(\mathscr{J},E_d)\to\mathscr{PC}(\mathscr{J},E_d)$ is given by

$$(Tw)(t)=\begin{cases}\left(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)w_0+\frac{1}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau, \quad t\in(0,t_1], \\ g_i(t,w(t_i^+)), \quad t\in(t_i,s_i], \\ g_i(s_i,w(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau.\end{cases}$$

Clearly the operator T is well-defined and for any $w\in\mathscr{PC}(\mathscr{J},E_d)$, we have
Case 1: For $t\in(0,t_1]$.
$D_0[Tw(t)(\Xi(t)-\Xi(0))^{1-\gamma},\widehat{0}]$

$$\begin{aligned} &\le D_0\left[\frac{w_0}{\Gamma(\gamma)}+\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau,\widehat{0}\right] \\ &\le\frac{w_0}{\Gamma(\gamma)}+\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}D_0[p(\tau,w(\tau)),\widehat{0}]d\tau \\ &\le\frac{w_0}{\Gamma(\gamma)}+\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}m^*(\tau)D_0[w(\tau),\widehat{0}]d\tau \\ &\quad+\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}n^*(\tau)d\tau \\ &\le\frac{w_0}{\Gamma(\gamma)}+\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}B(\gamma,\alpha)(\Xi(t)-\Xi(0))^{\alpha+\gamma-1}M^*D_0[w(\tau),\widehat{0}] \\ &\quad+\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha+1)}(\Xi(t)-\Xi(0))^{\alpha}N^* \\ &\le\frac{w_0}{\Gamma(\gamma)}+\frac{M^*B(\gamma,\alpha)}{\Gamma(\alpha)}(\Xi(t)-\Xi(0))^{\alpha}D_0[w(\tau),\widehat{0}]+\frac{N^*}{\Gamma(\alpha+1)}(\Xi(t)-\Xi(0))^{\alpha+1-\gamma}. \end{aligned}$$

Case 2: For $t \in (t_i, s_i]$.

$$D_0[Tw(t)(\Xi(t)-\Xi(0))^{1-\gamma},\widehat{0}] \leq (\Xi(t)-\Xi(0))^{1-\gamma} D_0[g_i(t,w(t_i^+)),\widehat{0}]$$

$$\begin{aligned}
&\leq (\Xi(t)-\Xi(t_i))^{1-\gamma} q(t) D_0[w(t),\widehat{0}] \\
&\leq Q D_0[w(t),\widehat{0}],
\end{aligned}$$

where $Q = (\Xi(t)-\Xi(t_i))^{1-\gamma} q(t)$.

Case 3: For $t \in (s_i, t_{i+1}]$.

$$D_0[Tw(t)(\Xi(t)-\Xi(s_i))^{1-\gamma},\widehat{0}]$$

$$\begin{aligned}
&\leq (\Xi(t)-\Xi(s_i))^{1-\gamma} D_0[g_i(s_i,w(s_i^+)),\widehat{0}] + \frac{(\Xi(t)-\Xi(s_i))^{1-\gamma}}{\Gamma(\alpha)} \\
&\quad \times \int_{s_i}^{t} \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} D_0[p(\tau,w(\tau)),\widehat{0}]d\tau \\
&\leq Q D_0[w(t),\widehat{0}] + \frac{M^* B(\gamma,\alpha)}{\Gamma(\alpha)} (\Xi(t_i)-\Xi(s_i))^{\alpha} D_0[w(t),\widehat{0}] \\
&\quad + \frac{N^*}{\Gamma(\alpha+1)} (\Xi(t_{i+1})-\Xi(s_i))^{\alpha+1-\gamma},
\end{aligned}$$

which gives T transforms the Ball $\mathscr{B}_\eta = \{w \in \mathscr{PC}(\mathscr{J}, E_d) | D_0[w,\widehat{0}] \leq \eta\}$, into itself. Next, we have to prove the operator $T : \mathscr{B}_\eta \to \mathscr{B}_\eta$ satisfies all the conditions of Schauder fixed point theorem. The following steps are done by the proof.

Step 1: T is continuous.

