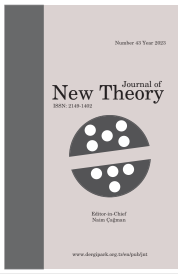
Journal of
New Theory
ISSN: 2149-1402

43 (2023) 73-82
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



A Note on Equivalence of G-Cone Metric Spaces and G-Metric Spaces

Abhishikta Das[1], Tarapada Bag[2]

Article Info
Received: 4 Apr 2023
Accepted: 21 Jun 2023
Published: 30 Jun 2023
doi:10.53570/jnt.1277026
Research Article

Abstract — This paper contains the equivalence between tvs-G cone metric and G-metric using a scalarization function ζ_p, defined over a locally convex Hausdorff topological vector space. This function ensures that most studies on the existence and uniqueness of fixed-point theorems on G-metric space and tvs-G cone metric spaces are equivalent. We prove the equivalence between the vector-valued version and scalar-valued version of the fixed-point theorems of those spaces. Moreover, we present that if a real Banach space is considered instead of a locally convex Hausdorff space, then the theorems of this article extend some results of G-cone metric spaces and ensure the correspondence between any G-cone metric space and the G-metric space.

Keywords *G-metric space, G-cone metric space, tvs-G cone metric space*

Mathematics Subject Classification (2020) 47H10, 54H25

1. Introduction

In 1906, French mathematician Frechet introduced metric space by generalizing the concept of the Euclidean distance function. Later, Hausdorff, in 1914, formalized the definition of metric space by a set of axioms inherited from the basic properties of Euclidean distance. After that, several generalized metric spaces, such as 2-metric [1], b-metric [2, 3], strong b-metric [4], D-metric [5], G-metric [6], S-metric [7], cone-metric [8], parametric S-metric [9], generalized parametric metric [10], and fuzzy metric [11] have been familiarized. Since 1922, when Banach proved the celebrated Banach fixed point theorem in complete metric spaces, several researchers have tried to generalize it. Sometimes this generalization is by changing the contraction condition or reforming it to some generalized metric spaces. Based on the types of self-mappings, such as contractive or expansive, single-valued or multivalued, fixed point theories have been developed on those spaces.

In 2006, G-metric space, one of those generalized metric spaces, was brought to light by Mustafa and Sims [6] to overcome elementary imperfection in the structure of D-metric spaces, defined by Dhage [5]. Immediately after, Guang and Xian [8] introduced the idea of cone metric in 2007, where they replaced the set of non-negative real numbers with an ordered real Banach space. Following them, Beg et al. [12] extended the concept of G-metric and cone metric to G-cone metric space in 2010. Consequently, many study on fixed point theory have been done in G-cone metric spaces.

[1]abhishikta.math@gmail.com(Corresponding Author); [2]tarapadavb@gmail.com
[1,2]Department of Mathematics, Siksha-Bhavana, Visva-Bharati, Santiniketan, West Bengal, India

This article aims to investigate the relation between the vector-valued and scalar-valued versions of fixed point theorems of generalized cone-metric spaces and G-metric spaces. We show that there is a relationship between G-metric and G-cone metric with the help of scalarization function ζ_p, defined on a locally convex Hausdorff topological vector space.

This article is presented as follows: Section 2 consists of some definitions and results used in the main result Section. Section 3 establishes the relation between G-metric and G-cone metric utilizing the tvs-G cone metric space definition. The final section contains some fixed point results ensuring their equivalence in general G-metric spaces and tvs-G cone metric spaces and discusses the need for further research.

2. Preliminaries

This section contains some definitions and results related to the main results of this study.

Let E be a topological vector space (tvs in short), θ be zero vector, and P be a nonempty convex, i.e., $P + P \subseteq P$ and $\mu P \subseteq P$, for $\mu \geq 0$, and pointed, i.e., $P \cap (-P) = \{\theta\}$, cone in E. For the cone $P \in E$, $\preceq$ is a partial ordering with respect to P given by $x \preceq \mu \Leftrightarrow \mu - x \in P$. $x \prec \mu$ stands for $x \preceq \mu$ but $x \neq \mu$ and $x \prec\prec \mu$ stands for $\mu - x \in \text{int}(P)$ where $\text{int}(P)$ denotes the interior of P.

Throughout the article, Y is a real locally convex Hausdorff TVS and P is closed, proper, and convex pointed cone with the non-empty interior, $p \in \text{int}(P)$, and $\preceq$ is a partial ordering with respect to P defined as above.

