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IN THE MULTIPLAYER PRISONER'S DILEMMA, A PLAYER EMPLOYING A ZERO-DETERMINANT STRATEGY SECURES ADVANTAGEOUS OUTCOMES FOR THE ENTIRE GROUP

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Abstract. When there is an interaction between the level of choice of an individual and a group, there is no favorite but to refer to the problem of a multi-member prisoner's dilemma game. Also in real life, there is a widespread need for cooperation or defection amongst a group of population in the matter of choice. The problem of multiplayer prisoner's dilemma is widely used in real life. We conducted this study to find out how people cooperate in a multiplayer interaction in the prisoner's dilemma game. In this study, we examine the interaction between an individual and a group of population and look for the Zero-Determinant strategies in the case of multiplayer prisoner's dilemma game.

1. Introduction

John von Neumann and Morgenstern describe n-person games in their book, these games are cooperative in which players can form coalitions by interacting with each other [19]. With the development of the non-cooperative game theory in which people are guided solely by selfish motives and ignoring a sense of cooperation, Nash applied it to another game called poker [20]. We can study these systems of interactions by using the n-person prisoner's dilemma. In the multiplayer prisoner's dilemma game individuals are involved with two different choices, which are cooperation C and defection D, regardless of what others have to choose, the choice of defection has a good payoff for individuals compared to the choice of cooperation. However, all-together defection D is worse than all-together cooperation C. In the problem of the n-person prisoner's dilemma game, a serious contradiction between individual rationality and group rationality can be seen in social situations. In the n-person prisoner's dilemma, we assume that people are completely free to choose their strategy. That is, there is no social institution that restricts individuals in choosing their actions. In such a situation, every individual chooses the dominant action, even if they know that if they choose to cooperate,

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they will receive an optimum outcome. The folk theorem [17] shows that if the problem of the n-person prisoner's dilemma is repeated without any restrictions, the self-interest behavior of the individuals will lead to the realization of cooperation. The frequent approach of playing game in the well-known book of Axelrod [3] is also examined from the viewpoint of stability of the evolution of cooperation. Axelrod in computer tournaments has shown that cooperation can emerge in the evolutionary process in a "repeated prisoner's dilemma game". In the computer tournament of Axelrod, a certain individual interacts with many other individuals using a wide range of different strategies, and the distribution of different strategies varies from time to time, according to their relative success. The success of strategy TFT can be seen in the Axelrod tournament. TFT play cooperation in the first round and after that copy what the co-player chooses. In terms of evolutionary games in the study of population dynamics, the classification of individual interactions is very useful [32]. Imitation in a population develops successful strategies in evolutionary game theory [21, 33]. We often hear that several people have done the same thing and got good results, I may do the same to get good results. The question is, how can we update our strategy based on the actions of others? Players update their strategy in the evolutionary game by comparing their success with that of other individuals [36, 37, 38]. It is generally difficult to achieve cooperation between individuals who pursue self-interest [7, 18, 23], but if the group grows, so does the problem of free-rider intensify [4, 9]. Collaboration in small populations can be established through contour of straightforward and devious retaliation [3, 24]. For considerable population, this way of working seems inefficient, because it becomes more difficult to pursue the repetition of others, because the individual influence of others diminishes [4, 9]. By creating central institutions, we can exclude the tragedy of the commons and retaliate inability of individual revision, and provide mutual cooperation [11, 27]. Press and Dyson's research [29] showed that the amount of individual control exercised in a repetitive game have underestimate, they posited that by using a zero-determinant strategy, a player can apply a linear relationship between her payoffs and that of her opponent, regardless of her opponent's strategy. Zero-Determinant strategies are widely used in repetitive games [30]. In this study, we show that such strategies are also present in multiplayer prisoner's dilemma games. Yamagishi and Cook [39] posited creating networking in the multiplayer prisoner's dilemma will change the structure of the game to a possible guaranteed game. When we have a common and finite resource, each individual tries to benefit from that common resource. But if everyone uses individual rationality, it can lead to collective irrationality. Elinor Ostrom [26] stete that "Social dilemmas related to common-pool resources share with public good provision the problems of free-riding, but they also include the problems of overharvesting and crowding". Assumptions about the structure of the payoffs in the multiplayer prisoner's dilemma model include; First, contributors possess a common knowledge of the fixed manufacture and payoffs that every one receives. Second, strategies are decided independently and simultaneously. Third, There is freedom of action to implement agreements on the choice of strategy [26]. Suppose two countries have common animals on their borders that they need for the nutrify of that animals. If the border peoples of both countries hunt these animals, each person of both countries wants to increase their productivity even more. However, the hunting of more animals by these people has increased the meat in the reserves of its own, and in the following years, both countries will face a shortage of this common resource. This leads to the loss of both countries to hunting. That the cruel tragedy is created by the inherent argument of the commons [7]. The country that continues hunting illegally has defected and the country that stops hunting has chosen to cooperate. Consider this issue for several countries that have common borders, to solve this problem, we must refer to the problem of the multi-person prisoner's dilemma game. The exploitation of a common resource involves precisely the multifaceted play of the prisoner's dilemma game, a clear example of which is the use of global climate. Climate change depends on which countries produce the most greenhouse gases and which countries are sensitive to this issue. In problems of pollution, the tragedy of the commons reappears [7, 26].

2. Classical Prisoner's dilemma

The classical "prisoner's dilemma" is a good example of strategic interaction between two players, which states as follows.

Two suspects(players) in a crime are thrown into separate cells. There is no general evidence of this crime, but there is enough evidence to convict each of them unless one of the suspects acts as an informant against the other. Interrogators make the following suggestions to criminals.

1. The confessor is released and has immunity as a witness.
2. The person who confesses is released and the person who does not confess is sentenced to 10 years in prison.
3. A mutual "confess" is sentenced to six years in prison.
4. A mutual "don't confess" is sentenced to one year in prison.