Let w_n be a sequence such that $w_n \to w$ in $C(\mathscr{J}, E_d)$. Then

Case i: For $t \in (0, t_1]$,

$$D_0[Tw_n(t)(\Xi(t)-\Xi(0))^{1-\gamma}, Tw(t)(\Xi(t)-\Xi(0))^{1-\gamma}]$$

$$\begin{aligned}
&\leq \frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} D_0[p(\tau,w_n(\tau)),p(\tau,w(\tau))]d\tau \\
&\leq \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} (\Xi(t_1)-\Xi(0))^{\alpha} D_0[p(t,w_n(t)),p(t,w(t))] \\
&\leq \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} (\Xi(t_1)-\Xi(0))^{\alpha} L_p D_0[w_n,w].
\end{aligned}$$

Case ii: For $t \in (t_i, s_i]$.

$$D_0[Tw_n(t)(\Xi(t)-\Xi(t_i))^{1-\gamma}, Tw(t)(\Xi(t)-\Xi(t_i))^{1-\gamma}]$$

$$\begin{aligned}
&\leq (\Xi(t)-\Xi(t_i))^{1-\gamma} D_0[g_i(t,w_n(t_i^+)), g_i(t,w(t_i^+))] \\
&\leq D_0[g_i(t,w_n(t_i^+)), g_i(t,w(t_i^+))] \\
&\leq L_{g_i} D_0[w_n(t_i^+), w(t_i^+)].
\end{aligned}$$

Case iii: For $t \in (s_i, t_{i+1}]$.

$$D_0[Tw_n(t)(\Xi(t)-\Xi(t_i))^{1-\gamma}, Tw(t)(\Xi(t)-\Xi(t_i))^{1-\gamma}]$$

$$\begin{aligned}
&\leq (\Xi(t)-\Xi(s_i))^{1-\gamma} D_0[g_i(s_i,w_n(t_i^+)), g_i(s_i,w(t_i^+))] \\
&\qquad + \frac{(\Xi(t)-\Xi(s_i))^{1-\gamma}}{\Gamma(\alpha)} \int_{s_i}^{t} \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} D_0[p(\tau,w_n(\tau)),p(\tau,w(\tau))]d\tau \\
&\leq D_0[g_i(t,w_n(t_i^+)), g_i(t,w(t_i^+))] + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} (\Xi(t_{i+1})-\Xi(s_i))^{\alpha} D_0[p(t,w_n(t)),p(t,w(t))] \\
&\leq L_{g_i} D_0[w_n,w] + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} (\Xi(t_{i+1})-\Xi(s_i))^{\alpha} L_p D_0[w_n,w].
\end{aligned}$$

Step 2: $T(\mathscr{B}_\eta)$ is uniformly bounded.

It is clear that, $T(\mathscr{B}_\eta) \subset \mathscr{B}_\eta$ is bounded.

Step 3: We have to prove that $T(\mathscr{B}_\eta)$ is equicontinuous.
If $t_1, t_2 \in \mathscr{J}$, $t_1 > t_2$ are bounded set of $C(\mathscr{J}, E_d)$ as in step 2. Then
Case i: For $t \in (0, t_1]$.
$D_0[(\Xi(t_1)-\Xi(0))^{1-\gamma}Tw(t_1), (\Xi(t_2)-\Xi(0))^{1-\gamma}Tw(t_2)]$

$$\begin{aligned}&\leq D_0[\frac{(\Xi(t_1)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}\Xi^{'}(\tau)(\Xi(t_1)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau,\\ &\quad\frac{(\Xi(t_2)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}\Xi^{'}(\tau)(\Xi(t_2)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau]\\ &\leq \frac{D_0[p(t,w(t)),\widehat{0}]}{\Gamma(\alpha)}B(\gamma,\alpha)[(\Xi(t_1)-\Xi(0))^{\alpha}+(\Xi(t_2)-\Xi(0))^{\alpha}].\end{aligned}$$