Consider the nonlinear scalarization function $\zeta_p : Y \to \mathbb{R}$ is defined by

$$\zeta_p(x) = \inf\{s \in \mathbb{R} : x \in ps - P\}, \quad \text{for all } x \in Y$$

Lemma 2.1. [13–17] For each $s \in \mathbb{R}$, $p \in \text{int}(P)$, and $x, x_1, x_2 \in Y$, the following conditions are satisfied:

i. $\zeta_p(x) \leq s \Leftrightarrow x \in ps - P$

ii. $\zeta_p(x) > s \Leftrightarrow x \notin ps - P$

iii. $\zeta_p(x) \geq s \Leftrightarrow x \notin ps - \text{int}(P)$

iv. $\zeta_p(x) < s \Leftrightarrow x \in ps - \text{int}(P)$

v. $\zeta_p(.)$ is continuous and positively homogeneous function on Y

vi. $x_2 \preceq x_1$ implies $\zeta_p(x_2) \leq \zeta_p(x_1)$

vii. $\zeta_p(x_1 + x_2) \leq \zeta_p(x_1) + \zeta_p(x_2)$.

Note 2.2. [14] Clearly $\zeta_p(\theta) = 0$. Moreover, the converse statement of *vi.* in Lemma 2.1 is not true necessarily. For example, let $E = \mathbb{R}^2$, $P = \{(x, y) \in X : x, y \geq 0\}$, and $p = (1, 1)$. Then, P is a closed, convex, proper, and pointed cone in Y with $\text{int}(P) \neq \phi$. For $r = 1$, it can be observed that $x_1 = (8, -15) \notin rp - \text{int}(P)$ and $x_2 = (0, 0) \in rp - \text{int}(P)$. By applying *iii.* and *iv.* of Lemma 2.1, we have $\zeta_p(x_1) < 1 \leq \zeta_p(x_2)$ while $x_1 \notin x_2 + P$.

Definition 2.3. [6] Let $\Im$ be a nonempty set and $\mathcal{G} : \Im \times \Im \times \Im \to [0, \infty)$ be a mapping that satisfies the following conditions:

(G1) $\mathcal{G}(x, \zeta, z) = 0$ if $x = \zeta = z$

(G2) $0 < \mathcal{G}(x, x, \zeta)$ if $x \neq \zeta$

(G3) $\mathcal{G}(x,x,\zeta) \leq \mathcal{G}(x,\zeta,z)$ if $\zeta \neq z$

(G4) $\mathcal{G}(x,\zeta,z) = \mathcal{G}(x,z,\zeta) = \mathcal{G}(\zeta,x,z) = \cdots$ (Symmetric in all three variables)

(G5) $\mathcal{G}(x,\zeta,z) \leq \mathcal{G}(x,\mu,\mu) + \mathcal{G}(\mu,\zeta,z)$

for all $x,\zeta,z,\mu \in \Im$. Then, $(\Im,\mathcal{G})$ is called a G-metric space.

Definition 2.4. [6] Let $(\Im,\mathcal{G}_1)$ and $(\Im,\mathcal{G}_2)$ be two G-metric spaces. A function $\mathcal{F} : (\Im,\mathcal{G}_1) \to (\Im,\mathcal{G}_2)$ is said to be

i. G-continuous at a point $\eta \in X$ if, for given $\alpha > 0$, there exists $\beta > 0$ such that, for all $\eta, b, z \in \Im$, $\mathcal{G}_2(\mathcal{F}\eta,\mathcal{F}b,\mathcal{F}z) < \alpha$ if $\mathcal{G}_1(\eta,b,z) < \beta$.

ii. G-sequentially continuous at a point $\eta \in X$ if $\{\eta_n\}$ is G-converges to η implies $\{\mathcal{F}(\eta_n)\}$ is G-converges to $\mathcal{F}(\eta)$.

Theorem 2.5. [6] Let $(\Im,\mathcal{G}_1)$ and $(\Im,\mathcal{G}_2)$ be two G-metric spaces. A function $\mathcal{F} : (\Im,\mathcal{G}_1) \to (\Im,\mathcal{G}_2)$ is G-continuous at a point $\eta \in \Im$ if and only if (iff) $\mathcal{F}$ is G-sequentially continuous at η.

Definition 2.6. [8] Let $\Im$ be a nonempty set, E be a real Banach space, P be a cone in E, and $\preceq$ is a partial ordering in E with respect to P. A mapping $M : \Im \times \Im \to E$ is called a cone metric on $\Im$ if it satisfies the following properties:

(M1) $M(y,\alpha) \succ \theta$, for all $y,\alpha \in \Im$, and $M(y,\alpha) = \theta$ iff $y = \alpha$

(M2) $M(y,\alpha) = M(\alpha,y)$, for all $y,\alpha \in \Im$

(M3) $M(y,\alpha) \preceq M(y,\eta) + M(\eta,\alpha)$, for all $y,\alpha,\eta \in \Im$

Moreover, the pair $(\Im, M)$ is called a cone metric space.