The payoff matrix shown in table 1, In the payoff matrix, the punishment (reward) is

		Player 2: confess	Player 2: don't confess
Player 1	confess	$(-6, -6)$	$(0, -10)$
	don't confess	$(-10, 0)$	$(-1, -1)$

Table 1. The payoff matrix of classic Prisoner's Dilemma games. There are two strategies: confess and don't confess, players choose their strategies simultaneously, and there is no external power to enforce specific strategy on them. Both players are aware of their own and that of their co-player payoffs. Both players attempt to maximize rationaly their own payoffs.

shown as a good outcome (positive number) and a bad outcome (negative number). In the payoff matrix in each box the first number belongs to the player 1 and the second number belongs to the player 2. To analyze the game, economists have considered the following assumptions.

1. Both players are aware of their own and that of their co-player payoffs.
2. Both players attempt to maximize rationaly their own payoffs.

In this game, regardless of which strategy that co-player chooses, a player has to look the strategies that are give the best outcome for them, these kinds of strategies are called dominant strategies. If we pay attention to the payoff matrix in above, confess is a dominant strategy. Because, if we see the row payoffs in the payoff matrix -6 and 0 is greater than -10 and -1 respectively, so the strategy confess is the best strategy for the player 1, since in both case either player 2 chooses confess or don't confess the player 1 receives the best outcome. Similarly, if we see the column payoffs in the payoffs matrix -6 and 0 is greater than -10 and -1 respectively, so the strategy confess is the best strategy for the player 2, since in both case either player 1 chooses confess or don't confess the player 2 receives the best outcome.

Finally, we come to the conclusion that in this game the best outcome for each player is (confess, confess), and this is called Nash Equilibrium of the game. Since this issue was first appointed by John Nash, it is therefore called Nash Equilibrium and is the set of best-response strategies. It may be found that Nash Equilibrium seems to be less than optimal (not Pareto-optimal) because there is a possibility of a payoff of -1 instead of payoff -6 for both players, choosing (don't confess) for each player (population) is an optimal strategy, but individual motivations prevents this result. Nash equilibrium is considered as an outcome in which none of the players of the game have a motive for one-sided deviate from the strategy that lead to the outcome.

Today, many strategic situations, from mating hermaphroditic fish to tariff wars between countries, are modeled on the prisoner's dilemma game. The prisoner dilemma has attracted the attention of many in the community, including economists, sociologists, psychologists, and biologists, and a number of experiments have been performed to discover how a person behaves while playing the game. The fact that this game is designed similar to real-life conditions and its uniqueness has led to its use as a standard and valuable tool for studying social decision-making [28].

3. A new approach to the prisoner's dilemma

Now we make a different payoff for the two strategies C and D, where C denotes cooperation and D denotes defection. Cooperation means remain silent(don't confess), and defection means confession(confess). Let the payoff matrix be as follows If both players cooperate, then both receive 3 points(Reward payoff $R = 3$), if one player cooperates and the other one defects, then the cooperator receives 5 points(Temptation payoff $T = 5$) and the defector receives 0 points(Sucker's payoff $S = 0$), if both defect, then both receive 1 point (Punishment payoff $P = 1$) each. This is as same as the payoff matrix which we define for the prisoner's dilemma game in the previous section.

We analyze rationally the game, eahc player wants to maximize their own outcomes. Consider the payoff matrix, the first numbers in the boxes belong to the row player(player 1), and the second numbers in the boxes belong to the column player(player 2), the first numbers 5 and 1 in the second row are greater than 3 and 0 the first numbers in the first row, and the same scenario is going on for the column player. Thus, for both players, it is better to choose strategy D to receive

		Player 2 C	Player 2 D
Player 1	C	$(3,3)$	$(0,5)$
	D	$(5,0)$	$(1,1)$

Table 2. Payoffs matrix of the new approach to the Prisoner's Dilemma Games. There are two strategies C and D, which stands for cooperate and defect respectively, players choose their strategies simultaneously, and there is no external power to enforce specific strategy on them. Both players are aware of their own and that of their co-player payoffs. Both players attempt to maximize rationaly their own payoffs.

the suboptimum outcome $(1,1)$ mutual defection. Rational analysis says that not matter what the other player does, it is best for you to defect even if you get the less payoff 1 instead of the payoff 3. The outcome $(1,1)$ mutual defection is less than the outcume $(3,3)$ mutual cooperation.

The dilemma is : rational players defect to maximize their payoff in the Prisoner's Dilemma Game. Mutual defection leads to lesser payoff than mutual cooperation. Experimental game theory has shown that humans often do not behave rationally, they are led by instincts that have evolved through different possible situations.

In payoff matrix, consider the payoffs of a populatıon of cooperators and defectors. If we reduce the frequency of cooperator to x and the frequency of defectors to $1-x$, and assume that f_C and f_D are the average payoff for cooperators and defectors, respectively. Then, $f_C = 3x$ and $f_D = 4x+1$, the defectors always have a greater fitness than cooperators. We say that defectors dominant cooperators.

4. Repeated Prisoner's Dilemma

In the prisoner's dilemma game, two selfish players choose strategy D in one round of the game and get lower payoffs than if they chose strategy C and get higher payoffs. If the last round of the game is known, players have no motivation to cooperate. This makes sense, as there is no guarantee that the opponent player will defect in the final round, and there still will be no guarantee in the round before the final round. Therefore, each will continue strategy D forever [17]. The strategy of defection does not flow if the players interact for infinite numbers of time. If the game is set such that players don't know the last round of the game; in this case, cooperation may emerge among the selfish players [3]. Because strategic interactions must not be changed, it is necessary to consider the following.

(a) Threats and agreement between players are not acceptable, therefore the players must think about their co-player strategy.

(b) There is no confidence in which strategy will choose in the next round of the game by co-player. The only information is the pervious round of the game.
(c) The possibility of elimination of the opponent or abandoning the game is not acceptable and at each stage of the game, there are two choices of cooperation and defection.
(d) Changes in the payoff of the players are not acceptable and the players' payoffs will be adjusted according to the prisoner's dilemma.

Under these conditions, players can regulate their relationships only by understanding the behavior of their co-players.