Case ii: For $t \in (t_i, s_i]$.
$D_0[(\Xi(t_1)-\Xi(0))^{1-\gamma}Tw(t_1), (\Xi(t_2)-\Xi(0))^{1-\gamma}Tw(t_2)]$

$$\begin{aligned}&\leq D_0[(\Xi(t_1)-\Xi(0))^{1-\gamma}g_i(t_1,w(t_i^+)),(\Xi(t_2)-\Xi(0))^{1-\gamma}Tw(t_2)g_i(t_2,w(t_i^+))],\\ &\leq D_0[g_i(t_1,w(t_i^+)),g_i(t_2,w(t_i^+))].\end{aligned}$$

Case iii: For $t \in (s_i, t_{i+1}]$.
$D_0[(\Xi(t_1)-\Xi(0))^{1-\gamma}Tw(t_1), (\Xi(t_2)-\Xi(0))^{1-\gamma}Tw(t_2)]$

$$\begin{aligned}&\leq D_0[\frac{(\Xi(t_1)-\Xi(s_i))^{1-\gamma}}{\Gamma(\alpha)}\int_{s_i}^{t_1}\Xi^{'}(\tau)(\Xi(t_1)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau,\\ &\quad\frac{(\Xi(t_2)-\Xi(s_i))^{1-\gamma}}{\Gamma(\alpha)}\int_{s_i}^{t_2}\Xi^{'}(\tau)(\Xi(t_2)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau],\\ &\to 0 \quad as \quad t_2\to t_1.\end{aligned}$$

As a sequence of step 1-2 together with the Arzela-Ascoli theorem states that T is continuous and compact on $\mathscr{B}_\eta$. Schauder's theorem states that T has a fixed point of w, which gives w is a solution of (1.1). This completes the proof. □

Theorem 3.4. *Assume that* $(H1)$-$(H2)$ *hold. If*

$$\Lambda=\max\left\{\frac{L_pB(\gamma,\alpha)}{\Gamma(\alpha)}(\Xi(t_1)-\Xi(0))^{\alpha},\left(L_{g_i}+\frac{L_pB(\gamma,\alpha)}{\Gamma(\alpha)}(\Xi(t_{i+1})-\Xi(s_i))^{\alpha}\right)\right\}<1.$$

Then, the problem (1.1) *has unique solution.*

Proof. Define a operator $T:\mathscr{PC}(\mathscr{J},E_d)\to\mathscr{PC}(\mathscr{J},E_d)$ is given by

$$(Tw)(t)=\begin{cases}\left(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)w_0+\frac{1}{\Gamma(\alpha)}\int_0^t\Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau, & t\in(0,t_1]\\ g_i(t,w(t_i^+)), \quad t\in(t_i,s_i] & \\ g_i(s_i,w(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^t\Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau. & \end{cases}$$

It is enough to prove T is a contraction mapping, we consider the following cases are done by the proof.
Case i: For $t \in (0, t_1]$.
$D_0[Tw(t)(\Xi(t)-\Xi(0))^{1-\gamma}, T\overline{w}(t)(\Xi(t)-\Xi(0))^{1-\gamma}]$

$$\begin{aligned}&\leq\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}D_0[p(\tau,w(\tau)),p(\tau,\overline{w}(\tau)]d\tau\\ &\leq\frac{(\Xi(t)-\Xi(0))^{1-\gamma}}{\Gamma(\alpha)}\int_0^t\Xi^{'}(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}L_pD_0[w,\overline{w}]d\tau\\ &\leq\frac{L_pB(\gamma,\alpha)}{\Gamma(\alpha)}(\Xi(t_1)-\Xi(0))^{\alpha}D_0[w,\overline{w}].\end{aligned}$$

Case ii: For $t \in (t_i, s_i]$.
$$D_0[Tw(t)(\Xi(t)-\Xi(t_i))^{1-\gamma}, T\overline{w}(t)(\Xi(t)-\Xi(t_i))^{1-\gamma}]$$
$$\leq (\Xi(t)-\Xi(0))^{1-\gamma} D_0[g_i(t,w(t_i^+)), g_i(t,\overline{w}(t_i^+))]$$
$$\leq L_{g_i} D_0[w,\overline{w}].$$