Definition 2.7. [12] Let $\Im$ be a nonempty set, E be a real Banach space, P be a cone in E, and $\preceq$ is a partial ordering in E with respect to P. A mapping $\mathcal{G} : \Im \times \Im \times \Im \to E$ satisfying, for all $x,\zeta,z,\mu \in \Im$,

i. $\mathcal{G}(x,\zeta,z) = \theta$ if $x = \zeta = z$

ii. $\theta \prec \mathcal{G}(x,x,\zeta)$ if $x \neq \zeta$

iii. $\mathcal{G}(x,x,\zeta) \preceq \mathcal{G}(x,\zeta,z)$ if $\zeta \neq z$

iv. $\mathcal{G}(x,\zeta,z) = \mathcal{G}(x,z,\zeta) = \mathcal{G}(\zeta,x,z) = \cdots$ (Symmetric in all three variables)

v. $\mathcal{G}(x,\zeta,z) \preceq \mathcal{G}(x,\mu,\mu) + \mathcal{G}(\mu,\zeta,z)$

is called a G-cone metric and $(\Im,\mathcal{G})$ is called a G-cone metric space.

Definition 2.8. [18] Let $\Im$ be a nonempty set, Y be a real Hausdorff tvs, and $\preceq$ is a partial ordering in Y with respect to a cone P. A vector-valued mapping $\mathcal{T} : \Im \times \Im \to \Im$ is called a tvs-cone metric if it satisfies

($\mathcal{T}$1) $\mathcal{T}(x,\eta) \succ \theta$, for all $x,\eta \in \Im$, and $\mathcal{T}(x,\eta) = \theta$ iff $x = \eta$

($\mathcal{T}$2) $\mathcal{T}(x,\eta) = \mathcal{T}(\eta,y)$, for all $x,\eta \in \Im$

($\mathcal{T}$3) $\mathcal{T}(x,\eta) \preceq \mathcal{T}(x,\alpha) + \mathcal{T}(\alpha,\eta)$, for all $x,\eta,\alpha \in \Im$

Moreover, the pair $(\Im,\mathcal{T})$ is called a tvs-cone metric space.

Definition 2.9. [19] Let $\Im$ be a nonempty set, Y be a tvs, $\preceq$ be a partial ordering in Y with respect to cone P, and $\mathcal{G} : \Im \times \Im \times \Im \to Y$ be a mapping satisfying the following conditions:

(G1) $\mathcal{G}(x,\zeta,z)=\theta$ if $x=\zeta=z$

(G2) $\theta \prec \mathcal{G}(x,x,\zeta)$ if $x\neq\zeta$, for all $x,\zeta\in\Im$

(G3) $\mathcal{G}(x,x,\zeta)\preceq\mathcal{G}(x,\zeta,z)$ if $\zeta\neq z$

(G4) $\mathcal{G}(x,\zeta,z)=\mathcal{G}(x,z,\zeta)=\mathcal{G}(\zeta,x,z)=\cdots$ (Symmetric in all three variables)

(G5) $\mathcal{G}(x,\zeta,z)\preceq\mathcal{G}(x,\mu,\mu)+\mathcal{G}(\mu,\zeta,z)$, for all $x,\zeta,z,\mu\in\Im$

Then, G is called a tvs-G cone metric, and the pair $(\Im,\mathcal{G})$ is called a tvs-G-cone metric space.

Definition 2.10. [19] Let $(\Im,\mathcal{G})$ be a tvs-G-cone metric space and $\{x_n\}$ be a sequence in $\Im$. Then,

i. $\{x_n\}$ is said to be tvs-G-cone convergent to x if, for all $\alpha\in Y$ with $0\prec\prec\alpha$, there is a $K\in\mathbb{N}$ such that $\mathcal{G}(x_m,x_n,x)\prec\prec\alpha$, for all $m,n\geq K$ and we write $\lim\limits_{n\to\infty}x_n=x$.

ii. $\{x_n\}$ is said to be a tvs-G-cone Cauchy if, for all $\alpha\in Y$ with $0\prec\prec\alpha$, there is a $K\in\mathbb{N}$ such that $G(x_m,x_n,x_k)\prec\prec c$, for all $m,n,k\geq K$.

iii. $\Im$ is called tvs-G-cone complete if every Cauchy sequence in $\Im$ converges to some element in $\Im$.