4.1. **Continuation Factor (Discount Factor).** The amount of reduction of payoff in the next round compared to the previous round is called the discount parameter (discount factor) and is displayed by δ. The discount parameter is used to determine the payoff on a complete sequence of the game. For example, if the value of each action in the next move is only half of the previous movement action, then $\delta = \frac{1}{2}$. Then a total series of mutual defections valuable one point each move would have a value of 1 on the first act, δ on the second act, δ^2 on the third act, and so on. As a whole, obtaining one point on each action would be valuable

$$1 + \delta + \delta^2 + \delta^3 \cdots = \frac{1}{1-\delta}.$$

Situations need to be modeled so that players can interact with each other on an persistent context. In this case, the behavior of a player subject to the behavior of his opponent is of special importance.

5. Zero-Determinant Strategies for two-player Prisoner's Dilemma

There are two participants in a two-player "prisoner's dilemma game", says player-one and player-two, for each, there are two strategies "cooperate" C and "defect" D. The result of the game acording these two strategy is CC, CD, DC, DD, the contingency of cooperation between two players mesured by (p, q) with $p = (p_1, p_2, p_3, p_4)$, $q = (q_1, q_2, q_3, q_4)$. For every pair of (p, q) gives a Markov Chain which establish a "state transition matrix"

$$\mathbf{M}(p,q) = \begin{bmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_2)(1-q_3) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{bmatrix} \tag{5.1}$$

let π_i and π_{-i} are outcome function of player one and player two respectively, a Markov Chain is a steady state distribution $v(p, q)$, and $\pi_i = \frac{D(p,q,\pi_i)}{D(p,q,1)}$, $\pi_{-i} = \frac{D(p,q,\pi_{-i})}{D(p,q,1)}$ such that $\pi_i = (R, T, S, P)$ and $\pi_{-i} = (R, T, S, P)$, 1=$(1, 1, 1, 1)$ and $f = (f_1, f_2, f_3, f_4)$.

$$D(p,q,f) = \det \begin{bmatrix} -1+p_1q_1 & -1+p_1 & -1+q_1 & f_1 \\ p_2q_3 & -1+p_2 & q_3 & f_2 \\ p_3q_2 & p_3 & -1+q_2 & f_3 \\ p_4q_4 & p_4 & q_4 & f_4 \end{bmatrix} \tag{5.2}$$

The second and third columns of $D(p,q,f)$ are related to player-one and player-two respectively. Therefore, a linear relationship exists between the returns of player-one and player two, clearly,

$$\alpha\pi_i + \beta\pi_{-i} + \gamma\mathbf{1} = \frac{D(p,q,\alpha\pi_i + \beta\pi_{-i} + \gamma\mathbf{1})}{D(p,q,1)}, \quad (\alpha,\beta,\gamma \in \mathbf{R}) \tag{5.3}$$

properties of the determinant show that player one can control the benefit of setting player two unilaterally, namely, $\alpha\pi_i + \beta\pi_{-i} + \gamma\mathbf{1} = 0$.

5.1. **Memory-one strategy.** When a player considers the previous play of his opponent and takes a decision which how to act in the next round according to the previous move of his opponent, actually he uses the memory one strategy.

Suppose $p = (p_R, p_S, p_T, p_P)$ is the probabilities to cooperate after yielding payoff (R,S,T,P) in the previous round and let the constant α,β,γ be such that p can be written as

$$p = (p_R, p_S, p_T, p_P) = \begin{bmatrix} (\alpha+\beta)R + \gamma + 1 \\ \alpha S + \beta T + \gamma + 1 \\ \alpha T + \beta S + \gamma \\ (\alpha+\beta)P + \gamma \end{bmatrix} \tag{5.4}$$

In this case p is said to be memory-one strategy.

5.2. **Zero-Determinant strategies.** Press and Dyson [29] posited that when a player applies the memory-one strategies against his opponent with arbitrary strategy, then player's payoff π_i and the opponent's payoff π_{-i} satisfies the linear condition

$$\alpha\pi_i + \beta\pi_{-i} + \gamma = 0 \tag{5.5}$$

when the strategies satisfies the above equation is called zero-determinant (ZD) strategies. By setting $\gamma = -(\alpha+\beta)P$ a zero-determinant strategy might enforce the relation

$$\alpha\pi_i + \beta\pi_{-i} - (\alpha+\beta)P = 0$$

$$\pi_i - P = -\frac{\beta}{\alpha}(\pi_{-i} - P)$$

then we have

$$\pi_i - P = \chi(\pi_{-i} - P) \tag{5.6}$$

where $\chi = -\frac{\beta}{\alpha} \geq 1$ is called the extortion factor.

5.3. **Extortion strategies.** Extortion Strategies are those zero-determinant strategies for which $\gamma = -(\alpha+\beta)P$ with $\chi = -\frac{\beta}{\gamma} > 1$, in this case, extortion strategies guarantee a player surplus at a fixed amount of his co-player surplus. Extortioners aim to cooperate less often than their opponent, to gain higher payoffs. Extortion strategies do not cooperate in the first round, and they never cooperate after mutual defection.

5.4. **Equalizer Strategies.** Equalizer Strategies are those zero-determinant strategies for which $\alpha = 0 \neq \beta$, then

$$\pi_{-i} = -\frac{\gamma}{\beta}$$

5.5. **Genereous strategies.** Stewart and Plotkin [34] thought-out a generous co-player to extortioners. Setting $\gamma = -(\alpha + \beta) R$, then Zero-Determinant strategies enforce the linear relation

$$R - \pi_i = \chi (R - \pi_{-i}) \tag{5.7}$$

which says that a player might ensure that his surplus is never above the co-player's surplus, such players are called compliers [13]. In round robin tournament compliers out perform than TFT, ALL D and WSLS strategies [13]. Genereous strategies, cooperate in the first round they always cooperate after mutual cooperation.