Case iii: For $t \in (s_i, t_{i+1}]$.
$$D_0[Tw(t)(\Xi(t)-\Xi(s_i))^{1-\gamma}, T\overline{w}(t)(\Xi(t)-\Xi(s_i))^{1-\gamma}]$$
$$\leq (\Xi(t)-\Xi(s_i))^{1-\gamma} D_0[g_i(s_i,w(t_i^+)), g_i(s_i,\overline{w}(t_i^+))]$$
$$+\frac{(\Xi(t)-\Xi(s_i))^{1-\gamma}}{\Gamma(\alpha)}\int_{s_i}^{t}\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}D_0[p(\tau,w(\tau)),p(\tau,\overline{w}(\tau)]d\tau$$
$$\leq \left(L_{g_i}+\frac{L_pB(\gamma,\alpha)}{\Gamma(\alpha)}(\Xi(t_{i+1})-\Xi(s_i))^{\alpha}\right)D_0[w,\overline{w}],$$

which gives $D_0[Tw, T\overline{w}] \leq \Lambda D_0[w,\overline{w}]$. Hence T is a contraction and there exists a unique solution. This completes the proof. □

4. Stability Results

In this section, we discuss a generalized Ulam-Hyers-Rassias stability (G-U-H-R) concept of Eqn.(1.1). Let $\zeta \geq 0$ and $\varphi \in \mathscr{PC}(\mathscr{J},\mathbb{R}^+)$ is nondecreasing. Then, we consider the following inequality

$$\begin{cases} D_0[{}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}u(t), p(t,u(t))] \leq \varphi(t), & t \in (s_i,t_{i+1}], \\ D_0[u(t), g_i(t,u(t_i^+))] \leq \zeta, & t \in (t_i,s_i]. \end{cases} \tag{4.1}$$

Definition 4.1. *The problem (1.1) is G-U-H-R stable with respect to (φ,ζ), if there exists $C_{p,g_i,\varphi} > 0$ such that for each solution $u \in \mathscr{PC}(\mathscr{J},E_d)$ of Eqn.(4.1), there exists a solution $z \in \mathscr{PC}(\mathscr{J},E_d)$ of Eqn.(1.1) with*

$$D_0[u(t),z(t)] \leq C_{p,g_i,\varphi}(\varphi(t)+\zeta), \quad t \in \mathscr{J}.$$

Remark 4.2. *A fuzzy function $u \in \mathscr{PC}(\mathscr{J},E_d)$ is a solution of Eqn.(4.1) if and only if there exists $G \in \mathscr{PC}(\mathscr{J},E_d)$ and a sequence G_i, $i = 1,2,...,m$ (which depends on u) such that*

(i) $D_0[G(t),\widehat{0}] \leq \varphi(t)$ *and* $D_0[G_i,\widehat{0}] < \zeta$, $i = 1,2,...,m$.

(ii) ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}u(t) = p(t,u(t)) + G(t)$, $t \in (s_i,t_{i+1}]$.

(iii) $u(t) = g_i(t,u(t_i^+)) + G_i$, $t \in (t_i,s_i]$.

Remark 4.3. *Let $u \in \mathscr{PC}(\mathscr{J},E_d)$ be a solution of Eqn.(4.1). Then, u is a solution of the following integral inequality*

$$\begin{cases} D_0[u(t), g_i(t,u(t_i^+))] \leq \zeta, \quad t \in (t_i,s_i], \quad i = 1,2\cdots,m, \\ D_0\left[u(t), \left(\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,z(\tau))d\tau\right] \\ \quad \leq \frac{1}{\Gamma(\alpha)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}\varphi(\tau)d\tau, \\ D_0\left[u(t), g_i(s_i,u(t_i^+)) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,z(\tau))d\tau\right] \\ \quad \leq \zeta + \frac{1}{\Gamma(\alpha)}\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}\varphi(\tau)d\tau, \quad t \in (s_i,t_{i+1}]. \end{cases} \tag{4.2}$$

Theorem 4.4. *Suppose that $p \in C([s_i,t_{i+1}],E_d)$ and $g_i \in C([t_i,s_i],E_d)$ satisfied $(H2)-(H5)$ and a fuzzy function $w \in \mathscr{PC}(\mathscr{J},E_d)$ satisfies Eqn.(3.6), there exists a unique solution $u : \mathscr{J} \to E_d$ of (3.1) with the initial condition $u(0) = w(0)$ such that*

$$D_0[u(t),w(t)] \leq \frac{(1+C_{\varphi)})(\varphi(t)+\zeta)}{1-\Lambda}, \quad t \in \mathscr{J}, \tag{4.3}$$

where $\Lambda = \max\{L_{g_i} + L_pC_{\varphi}\}$.