3. Main Result

In the following, we consider Y as a real locally convex Hausdorff tvs, P as a closed, proper, and convex pointed cone in Y with non-empty interior, $p\in\operatorname{int}(P)$, and $\preceq$ as a partial ordering with respect to P defined as above.

Definition 3.1. A tvs-G-cone metric space $(\Im,\mathcal{G})$ is said to be symmetric if, for all $\alpha,y\in\Im$, $\mathcal{G}(\alpha,y,y)=\mathcal{G}(y,\alpha,\alpha)$.

Note 3.2. In particular, if we take E as a real Banach space, then the definition of tvs-G-cone metrics is reduced to G cone metrics of Beg et al. [12]. Hence, for examples of symmetric and non-symmetric tvs-G-cone metric spaces, please see Examples 2.4 and 2.5 of Beg et al. [12].

Definition 3.3. Let $(\Im,\mathcal{G}_1)$ and $(\Im,\mathcal{G}_2)$ be two tvs-G-cone metric spaces and $\mathcal{F}:(\Im,\mathcal{G}_1)\to(\Im,\mathcal{G}_2)$ be a function. Then, $\mathcal{F}$ is

i. tvs-G-cone continuous at a point $\xi\in\Im$ if, for any $\alpha\succ\succ\theta$, there exists $\beta\succ\succ\theta$ such that, for all $\xi,y,c\in\Im$, $\mathcal{G}_2(\mathcal{F}\xi,\mathcal{F}y,\mathcal{F}c)\prec\prec\alpha$ if $\mathcal{G}_1(\xi,y,c)\prec\prec\beta$.

ii. tvs-G-cone sequentially continuous at a point $\xi\in\Im$ if $\{\xi_n\}$ is tvs-G-cone converges to ξ implies $\{\mathcal{F}(\xi_n)\}$ is tvs-G-cone converges to $\mathcal{F}(\xi)$.

In the following theorem, we show that there is a relationship between a tvs-G cone metric space and a G-metric space.

Theorem 3.4. Let $\Im$ be a nonempty set and $(\Im,\mathcal{G})$ be a tvs-G cone metric space. Define a mapping $M_{\mathcal{G}}:\Im\times\Im\times\Im\to\mathbb{R}_{\geq 0}$ by $M_{\mathcal{G}}=\zeta_p\ o\ \mathcal{G}$ where $p\in\operatorname{int}(P)$ in Y. Then, $M_{\mathcal{G}}$ is a G-metric on $\Im$.

Proof.
Let $\Im$ be a nonempty set and $(\Im,\mathcal{G})$ be a tvs-G cone metric space. Define a mapping $M_{\mathcal{G}}:\Im\times\Im\times\Im\to\mathbb{R}_{\geq 0}$ by $M_{\mathcal{G}}=\zeta_p\ o\ \mathcal{G}$ where $p\in\operatorname{int}(P)$ in Y.

i. If $x=\mu=z$, then $\mathcal{G}(x,\mu,z)=\theta$. Therefore, $M_{\mathcal{G}}(x,\mu,z)=\zeta_p(\mathcal{G}(x,\mu,z))=0$, since $\zeta_p(\theta)=0$. Thus, (G1) holds.

ii. If $\alpha\neq y$, then $\mathcal{G}(\alpha,\alpha,y)\succ\theta$. Thus,

$$M_{\mathcal{G}}(x,y,z)=(\zeta_p o\mathcal{G})(x,x,y)=\zeta_p(\mathcal{G}(x,x,y))\geq\zeta_p(\theta)=0$$

Hence, (G2) holds for $M_{\mathcal{G}}$.

iii. Since $\mathcal{G}(\alpha,\alpha,y) \preceq \mathcal{G}(\alpha,y,c)$, for $\alpha, y \neq c \in \Im$, thus $\zeta_p(\mathcal{G}(\alpha,\alpha,y)) \leq \zeta_p(\mathcal{G}(\alpha,y,c))$ and hence $M_{\mathcal{G}}(\alpha,\alpha,y) \leq M_{\mathcal{G}}(\alpha,y,c)$, for $\alpha, y \neq c \in \Im$. Therefore, $M_{\mathcal{G}}$ satisfies the condition (G3).

iv. (G4) is valid for $M_{\mathcal{G}}$, since $\mathcal{G}$ is symmetric in all three variables implies $M_{\mathcal{G}}$ is.