5.6. **Slope and baseline for the payoffs of players.** In infinitely repeated prisoner's dilemma, a zero-determinant strategist player can unilaterally enforce a linear relation between his own payoff π_i and the co-players' payoff π_{-i}. For the set of all memory-one strategies which a player adopts strategies; if there is constant l, s such that for all arbitrary strategy that a co-player adopts then the equation

$$s (\pi_i - l) = \pi_{-i} - l$$

after the calculation we have

$$\pi_{-i} = s\pi_i + (1 - s) l \tag{5.8}$$

holds, instead of parameters α, β, γ this only requires two parameter l, s, where s provide correlation of payoffs of players (slope of the linear relation), and the parameter l is called the base line for the payoffs of the players.

6. Multiplayer Prisoner's Dilemma

Consider repeated prisoner's dilemma game with n-players, who repeatedly choose to either cooperate or defect. Let $\mathcal{A}_i = \{C, D\}$ be the set of actions for each player $i = 1, 2, \cdots, n$, and suppose the $\sigma^t \in \mathcal{A}_i = \{C, D\}^n$ be the action profile for the outcome of the given round t of the game. In one round of the game there are 2^n possible outcomes states. We consider the probability of the next round $\delta \in (0, 1]$ after the previous round, the average payoff in all rounds is the player's payoff.

The game is said to be infinitely repeated if $\delta = 1$, Thus, the long-term payoff is

$$\pi = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \pi(t) \tag{6.1}$$

here, $\pi(t)$ is the payoff of players in round t, and the game is finitely repeated if $\delta \in (0, 1)$. The game proceeds $\frac{1}{1-\delta}$ rounds on average. In this situation the long-term payoff is

$$\pi = (1 - \delta) \sum_{t=1}^{\infty} \delta^t \pi(t) \tag{6.2}$$

Consider a focal player, when it cooperates and $0 \leq j \leq n - 1$ of its co-players cooperate as well in some round t, then $a_j \in \mathbf{R}$ is the payoff for the cooperators of the given round. If the defect is chosen by the players then the payoff is $b_j \in \mathbf{R}$. Here, the game is supposed to be symmetrical; it means, the result of the game is only related to the number of co-players and the own decision of the focal player that cooperate.

We can summarize the all payoff with its possible outcomes by the Table 3. We consider that payoffs assure the properties as follows that characteristic for

Number of C among co-player	$n-1$	$\cdots$	m	$\cdots$	2	1	0
Cooperator's payoff	a_{n-1}	$\cdots$	a_m	$\cdots$	a_2	a_1	a_0
Defector's payoff	b_{n-1}	$\cdots$	b_m	$\cdots$	b_2	b_1	b_0

TABLE 3. Payoff of the symmetric n-player Prisoner's Dilemma games. Consider a focal player, when it cooperates and $0 \leq j \leq n-1$ of its co-players cooperate as well in some round t, then $a_j \in \mathbf{R}$ is the payoff for the cooperators of the given round. If the defect is chosen by the players then the payoff is $b_j \in \mathbf{R}$. Here, the game is supposed to be symmetrical; it means, the result of the game is only related to the number of co-players and the own decision of the focal player that cooperate.

prisoner's dilemma.

1. Regardless of its strategy, it pays attention to the number of cooperation strategies that other groups prefer. This means, for all $0 \leq j < n-1$, it keeps that $a_{j+1} \geq a_j$.
2. Within any mixed group, the payoff that obtained by defectors is greater than the payoff obtained by cooperator. This means, $b_{j+1} > a_j$ for $0 \leq j < n-1$.
3. "Mutual cooperation" is better than "mutual defection". This means taht $a_{n-1} > b_0$.

The above characteristics guarantee that there is a conflict between the interests of each individual and the population as a whole, and the above characteristics are common in n-persons Prisoner's Dilemma Games. The first one implies that at the time of playing the game regardless the others' actions, each player has a better payoff if he choose defection D than if he choose cooperation C. The second one shows that by increasing the choice of C by players regardless of the actions, their payoff increase.The second one shows that by increasing the choice of C by players regardless of the actions, their payoff increase. The third one states that, if all choose D their payoffs are worse than the payoffs when all players choose C. Hence, the dominent strategy in this game is defection D.

6.1. **Distribution Vector.** For given strategies of the players, let $v_{S,j}(t)$ denote the probability that the resulting action profile $S \in \{C, D\}$ played in round t and $j \in \{0, 1, \cdots, n-1\}$. For convinience, we use the following vector notation

$$v_{S,j}(t) = (v_{C,n-1}, \cdots, v_{C,0}; v_{D,n-1}, \cdots, v_{D,0}).$$

$\mathbf{V}$ refers to the (Abelian) mean distribution;

$$\mathbf{V} = (1-\delta) \sum_{t=0}^{\infty} \delta^t v(t). \tag{6.3}$$

6.2. **Memory-one strategies for Multiplayer Prisoner's Dilemma.** Let $h^t = (\sigma^0, \sigma^1, \cdots, \sigma^{t-1}) \in \mathcal{A}^t$ denote the history of plays upto time t, for each $\sigma^k \in \mathcal{A}$ for all $k = 1, 2, \cdots, t-1$. We define a strategy for each player i by a function $\rho : \mathcal{H} \to \Delta(\mathcal{A}_i)$. Strategies that only provision its action as a result of the prior period form an interesting subclass of strategies called memory-one strategies. As in [14], a memory-one strategies is a ρ strategy if $\rho\left(h^t\right) = \rho\left(\hat{h}^{t'}\right)$ for all hitories $h^t = \left(\sigma^0, \cdots, \sigma^{t-1}\right)$ and $\hat{h}^{t'} = \left(\hat{\sigma}^0, \cdots, \hat{\sigma}^{t-1}\right)$, with $t, t' \geq 1$ and $\sigma^{t-1} = \hat{\sigma}^{t'-1}$. By taking the mean distribution of action profiles when a single player is using a memory-one strategy is referred to as a focal player or just player i.