Proof. Consider the space of piecewise continuous function

$$S = \{w : \mathscr{J} \to E_d | w \in \mathscr{PC}(\mathscr{J}, E_d)\},$$

with a generalized metric on S. Now, let us consider

$$D_S[w,\overline{w}] = \inf\{C' + C'' \in [0,\infty) | D_0[w(t),\overline{w}(t)] \leq C' + C''(\varphi(t) + \zeta), \quad t \in \mathscr{J}\},$$

obviously, (S, D_S) is a complete generalized metric space, where

$$\begin{aligned} C' &\in \{C \in [0,+\infty) | D_0[w(t),\overline{w}(t)] \leq C\varphi(t), \quad \text{for all} \quad t \in (s_i, t_{i+1}]\}, \\ C'' &\in \{C \in [0,+\infty) | D_0[w(t),\overline{w}(t)] \leq C\zeta(t), \quad \text{for all} \quad t \in (t_i, s_i]\}. \end{aligned}$$

Define an operator $T : S \to S$ by

$$(Tw)(t) = \begin{cases} \dfrac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)} w_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau, w(\tau)) d\tau, \quad t \in (0,t_1], \\ g_i(t, w(t_i^+)), \quad t \in (t_i, s_i], \\ g_i(s_i, w(t_i^+)) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau, w(\tau)) d\tau. \end{cases} \tag{4.4}$$

Clearly, the operator T is a well-defined operator. Next, we show that T is strictly contractive on S. From the definition of the space (S, D_S), for any $w, \overline{w} \in S$, it is possible to find $C', C'' \in [0,\infty)$ such that

$$D_0[w(t),\overline{w}(t)] \leq \begin{cases} C'\varphi(t), \quad t \in (s_i, t_{i+1}] \quad k = 0,1,...,m, \\ C''\zeta(t), \quad t \in (t_i, s_i], \quad k = 1,2,...,m, \end{cases}$$

and from the definition of operator T. By using $(H2)$, $(H3)$, and $(H5)$, we get
Case 1: For $t \in (0, t_1]$.

$$\begin{aligned} D_0[Tw(t), T\overline{w}(t)] &= D_0\Bigg[\frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau, w(\tau)) d\tau \\ &\qquad , \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} p(\tau, \overline{w}(\tau)) d\tau\Bigg] \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} D_0[p(\tau, w(\tau)), p(\tau, \overline{w}(\tau))] d\tau \\ &\leq \frac{L_p}{\Gamma(\alpha)} \int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} D_0[w,\overline{w}] d\tau \\ &\leq LpC' C_\varphi \varphi(t). \end{aligned}$$

Case 2: For $t \in (t_i, s_i]$.
By $(H3)$, we get

$$\begin{aligned} D_0[Tw(t), T\overline{w}(t)] &= D_0[g_i(t, w(t_i^+)) g_i(t, \overline{w}(t_i^+))] \\ &\leq L_{g_i} D_0[w,\overline{w}] \\ &\leq L_{g_i} C'' \zeta(t). \end{aligned}$$

Case 3: For $t \in (s_i, t_{i+1}]$.