v. For all $\alpha, y, a, \mu \in \Im$, we have $\mathcal{G}(\alpha,y,a) \preceq \mathcal{G}(\alpha,\mu,\mu) + \mathcal{G}(\mu,y,a)$ which implies

$$\zeta_p(\mathcal{G}(\alpha,y,a)) \leq \zeta_p(\mathcal{G}(\alpha,\mu,\mu) + \mathcal{G}(\mu,y,a)) \leq \zeta_p(\mathcal{G}(\alpha,\mu,\mu)) + \zeta_p(\mathcal{G}(\mu,y,a))$$

Therefore, $M_{\mathcal{G}}(\alpha,y,a) \leq M_{\mathcal{G}}(\alpha,\mu,\mu) + M_{\mathcal{G}}(\mu,y,a)$, for all $\alpha, y, a, \mu \in \Im$ and thus (G5) is valid.

Hence, $M_{\mathcal{G}}$ is a G-metric on $\Im$ and the pair $(\Im, M_{\mathcal{G}})$ is a G-metric space. □

Corollary 3.5. If $\mathcal{G}$ is a G-cone metric on $\Im$ in the sense of Beg et al. [12], then $M_{\mathcal{G}} = \zeta_p \; o \; \mathcal{G}$ is a G-metric on $\Im$.

Proof.
In the above theorem, in particular, if we take Y as a real Banach space, then the result can be concluded from the above. □

Next theorem establishes the relation between the notions of convergence of sequences in tvs-G cone metric spaces and G-metric spaces.

Theorem 3.6. Suppose that $\mathcal{G}$ is a tvs-G cone metric and $M_{\mathcal{G}}$ is a G-metric on $\Im$ where $M_{\mathcal{G}}$ is defined in Theorem 3.4. Then,

i. A sequence $\{\eta_n\}$ converges in $(\Im, \mathcal{G})$ iff $\{\eta_n\}$ converges in $(\Im, M_{\mathcal{G}})$.

ii. A sequence $\{\eta_n\}$ is a Cauchy sequence in $(\Im, \mathcal{G})$ iff $\{\eta_n\}$ is Cauchy in $(\Im, M_{\mathcal{G}})$.

iii. $(\Im, \mathcal{G})$ is complete iff $(\Im, M_{\mathcal{G}})$ is complete.

Proof.
i. $(\Rightarrow)$: Assume that $\{\eta_n\}$ converges to η in $(\Im, \mathcal{G})$. Let $\epsilon > 0$ be arbitrary. For any $p \succ\succ \theta$ in Y, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\begin{aligned} \mathcal{G}(\eta_n,\eta_n,\eta) \prec\prec p\epsilon \quad &\Longrightarrow \quad \mathcal{G}(\eta_n,\eta_n,\eta) \in p\epsilon - \operatorname{int}(P) \\ &\Longrightarrow \quad \zeta_p(\mathcal{G}(\eta_n,\eta_n,\eta)) < \epsilon \\ &\Longrightarrow \quad (\zeta_p o \mathcal{G})(\eta_n,\eta_n,\eta)) < \epsilon \end{aligned}$$

That is, $M_{\mathcal{G}}(\eta_n,\eta_n,\eta) < \epsilon$, for all $m, n \geq N$, which implies $\{\eta_n\}$ converges to η in $(\Im, M_{\mathcal{G}})$.

$(\Leftarrow)$: Assume that a sequence $\{\eta_n\}$ converges to η in $(\Im, M_{\mathcal{G}})$. Let $c \succ\succ \theta$ in Y be arbitrary. Take $p \in intP$ and $\epsilon > 0$ be such that $p\epsilon \prec\prec c$. Since $\{\eta_n\}$ converges to η in $(\Im, M_{\mathcal{G}})$, thus there exists $N \in \mathbb{N}$ such that, for all $m, n \geq N$,

$$\begin{aligned} M_{\mathcal{G}}(\eta_n,\eta_n,\eta) < \epsilon \quad &\Longrightarrow \quad (\zeta_p o \mathcal{G})(\eta_n,\eta_n,\eta) < \epsilon \\ &\Longrightarrow \quad \zeta_p(\mathcal{G}(\eta_n,\eta_n,\eta)) < \epsilon \\ &\Longrightarrow \quad \mathcal{G}(\eta_n,\eta_n,\eta) \in p\epsilon - \operatorname{int}(P) \end{aligned}$$

which implies $\mathcal{G}(\eta_n,\eta_n,\eta) \prec\prec p\epsilon \prec\prec c$, for all $m, n \geq N$, and thus $\{\eta_n\}$ converges to η in $(\Im, \mathcal{G})$.