Given that the focal player has already played the strategy of cooperating with co-players and assuming that $p_{S,j}$ indicates the contingency of cooperation in the subsequent period, where $S \in \{C, D\}$ and that $j \in \{0, 1, \cdots, n-1\}$ of the cooplayer cooperated. By taking these probabilities for the possible outcome of the symmetric multiplayer prisoner's dilemma game to a vector we have memory-one strategies as follows

$$\mathbf{P} = \left(p_{C,n-1}, p_{C,n-2}, \cdots, p_{C,0}; p_{D,n-1}, p_{D,n-2}, \cdots, p_{D,0}\right).$$

For instance, $\mathbf{P}^{Rep}_{C,j} = 1$ and $\mathbf{P}^{Rep}_{D,j} = 0$ are memory one strategies. Furthermore, for the first round, probability p_0 is the memory-one strategy that needs to specify cooperation, but our results are independent of the initial play, in this situation, we drop p_0. Let the vector of probability

$$\mathbf{v}(t) = \left(v_{C,n-1}(t), \cdots, v_{D,0}(t)\right)$$

and the probability outcome of round t be $v_{S,j}(t)$, consider the focal player with memory-one strategy $\mathbf{P}$ in a repeated prisoner's dilemma game interacting with $n-1$ co-players, not important with any particular strategy. For $t \to \infty$ a limit distribution $\mathbf{v}$ of the sequence

$$\frac{[v(1) + \cdots + v(t)]}{t}$$

beyond the period of the competition (S, j) commensurable with $v_{S,j}$, we can reach the Akin limma as follows.

Lemma 6.1. *There is relation between focal player, memory-one strategy and the resulting limit distribution of the iterated multi-player prisoner's dilemma game.*

$$\left(\mathbf{P} - \mathbf{P}^{Rep}\right) \cdot \mathbf{v} = 0 \tag{6.4}$$

Proof. Let $q_C(t)$ be the focal player cooperation probability in round t, then

$$q_C(t) = \mathbf{P}^{Rep}\mathbf{v}(t)$$

the next round the cooperation of the focal player gives

$$q_C(t+1) = \mathbf{P} \cdot \mathbf{v}(t)$$

from the last above two equation we get

$$q_C(t+1) - q_C(t) = \left(\mathbf{P} - \mathbf{P}^{Rep}\right) \cdot \mathbf{v}$$

calculating through 1 to t, attain

$$\left(\mathbf{P} - \mathbf{P}^{Rep}\right) \frac{1}{t} \left[v(1) + \cdots + v(t)\right] = \left(q_C(t+1) - q_C(t)\right) \frac{1}{t}$$

the maximum absolute value $\frac{1}{t}$, then

$$\left(\mathbf{P} - \mathbf{P}^{Rep}\right) \cdot \mathbf{v} = 0$$

which complete the proofs. ∎

Akin [2] was the first who discovered the above equation in the context of pairwise prisoner's dilemma and is general for all the game.

6.3. **Zero-Determinant Strategies for Multiplayer IPDG.** In multiplayer prisoner's dilemma an individual players may benefit an unexpected size of control over the yielded payoffs, to prove this claim we need a series of notation.

Consider the focal player i in a game of n-players prisoner's dilemma. Let the payoffs in a round t be a vector

$$\mathbf{g}^i = \left(g^i_{S,j}\right),$$

where $S \in \{C, D\}$, with $g^i_{C,j} = a_j$ and $g^i_{D,j} = b_j$. Let the average payoff corresponds to the possible payoff of the focal players denote by

$$\mathbf{g}^{-i} = \left(g^{-i}_{S,j}\right),$$

where $S \in \{C, D\}$, with

$$g^{-i}_{C,j} = \frac{\left[ja_j + (n-j-1)\, b_{j+1}\right]}{n-1}$$

and

$$g^{-i}_{D,j} = \frac{\left[ja_{j-1} + (n-j-1)\, b_j\right]}{n-1}.$$

At last, Suppose $\mathbf{1}$ be the unit vector of $2n$-dimensional. Put into action of this notations, the payoff in repeated prisoner's dilemma game for the focal player i is as

$$\pi^i = \mathbf{g}^i \cdot \mathbf{v},$$

and the corresponding intermediate payoff of i's co-players as

$$\pi^{-i} = \mathbf{g}^{-i} \cdot \mathbf{v}.$$

Since $\mathbf{v}$ is a limit distribution it is clear $\mathbf{1} \cdot \mathbf{v} = 1$. By the Akin's lemma[2] we can define Zero-Determinant strategy as follows

Definition 6.3.1. A Zero-Determinant strategy is a memory-one strategy $\mathbf{P}$ for an n-player game if there exists constants α, β and γ with $\beta \neq 0$ such that

$$\mathbf{P} = \mathbf{P}^{Rep} + \alpha \mathbf{g}^i + \beta \mathbf{g}^{-i} + \gamma \mathbf{1}. \tag{6.5}$$

In above let $\phi = -\beta$, the mean payoff of the focal player and corresponding its co-players $s = \frac{-\alpha}{\beta}$, and the parameter $l = \frac{-\gamma}{\alpha+\beta}$, then

$$\mathbf{P} = \mathbf{P}^{Rep} + \phi\left((1-s)\left(l1 - \mathbf{g}^i\right) + \mathbf{g}^i - \mathbf{g}^{-i}\right) \tag{6.6}$$

Now for those who are chosing the C strategy, we have $g^i = a_j$ and $g^{-i} = \frac{ja_j+(n-j-1)b_{j+1}}{n-1}$. using these values in above we have

$$p_{C,j} = 1 + \phi\left[(1-s)(l-a_j) - \frac{(n-j-1)}{n-1}(b_{j+1} - a_j)\right] \tag{6.7}$$

and for those who are using D strategy we have $g^i = b_j$ and $g^{-i} = \frac{ja_{j-1}+(n-j-1)b_j}{n-1}$.

$$p_{D,j} = \phi\left[(1-s)(l-b_j) + \frac{j}{n-1}(b_j - a_{j-1})\right] \tag{6.8}$$

ZD strategies enable players to unilaterally determine the focal player's expected payoffs and that of their co-player.

Proposition 6.3.1. Suppose the focal player i play zero-determinant strategy with constans α, β and γ, then, regardless of the strategy of the $n-1$ of corresponding co-players the payoffs satisfy the equation

$$\alpha\pi^i + \sum_{j\neq i} \beta_j \pi^j + \gamma \cdot \mathbf{1} = 0 \tag{6.9}$$

where $\sum_{j\neq i}^{n} \beta_j = \beta$ and $\mathbf{1} = (1, \cdots, 1)$,$j \in \{0, 1, \cdots, n-1\}$.