By $(H2)-(H5)$, we have

$$\begin{aligned}
D_0[Tw(t),T\overline{w}(t)] &= D_0\Bigg[g_i(s_i,w(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau,\\
&\qquad g_i(s_i,\overline{w}(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,\overline{w}(\tau))d\tau\Bigg]\\
&\le D_0[g_i(s_i,w(t_i^+)),g_i(s_i,\overline{w}(t_i^+))]\\
&\quad +D_0\left[\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}(p(\tau,w(\tau)),p(\tau,\overline{w}(\tau)))d\tau\right]\\
&\le L_{g_i}D_0[w,\overline{w}]+\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}\varphi(\tau)d\tau\\
&\quad \times D_0[p(\tau,w(\tau)),p(\tau,\overline{w}(\tau))]\\
&\le (L_{g_i}+L_pC_\varphi)(C'+C'')(\varphi(t)+\zeta))\\
&\le \max_{i\in\{1,2,\cdots,m\}}(L_{g_i}+L_pC_\varphi)(C'+C'')(\varphi(t)+\zeta)\\
&= \Lambda(C'+C'')(\varphi(t)+\zeta), \quad t\in\mathscr{J},
\end{aligned}$$

where $\Lambda=\max_{i\in\{1,2,\cdots,m\}}(L_{g_i}+L_pC_\varphi)$. This implies that

$$D_S[Tw,T\overline{w}]\le \Lambda D_S[w,\overline{w}], \quad \text{for any} \quad w,\overline{w}\in S.$$

Hence T is strictly contractive. Now, we take $w_0\in S$ and by using the piecewise continuous property of w_0 and Tw_0, it is possible to find $0<G_i<\infty$ so that

$$\begin{aligned}
D_0[Tw_0(t),w_0(t)] =&D_0\Bigg[\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0+\frac{1}{\Gamma(\alpha)}\int_0^t\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}\\
&\qquad \times p(\tau,w_0(\tau))d\tau,w_0(t)\Bigg]\\
&\le G_1\varphi(t)\le G_1(\varphi(t)+\zeta), \quad t\in[0,t_1].
\end{aligned}$$

Also,

$$\begin{aligned}
D_0[Tw_0(t),w_0(t)] &= D_0[g_i(s_i,w(t_i^+)),w_0(t)]\\
&\le G_2\zeta\le G_2(\varphi(t)+\zeta), \quad t\in(t_i,s_i],
\end{aligned}$$

and

$$\begin{aligned}
D_0[Tw_0(t),w_0(t)] &= D_0[g_i(t,w(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^{t}\Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w_0(\tau))d\tau,w_0(t)]\\
&\le G_3(\varphi(t)+\zeta), \quad t\in(s_i,t_{i+1}].
\end{aligned}$$

Since p, g_i and w_0 are bounded on $\mathscr{J}$ and $\varphi(t)+\zeta>0$, it follows that $D_S[Tw_0,w_0]\le\max_{i=1,2,..,m}\{G_1,G_2,G_3\}<\infty$. According to Banach fixed point theorem, there exists a fixed point of fuzzy continuous function $S:\mathscr{J}\to E_d$ such that $T^nw_0\to w_0\in(S,D_S)$ as $n\to\infty$ and $Tw_0=w_0$, that is., w_0 satisfies Eqn.(3.1) for all $t\in\mathscr{J}$. For finally, we check that $C_w\in(0,\infty)$ so that $D_0[w_0(t),w(t)]\le C_w(\varphi(t)+\zeta)$, for any $t\in\mathscr{J}$. Since w,w_0 are bounded on $\mathscr{J}$, which gives, $\min_{t\in\mathscr{J}}(\varphi(t)+\zeta)>0$. Thus $D_S[w_0,w]<\infty$, $w\in S$, which gives $S=\{w\in S|D_S(w_0,w)<\infty\}$, we get u is the unique solution continuous function.
In this same process, we prove Eqn.(4.3) holds. A function $w\in\mathscr{PC}(\mathscr{J},E_d)$ is a solution of Eqn. (4.1) on $\mathscr{J}$, then there exists a function $G\in\mathscr{PC}(\mathscr{J},E_d)$ and a sequence G_i (which depends on w) such that

$$\begin{cases}
D_0[G(t),\widehat{0}]\le\varphi(t), \quad \textit{and}\\
D_0[G_i,\widehat{0}]\le\zeta, \quad i=1,2,..m\\
{}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\Xi}w(t)=p(t,w(t))+G(t), \quad t\in(s_i,t_{i+1}]\\
w(t)=g_i(t,w(t_i^+))+G_i, \quad t\in(t_i,s_i].
\end{cases} \tag{4.5}$$