Proof of *ii* can be derived in a similar way of *i*, and *iii* is immediate consequence of *i* and *ii*. □

Theorem 3.7. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two tvs-G-cone metrics on $\Im$ and M_{G_1}, respectively, and M_{G_2} is the induced G-metrics on $\Im$, as defined in Theorem 3.4. Then, a function $\mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2)$ is tvs-G-cone continuous iff $\mathcal{T} : (\Im, M_{\mathcal{G}_1}) \to (\Im, M_{\mathcal{G}_2})$ is G-continuous.

Proof.
($\Leftarrow$): Assume that $\mathcal{T}$ is G-continuous. Let $c \succ\succ \theta$ in Y. Take $p \in \text{int}(P)$ and $\epsilon > 0$ be such that $p\epsilon \prec\prec c$. Then, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$M_{\mathcal{G}_2}(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } M_{\mathcal{G}_1}(a, b, c) < \delta \tag{1}$$

For $p \in \text{int}(P)$ and $\delta > 0$, there exists an $e \succ\succ \theta$ be such that $p\delta \prec\prec e$. From Relation 1, it follows that

$$\begin{aligned}(\zeta_p o \mathcal{G}_2)(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } (\zeta_p o \mathcal{G}_1)(a, b, c) < \delta &\Longrightarrow \zeta_p(\mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c)) < \epsilon \text{ if } \zeta_p(\mathcal{G}_1(a, b, c)) < \delta \\ &\Longrightarrow \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \in p\epsilon - \text{int}(P) \text{ if } G_1(a, b, c) \in p\delta - \text{int}(P) \\ &\Longrightarrow \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \prec\prec p\epsilon \prec\prec c \text{ if } \mathcal{G}_1(a, b, c) \prec\prec p\delta \prec\prec e\end{aligned}$$

Therefore, $\mathcal{T}$ is tvs-G-cone continuous.

($\Rightarrow$): Assume that $\mathcal{T}$ is tvs-G-cone continuous. Let $\epsilon > 0$. Then, for any $p \in \text{int}(P)$, there exists a $\delta > 0$ such that

$$\begin{aligned}\mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \prec\prec p\epsilon \text{ if } \mathcal{G}_1(a, b, c) \prec\prec p\delta &\Longrightarrow \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \in p\epsilon - \text{int}(P) \text{ if } \mathcal{G}_1(a, b, c) \in p\delta - \text{int}(P) \\ &\Longrightarrow \mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) \in p\epsilon - \text{int}(P) \text{ if } \mathcal{G}_1(a, b, c) \in p\delta - \text{int}(P) \\ &\Longrightarrow \zeta_p(\mathcal{G}_2(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c)) < \epsilon \text{ if } \zeta_p(\mathcal{G}_1(a, b, c)) < \delta \\ &\Longrightarrow (\zeta_p \ o \ \mathcal{G}_2)(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } (\zeta_p \ o \ \mathcal{G}_1)(a, b, c) < \delta \\ &\Longrightarrow M_{\mathcal{G}_2}(\mathcal{T}a, \mathcal{T}b, \mathcal{T}c) < \epsilon \text{ if } M_{\mathcal{G}_1}(a, b, c) < \delta.\end{aligned}$$

Hence, $\mathcal{T}$ is G-continuous. □

Theorem 3.8. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two tvs-G-cone metrics on $\Im$ and $\mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2)$ be a function. Then, $\mathcal{T}$ is tvs-G-cone continuous on $\Im$ iff $\mathcal{T}$ is tvs-G-cone sequentially continuous on $\Im$.

Proof.
For $p \in \text{int}(P)$ in Y, the mapping $M_{\mathcal{G}_i} = \zeta_p o \mathcal{G}_i$ such that $i \in \{1, 2\}$ are the induced G-metrics on $\Im$. Then,

$$\begin{aligned}\mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2) \text{ is tvs-G-cone continuous on } \Im &\iff \mathcal{T} : (\Im, M_{\mathcal{G}_1}) \to (\Im, M_{\mathcal{G}_2}) \text{ is G-cone continuous on } \Im \\ &\iff \mathcal{T} : (\Im, M_{\mathcal{G}_1}) \to (\Im, M_{\mathcal{G}_2}) \text{ is G-cone sequentially continuous on } \Im \\ &\iff \mathcal{T} : (\Im, \mathcal{G}_1) \to (\Im, \mathcal{G}_2) \text{ is tvs-G-cone sequentially continuous on } \Im\end{aligned}$$

□

Theorem 3.9. Let $\mathcal{G}$ be a tvs-G-cone metric space on $\Im$. Then, a mapping $p : \Im \times \Im \to Y$ defined by $p(x, \xi) = \mathcal{G}(x, x, \xi) + \mathcal{G}(x, \xi, \xi)$, for all $x, \xi \in \Im$, is a tvs-cone-metric on $\Im$.