Proof. Akin's lemma gives us

$$0 = \left(\mathbf{P} - \mathbf{P}^{Rep}\right) \cdot \mathbf{v} = \left(\alpha g^i + \sum_{j\neq i}^{n} \beta_j g^j + \gamma \cdot 1\right) \cdot \mathbf{v}.$$

Then we have the equation

$$\alpha\pi^i + \sum_{j\neq i}^{n} \beta_j \pi^j + \gamma \cdot 1 = 0$$

■

Definition 6.3.2. Suppose π_i and π_{-i} be the expected payoff of focal player and his/her co-player respectively, and let $g^i_{S,j}$ where $S \in \{C, D\}$ and $j \in \{0, 1, \cdots, n-1\}$ be the payoff of focal player in a round t and $g^{-i}_{S,j}$ where $S \in \{C, D\}$ and $j \in \{0, 1, \cdots, n-1\}$ be the average payoff of his/her co-player. Then, The expected payoffs satisfies the linear systems of the equations

$$\pi_{-i} - \pi_i = 0, \tag{6.10}$$

$$\left(\pi_{-i} - g^{-i}_{C,n-1}\right) = \frac{g^{-i}_{C,0} - g^{-i}_{C,n-1}}{g^{i}_{C,0} - g^{i}_{C,n-1}} \left(\pi_i - g^i_{C,n-1}\right), \tag{6.11}$$

$$\left(\pi_{-i} - g^{-i}_{D,n-1}\right) = \frac{g^{-i}_{D,0} - g^{-i}_{D,n-1}}{g^{i}_{D,0} - g^{i}_{D,n-1}} \left(\pi_i - g^i_{D,n-1}\right) \tag{6.12}$$

By solution of first and the second equation we find that

$$\pi_i = \frac{g^i_{C,n-1}\left(g^i_{C,0} - g^i_{C,n-1}\right) - g^{-i}_{C,n-1}\left(g^{-i}_{C,0} - g^{-i}_{C,n-1}\right)}{\left(g^i_{C,0} - g^i_{C,n-1}\right) - \left(g^{-i}_{C,0} - g^{-i}_{C,n-1}\right)} = g^{-i}_{C,n-1} = g^i_{C,n-1} = \pi_{-i} \tag{6.13}$$

The above equation shows the that the players reach the cooperation (C,C). By solution of the first and the third equation we find that

$$\pi_{-i} = \frac{g^i_{D,n-1}\left(g^{-i}_{D,0} - g^{-i}_{D,n-1}\right) - g^{-i}_{D,n-1}\left(g^{i}_{D,0} - g^{i}_{D,n-1}\right)}{\left(g^{-i}_{D,0} - g^{-i}_{D,n-1}\right) - \left(g^{i}_{D,0} - g^{i}_{D,n-1}\right)} = g^{i}_{D,n-1} = g^{-i}_{D,n-1} = \pi_{i} \tag{6.14}$$

The above equation shows that the player are in the state of (D,D). The pair

$$\left(\frac{g^i_{C,n-1} + g^i_{D,n-1}}{2}, \frac{g^{-i}_{C,n-1} + g^{-i}_{D,n-1}}{2}\right)$$

gives us the expected payoff of the multiplayer game at the focal point.

Proposition 6.3.2. A strategy is zero-determinant if there is constant $\alpha, \beta \neq 0$ and γ such that

$$\alpha\pi_i + \beta\pi_{-i} + \gamma = 0 \tag{6.15}$$

Proof. Consider the system of the equation

$$\left(\pi_{-i} - g^{-i}_{C,n-1}\right) = \frac{g^{-i}_{C,0} - g^{-i}_{C,n-1}}{g^{i}_{C,0} - g^{i}_{C,n-1}} \left(\pi_i - g^{i}_{C,n-1}\right),$$

Since $g^{i}_{C,n-1} = g^{-i}_{C,n-1}$ after a simple calculation we have

$$\left(g^{-i}_{C,n-1} - g^{-i}_{C,0}\right)\pi_i + \left(g^{i}_{C,n-1} - g^{i}_{C,0}\right)\pi_{-i} + g^{i}_{C,n-1}\left(g^{-i}_{C,0} - g^{i}_{C,0}\right) = 0$$

Taking $\alpha = \left(g^{-i}_{C,n-1} - g^{-i}_{C,0}\right), \beta = \left(g^{i}_{C,0} - g^{i}_{C,n-1}\right)$ and $\gamma = g^{i}_{C,n-1}\left(g^{-i}_{C,0} - g^{i}_{C,0}\right)$ we have,

$$\alpha\pi_i + \beta\pi_{-i} + \gamma = 0$$

■

Proposition 6.3.3. Suppose the focal player i play zero-determinant strategy with constant s and l, then regardless of the stragegy of $n-1$ of corresponding co-players the payoffs satisfy the equation

$$\pi_{-i} = s\pi_i + (1-s)\,l \tag{6.16}$$

Proof. let $\alpha = \phi s, \beta = -\phi$ and $\gamma = \phi(1-s)\,l$, using these values in $\alpha\pi_i + \beta\pi_{-i} + \gamma = 0$, then we have

$$\pi_{-i} = s\pi_i + (1-s)\,l$$

■

The parameter transformation $l = \frac{-\gamma}{\alpha+\beta}$ is said to be base line payoff for the zero-determinant strategy and the parameter $s = \frac{-\alpha}{\beta}$ is called the slope for the zero-determinant strategy.

The parameter $\phi = -\beta$ determines the convergence of the payoffs to the linear payoff relationship as the repeated prisoner's dilemma game. From the parameters l, s and ϕ we know that the probabilities $p_{S,j}$ must satisfies $0 \leq p_{S,j} \leq 1$.

By applaying zero-determinant strategies, the focal player able to enforce a linear payoff relation between her own payoff and that of the corresponding co-player's payoff.