It follows from Lemma 3.2, one has

$$\begin{cases} w(t)= & \frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0+\frac{1}{\Gamma(\alpha)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} \\ & \quad\times[p(\tau,w(\tau))+G(t)]d\tau, \quad t\in(0,t_1] \\ w(t)= & g_i(t,w(t_i^+))+G_i, \quad t\in(t_i,s_i], \\ w(t)= & [g_i(s_i,w(t_i^+))+G_i]+\frac{1}{\Gamma(\alpha)}\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1} \\ & \quad\times[p(\tau,w(\tau))+G(t)]d\tau, t\in(s_i,t_{i+1}]. \end{cases} \tag{4.6}$$

Thus, by (H5) and from the first inequalities of Eqn. (4.5), we get

$$\begin{cases} D_0[w(t),\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0+\frac{1}{\Gamma(\alpha)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau]\le C_\varphi\varphi(t), \\ D_0[w(t),g_i(t,w(t_i^+))]\le\zeta, \quad t\in(t_i,s_i] \\ D_0[w(t),g_i(t,w(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau]\le\zeta+C_\varphi\varphi(t), \\ t\in(s_i,t_{i+1}]. \end{cases} \tag{4.7}$$

By $(H5)$, Remark 4.2 and Eqn. (4.7), one derives
Case 1: For $t\in(0,t_1]$.

$$\begin{aligned} &D_0[w(t),\frac{(\Xi(t)-\Xi(0))^{\gamma-1}}{\Gamma(\gamma)}w_0+\frac{1}{\Gamma(\alpha)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau] \\ &\qquad\le\frac{1}{\Gamma(\alpha)}\int_0^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}D_0[G(t),\widehat{0}]d\tau \\ &\qquad\le\varepsilon C_\varphi\varphi(t). \end{aligned}$$

Case 2: For $t\in(t_i,s_i]$.

$$\begin{aligned} &D_0[w(t),g_i(t,w(t_i^+))]=D_0[g_i(t,w(t_i^+))+G_i,g_i(t,w(t_i^+))] \\ &\qquad\le D_0[G_i,\widehat{0}] \\ &\qquad\le\zeta. \end{aligned}$$

Case 3: For $t\in(s_i,t_{i+1}]$.

$$\begin{aligned} &D_0\left[w(t),g_i(s_i,w(t_i^+))+\frac{1}{\Gamma(\alpha)}\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}p(\tau,w(\tau))d\tau\right] \\ &\qquad\le D_0[G_i,\widehat{0}]+D_0\left[\frac{1}{\Gamma(\alpha)}\int_{s_i}^t \Xi'(\tau)(\Xi(t)-\Xi(\tau))^{\alpha-1}G(\tau)d\tau\right] \\ &\qquad\le D_0[G_i,\widehat{0}]+D_0[G(t),\widehat{0}] \\ &\qquad\le\varepsilon\zeta+C_\varphi\varphi(t)) \\ &\qquad\le(1+C_\varphi)(\varphi(t)+\zeta). \end{aligned}$$

Thus, $D_S[w,Tw]\le(1+C_\varphi)$, it follows that $D_S[w,u]\le\frac{D_S[Tw,w]}{1-\Lambda}\le\frac{(1+C_\varphi)}{1-\Lambda}$.
Because, Eqn.(4.3) is true for all $t\in\mathscr{J}$. Hence Eqn.(1.1) is G-U-H-R stable. □

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References

[1] M. Benchohra, J.J. Nieto, A. Ouahab, *Fuzzy solutions for impulsive differential equations*, Commun. Appl. Anal., **11**(2007), 379-394.

[2] M. Benchohra, J. Henderson, S.L. Ntouyas, *Impulsive Differential Equations and Inclusions*, New York, Hindawi Publishing Corporation, **2**(2006).

[3] M. Feckan, Y. Zhou, J. Wang, *On the concept and existence of solutions for impulsive fractional differential equations*, Commun. Nonlinear. Sci. Numer. Simul., **17**(7)(2012), 3050-3060.