Proof.
Let $\mathcal{G}$ be a tvs-G-cone metric space on $\Im$. Define a mapping $p : \Im \times \Im \to Y$ by $p(x, \xi) = \mathcal{G}(x, x, \xi) + \mathcal{G}(x, \xi, \xi)$, for all $x, \xi \in \Im$.

i. Clearly $p(x, \xi) \succ \theta$, for all $x, \xi \in \Im$ and

$$\begin{aligned}p(x, \xi) = \theta &\iff \mathcal{G}(x, x, \xi) + \mathcal{G}(x, \xi, \xi) = \theta \\ &\iff \mathcal{G}(x, x, \xi) = \theta \text{ and } \mathcal{G}(x, \xi, \xi) = \theta \\ &\iff x = \xi\end{aligned}$$

Therefore, ($\mathcal{T}$1) holds.

ii. ($\mathcal{T}$2) holds trivially since $\mathcal{G}$ is symmetric in its all three variables.

iii. $\mathcal{G}$ satisfies the inequality $\mathcal{G}(x,\xi,z) \preceq \mathcal{G}(x,a,a)+\mathcal{G}(a,\xi,z)$, for all $x,\xi,z,a \in \Im$. Therefore, for all $x,\xi,a \in \Im$,

$$\begin{aligned} p(x,\xi) = \mathcal{G}(x,x,\xi)+\mathcal{G}(x,\xi,\xi) &= \mathcal{G}(\xi,x,x)+\mathcal{G}(x,\xi,\xi), \quad \text{(since } \mathcal{G} \text{ is symmetric)} \\ &\preceq \mathcal{G}(\xi,a,a)+\mathcal{G}(a,x,x)+\mathcal{G}(x,a,a)+\mathcal{G}(a,\xi,\xi) \\ &= \mathcal{G}(x,x,a)+\mathcal{G}(x,a,a)+\mathcal{G}(\xi,a,a)+\mathcal{G}(a,\xi,\xi) \\ &= p(x,a)+p(a,\xi) \end{aligned}$$

Thus, p satisfies the condition $(\mathcal{T}3)$.

This shows that p is a tvs-cone-metric on $\Im$. □

Afterward, we show that fixed point theorems on tvs-G-cone metric spaces can be presented via G-metric spaces with the help of the scalarization function ζ_p.

Theorem 3.10. Suppose that $\mathcal{G}$ is a complete tvs-G cone metric and $M_{\mathcal{G}}$ be the induced G-metric on $\Im$. If $\mathcal{T}:\Im \to \Im$ is a mapping satisfying either

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)+n\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z) \tag{2}$$

or

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,x)+n\mathcal{G}(\xi,\xi,\mathcal{T}\xi)+r\mathcal{G}(z,z,\mathcal{T}z) \tag{3}$$

for all $x,\xi,z \in \Im$ where $0<l+m+n+r<1$, then $\mathcal{T}$ has a unique fixed point.

Proof.
For any $p \in \text{int}(P)$ in Y, we consider the function ζ_p. Then, by Theorem 3.4, $M_{\mathcal{G}} = \zeta_p o \mathcal{G}$ is a G-metric on $\Im$. Since $(\Im,\mathcal{G})$ is tvs-G-cone complete, then $(\Im, M_{\mathcal{G}})$ is also G-complete by the Theorem 3.6. Let $\mathcal{T}$ satisfies Condition 2. Then, for all $x,\xi,z \in \Im$, Lemma 2.1 implies if

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)+n\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)$$

then

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \zeta_p(l\mathcal{G}(x,\xi,z)+m\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)+n\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z))$$

Thus,

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq l\zeta_p(\mathcal{G}(x,\xi,z))+m\zeta_p(\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x))+n\zeta_p(\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi))+r\zeta_p(\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z))$$

Hence,

$$(\zeta_p o\mathcal{G})(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \leq l(\zeta_p o\mathcal{G})(x,\xi,z)+m(\zeta_p o\mathcal{G})(x,\mathcal{T}x,\mathcal{T}x)+n(\zeta_p o\mathcal{G})(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+r(\zeta_p o\mathcal{G})(z,\mathcal{T}z,\mathcal{T}z)$$