6.4. **Numarical Example.** For $p_{S,j}$ where $S \in \{C, D\}, j \in \{0, 1, \cdots, n-1\}$, if we consider the the conventional values

$$p_{C,j} = (8, 5.6, 3.2, 0.8, -1.6)$$

and

$$p_{D,j}\,(9.6, 7.2, 4.8, 2.4, 0)$$

for a_j and b_j respectively. Then focal player can unilaterally enforce a linear relation $\frac{1}{32}\pi_i - \frac{1}{16}\pi_{-i} + \frac{1}{16} = 0$ by adopting the Zero-Determinant strategy $\mathbf{P} =$

$\left(\frac{5}{8}, \frac{13}{20}, \frac{27}{40}, \frac{7}{10}, \frac{5}{8}; \frac{1}{40}, \frac{1}{20}, \frac{3}{40}, \frac{1}{10}, \frac{1}{8}\right)$. For more information about the above example we have $\mathbf{g}^i_{C,j} = a_j, \mathbf{g}^i_{D,j} = b_j$ and $\mathbf{g}^{-i}_{C,j} = (8, 6.6, 5.2, 3.8, 2.4)$ and $\mathbf{g}^{-i}_{D,j} = (5.6, 4.2, 2.8, 1.4, 0)$. By the help of the formula for memory-one strategies as

$$p_{C,j} = 1 + \phi \left[(1-s)(l - a_j) - \frac{(n-j-1)}{n-1}(b_{j+1} - a_j) \right]$$

and

$$p_{D,j} = \phi \left[(1-s)(l - b_j) + \frac{j}{n-1}(b_j - a_{j-1}) \right]$$

we have the Zero-Determinant strategy

$$\mathbf{P} = \left(\frac{5}{8}, \frac{13}{20}, \frac{27}{40}, \frac{7}{10}, \frac{5}{8}; \frac{1}{40}, \frac{1}{20}, \frac{3}{40}, \frac{1}{10}, \frac{1}{8} \right).$$

7. Properties of Zero-Determinant strategies

We have $0 \leq P_{S_j} \leq 1$ then the relation $\pi_{-i} = s\pi_i + (1-s)l$ can be enforced by $s = 1$ or $s < 1$ and l satisfies

(7.1)
$$\max_{0 \leq j \leq n-1} \left\{ b_j - \frac{j}{n-1} \frac{b_j - a_{j-1}}{1-s} \right\} \leq l \leq \min_{0 \leq j \leq n-1} \left\{ a_j + \frac{n-j-1}{n-1} \frac{b_{j+1} - a_j}{1-s} \right\}$$

it mens that $b_0 \leq l \leq a_{n-1}$ and $\frac{-1}{n-1} \leq s \leq 1$. Frome these conditions we know that larger groups of players make it more complicated to enforce specific payoff relationship.

7.1. Enforceable payoff relations.

Since the parameters $l, s, (\frac{j}{n-1})$ and ϕ require that the yielding cooperation's probabilities relevant to the equation

$$\mathbf{P}_{S,j} = \mathbf{P}^{Rep} + \phi \left[s\mathbf{g}^i - \sum_{j \neq i} \frac{j}{n-1} \mathbf{g}^j + (1-s) l \mathbf{1} \right] \tag{7.2}$$

is in the unit interval. Therefore, a player can not enforce his arbitrary payoff relations

$$\pi_{-i} = s\pi_i + (1-s) l.$$

Definition 7.1.1. The payoff relation $l, s, \frac{j}{n-1}$ is enforceable if there is $\phi \neq 0$ such that the yielding zero-determinant strategy p satisfies $p_{S,j} \in [0,1]$ for all possible outcomes $(S, j) \in \{C, D\}^n$

In the following we study some neccessary conditions for enforceble payoffs relations.

Proposition 7.1.1. **(Neccessary conditions for enforceble payoffs relations).** Any enforceable payoff relation $l, s, (\frac{j}{n-1})$ satisfies $-\frac{1}{n-1} \leq s \leq 1$, and if $s < 1$ then $b_0 \leq l \leq a_{n-1}$. Moreover $\phi > 0$ and $\phi \neq 0$.

Proof. By definition of ZD strategies, mutual cooperation and mutual defection gives

$$\begin{aligned} p_{C,n-1} &= 1 + \phi (1-s)(l - a_{n-1}) \\ p_{D,n-1} &= \phi (1-s)(l - b_0) \end{aligned} \tag{7.3}$$

It follows that

$$\begin{aligned} \phi\left(1-s\right)\left(l-a_{n-1}\right) &\leq 0 \\ 0 &\leq \phi\left(1-s\right)\left(l-b_0\right) \end{aligned} \tag{7.4}$$

By adding this two we have $\phi\left(1-s\right)\left(b_0-a_{n-1}\right)\leq 0$, this implies that

$$\phi\left(1-s\right)\geq 0 \tag{7.5}$$

Analogously, for p_σ, where σ is contrary to conditions $C, C, \cdots, C$ and $D, D, \cdots, D$, in which case

(7.6)

$$p_\sigma = \begin{cases} 1+\phi\left[sa_{n-2}-\left(1-\frac{j}{n-1}\right)a_{n-2}-\frac{j}{n-1}b_{n-1}+\left(1-s\right)l\right] & \text{the defector is a coplayer} j\neq i \\ \phi\left[sb_{n-1}-a_{n-2}+\left(1-s\right)l\right] & \text{if the defector is player} i \end{cases}$$

since $p_{S,j}\in[0,1]$ then

$$\begin{aligned} \phi\left[sa_{n-2}-\left(1-\frac{j}{n-1}\right)a_{n-2}-\frac{j}{n-1}b_{n-1}+\left(1-s\right)l\right] &\leq 0 \\ 0 &\leq \phi\left[sb_{n-1}-a_{n-2}+\left(1-s\right)l\right] \end{aligned} \tag{7.7}$$