[4] T.L. Guo, W. Jiang, *Impulsive fractional functional differential equations*, Comput. Math. Appl., **64**(2012), 3414-3424.

[5] N.V. Hoa, D. O'Regan, *A remark on ψ-Hilfer fractional differential equations with non-instantaneous impulses*, Math. Methods. Appl. Sci., **43**(2020), 3354-3368.

[6] N.V. Hoa, T.V. An, *Fuzzy differential equations with Riemann-Liouville generalized fractional integrable impulses*, Fuzzy Sets Syst., 2021.

[7] D. Luo, Z. Luo, *Existence and Hyers-Ulam stability results for a class of fractional order delay differential equations with non-instantaneous impulses*, Math. Slovaca., **70**(5)(2020), 1231-1248.

[8] J.V.D.C. Sousa, K.D. Kucche, E.C. de Oliveira, *Stability of ψ-Hilfer impulsive fractional differential equations*, Appl. Math. Lett., **88**(2019), 73-80.

[9] J.V.D.C. Sousa, D.D.S. Oliveira, E.C. de Oliveira, *On the existence and stability for non-instantaneous impulsive fractional integro-differential equations*, Math. Methods. Appl. Sci., **42**(2019), 1249-1261.

[10] K.M. Furati, M.D. Kassim, *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl., **64**(6)(2012), 1616-1626.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, Singapore, World Scientific, 2000.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[13] I. Podlubny, *Fractional Differential Equation*, Academic Press, San Diego, 1999.

[14] J.V.D.C. Sousa, E.C. de Oliveira, *On the ψ-Hilfer fractional derivative*, Commun. Nonlinear. Sci. Numer. Simul., **60**(2018), 72-91.

[15] B. Bede, L. Stefanini, *Generalized differentiability of fuzzy-valued functions*, Fuzzy sets syst., **230**(2013), 119-141.

[16] V. Lakshikantham, R.N. Mohapatra, *Theory of fuzzy differential equations and applications*, London, CRC Press, 2003.

[17] L. Sajedi, N. Eghbali, H. Aydi, *Impulsive coupled system of fractional differential equations with Caputo-Katugampola fuzzy fractional derivative*, Hindawi. J. Math., **13**(2021).

[18] X. Chen, H. Gu, X. Wang, *Existence and uniqueness for fuzzy differential equation with Hilfer-Katugampola fractional derivative*, Adv. Differ. Equ., **2020**(2020), 241.

[19] N.V. Hoa, H. Vu, T.M. Duc, *Fuzzy fractional differential equations under Caputo-Katugampola fractional derivative approach*, Fuzzy Sets Syst., **375**(2019), 70-99.

[20] N.V. Hoa, *Fuzzy fractional functional differential equations under Caputo gH-differentiability*, Commun. Nonlinear. Sci. Numer. Simul., **22**(1-3)(2015), 1134-1157.

[21] D.F. Luo, T. Abdeljawad, Z.G. Luo, *Ulam-Hyers stability results for a novel nonlinear nabla caputo fractional variable-order difference system*, Turk. J. Math., **45**(1)(2021), 456-70.

[22] D. Luo, K. Shah, Z. Luo, *On the novel Ulam-Hyers stability for a class of nonlinear ψ-Hilfer fractional differential equation with time-varying delays*, Mediterr. J. Math., **16**(5)(2019), 112.

[23] S. Rashid, F. Jarad, K.M. Abualnaja, *On fuzzy Volterra-Fredholm integrodifferential equation associated with Hilfer-generalized proportional fractional derivative*, AIMS Math., **6**(2021), 10920-10946.

[24] H. Vu, J.M. Rassias, N.V. Hoa, *Ulam-Hyers-Rassias stability for fuzzy fractional integral equations*, Iran. J. Fuzzy syst., **17**(2020), 17-27.

[25] X. Wang, D. Luo, Q. Zhu, *Ulam-Hyers stability of Caputo type fuzzy fractional differential equation with time-delays*, Chaos. Solitons. Fractals., **156**(2022), 111822.