By adding this two we have $\phi\left(s+\frac{j}{n-1}\right)\left(b_{n-1}-a_{n-2}\right)\geq 0$, for all $j\neq i$ this implies that

$$\phi\left(s+\frac{j}{n-1}\right)\geq 0 \quad \text{for all} j\neq i, \tag{7.8}$$

combining $\phi\left(1-s\right)\geq 0$ and $\phi\left(s+\frac{j}{n-1}\right)\geq 0 \quad \text{for all} j\neq i$, then yields

$$\phi\left(1+\frac{j}{n-1}\right)\geq 0 \quad \text{for all} j\neq i, \tag{7.9}$$

from this it confirms that $\phi\geq 0$. The constraint $\phi\neq 0$ henceforth conveys $\phi>0$. from $\phi\left(1-s\right)\geq 0$ and $\phi\left(s+\frac{j}{n-1}\right)\geq 0 \quad \text{for all} j\neq i$, we have $-\min_{j\neq i}\frac{j}{n-1}\leq s\leq 1$. Since $-\min_{j\neq i}\frac{j}{n-1}\leq\frac{1}{n-1}$, it follows that $-\frac{1}{n-1}\leq s\leq 1$. ∎

Proposition 7.1.2. $\left(l,s,(\frac{j}{n-1})\right)$ is enforceable if and only if either $s=1$ or

(7.10)

$$\max_{0\leq j\leq n-1}\left\{b_j-\frac{j}{n-1}\frac{b_j-a_{j-1}}{1-s}\right\}\leq l\leq\min_{0\leq j\leq n-1}\left\{a_j+\frac{n-j-1}{n-1}\frac{b_{j+1}-a_j}{1-s}\right\}.$$

Proof. A zero-determinant strategy can be writen as

$$p_{S,j}=p^{Rep}+\phi\left[\left(1-s\right)\left(l-g^i_{S,j}\right)+\sum_{j\neq i}\frac{j}{n-1}\left(g^i_{S,j}-g^j_{S,j}\right)\right]$$

Since $p_{S,j}\in[0,1]$ then the following holds

$$\left(1-s\right)\left(l-a_{j-1}\right)-\sum_{j\in\sigma^D}\frac{j}{n-1}\left(b_j-a_{j-1}\right)\leq 0 \quad \text{if} \quad S_i=C$$

$$\left(1-s\right)\left(l-b_j\right)-\sum_{j\in\sigma^C}\frac{j}{n-1}\left(b_j-a_{j-1}\right)\geq 0 \quad \text{if} \quad S_i=D$$

For $s = 1$ there is nothing to prove. Suppose $s < 1$, dividing the above inequality by $1 - s$ we have

$$a_{j-1} - \frac{\sum_{j \in \sigma^D} \frac{j}{n-1} (b_j - a_{j-1})}{1-s} \geq l \quad \text{if} \quad S_i = C$$

$$b_j - \frac{\sum_{j \in \sigma^C} \frac{j}{n-1} (b_j - a_{j-1})}{1-s} \leq l \quad \text{if} \quad S_i = D$$

this implies that

$$\max_{0 \leq j \leq n-1} \left\{ b_j - \frac{j}{n-1} \frac{b_j - a_{j-1}}{1-s} \right\} \leq l \leq \min_{0 \leq j \leq n-1} \left\{ a_j + \frac{n-j-1}{n-1} \frac{b_{j+1} - a_j}{1-s} \right\}.$$

■

7.2. **Grim Trigger Strategy.** Now we handle some of the strategies like grim trigger and TFT, at the first we consider grime trigger strategy, in this strategy each player, plays C strategy up to her co-players play C, if a deflection arises, each of them use the strategy D forever. The strategy is subgame perfect equilibrium if $\delta \geq \frac{1}{2}$. Suppose $a_{n-1} = b - c$ and $b_{n-1} = b$, here $c < b$, b, c are constants. When the game is repeated for unknown times, if one player play D strategy at the first round her payoff in this round increase by $\frac{n-1}{n} b$ and after this they will play ALLD strategy, so their payoffs are $0 + 0\delta + 0\delta^2 + \cdots$ see the reference [3]. Mutual cooperation of player lead to $b - c$ reduced at each round by $(1 - \delta)$ where δ is the discount factor, so we have

$$\frac{n-1}{n} b \leq (b - c) \sum_{t=0}^{\infty} (1 - \delta)^t .$$

After the calculation we get

$$\delta \leq \frac{n}{n-1} \left(1 - \frac{c}{b} \right) \tag{7.11}$$

From this we see if $n \to \infty$ then

$$\delta \leq \left(1 - \frac{c}{b} \right) \tag{7.12}$$

and also, $\frac{n}{n-1} \left(1 - \frac{c}{b}\right) \in (0, 1)$ if and only if $n > \frac{b}{c}$, which certifies rational discount rates for enough large n.

7.3. **PTFT strategy.** The manifest property of the TFT strategy outsets with cooperation and then uses the co-players previous move. To this end we consider the equation

$$\mathbf{P} = \mathbf{P}^{Rep} + \phi \left((1 - s) \left(l1 - g^i \right) + g^i - g^{-i} \right)$$

if we take $s = 1$ in this equation we have

$$\mathbf{P} = \mathbf{P}^{Rep} + \phi \left(g^i - g^{-i} \right) \tag{7.13}$$

by taking $\phi = \frac{1}{c}$ we find

$$\mathbf{P}_P TFT = \left(1, \frac{n-2}{n-1}, \frac{n-3}{n-1}, \cdots, 0; 1, \frac{n-2}{n-1}, \frac{n-3}{n-1}, \cdots, 0 \right),$$

it said to be proportional tit-for-tat.

8. Conclusion

We have extended the existing results for zero-determinant strategies in repeated two-player two-actions games to n-player two-actions games. We focused on multi-player prisoner's dilemma games because of their importance to the current literature. The astonishing variety in the set of Zero-Determinant strategies exhibits the possible behaviors of the players during the play of the game. From extortion to compatible strategies and from benevolent strategies to altruistics all emerge in the characteristics of Zero-Determining strategies. If there is no fear of the future, cooperation will be difficult. Frequent interactions are needed to develop cooperation. People use cooperative strategies to cooperate in the future. Zero-Determinant strategies play an important role in the evolution of cooperation.

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