Therefore,

$$M_{\mathcal{G}}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \leq lM_{\mathcal{G}}(x,\xi,z)+mM_{\mathcal{G}}(x,\mathcal{T}x,\mathcal{T}x)+nM_{\mathcal{G}}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)+rM_{\mathcal{G}}(z,\mathcal{T}z,\mathcal{T}z)$$

This shows that $\mathcal{T}$ satisfies Condition 2.1 of Theorem 2.1 [20]. Since $(\Im, M_{\mathcal{G}})$ is a complete G-metric space, the existence and uniqueness of fixed point $\mathcal{T}$ follows from the Theorem 2.1 [20] in G-metric spaces. Consequently, $\mathcal{T}$ has a unique fixed point in $(\Im,\mathcal{G})$. Similarly, we can draw the conclusion if $\mathcal{T}$ satisfies Condition 3. □

Note 3.11. In particular, when we take $Y=E$, a real Banach space, the above theorem reduces to the theorem of Beg et al. [12].

Theorem 3.12. Suppose that $(\Im,\mathcal{G})$ is a complete tvs-G cone metric space and $M_{\mathcal{G}}$ is the induced

G-metric on $\Im$. If $\mathcal{T}$ is a self mapping on $\Im$ satisfying either of the following conditions

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \preceq \kappa\{\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x)\} \tag{4}$$

or

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \preceq \kappa\{\mathcal{G}(x,x,\mathcal{T}\xi)+\mathcal{G}(\xi,\xi,\mathcal{T}x)\} \tag{5}$$

for all $x,\xi \in \Im$ where $0<\kappa\leq \frac{1}{2}$, then $\mathcal{T}$ has a unique fixed point in $\Im$ and $\mathcal{T}$ is tvs-G-cone continuous on $\Im$.

Proof.
For any $p \in \mathrm{int}(P)$ in Y, we consider the function ζ_p. Then, by Theorem 3.4, $M_{\mathcal{G}} = \zeta_p o\mathcal{G}$ is a G-metric on $\Im$. If, for all $x,\xi\in\Im$, $\mathcal{T}$ satisfies Condition 4, then Lemma 2.1 gives

$$\begin{aligned}
\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \preceq \kappa\{\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x)\} &\Longrightarrow \zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \zeta_p(\kappa\{\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x)\})\\
&\Longrightarrow \zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \kappa\zeta_p(\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi)+\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x))\\
&\Longrightarrow \zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \kappa\{\zeta_p(\mathcal{G}(x,\mathcal{T}\xi,\mathcal{T}\xi))+\zeta_p(\mathcal{G}(\xi,\mathcal{T}x,\mathcal{T}x))\}\\
&\Longrightarrow (\zeta_p o\mathcal{G})(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi) \leq \kappa\{(\zeta_p o\mathcal{G})(x,\mathcal{T}\xi,\mathcal{T}\xi)+(\zeta_p o\mathcal{G})(\xi,\mathcal{T}x,\mathcal{T}x)\}\\
&\Longrightarrow M_{\mathcal{G}}(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}\xi)) \leq \kappa\{M_{\mathcal{G}}(x,\mathcal{T}\xi,\mathcal{T}\xi)+M_{\mathcal{G}}(\xi,\mathcal{T}x,\mathcal{T}x)\}
\end{aligned}$$

This shows that $\mathcal{T}$ satisfies Condition 2.49 [20] in G-metric space $(\Im, M_{\mathcal{G}})$. Since $(\Im,\mathcal{G})$ is tvs-G-cone complete, by the Theorem 3.6, $(\Im, M_{\mathcal{G}})$ is G-complete. Therefore, by the Theorem 2.8, $\mathcal{T}$ has a unique fixed point in $(\Im, M_{\mathcal{G}})$ and $\mathcal{T}$ is G-continuous. Hence, $\mathcal{T}$ has a unique fixed point in $(\Im,\mathcal{G})$ and $\mathcal{T}$ is tvs-G-cone continuous by the Theorem 3.7. If $\mathcal{T}$ satisfies the Condition 5, then the conclusion can be drawn in a similar way. $\square$

Theorem 3.13. Let $\mathcal{G}$ be a complete tvs-G cone metric and $M_{\mathcal{G}}$ be the induced G-metric on $\Im$. If $\mathcal{T}$ is a self mapping on $\Im$ satisfying either of the conditions

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq \kappa \max\{\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x),\ \mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ \mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)\} \tag{6}$$

or

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq \kappa \max\{\mathcal{G}(x,x,\mathcal{T}x),\ \mathcal{G}(\xi,\xi,\mathcal{T}\xi),\ \mathcal{G}(z,z,\mathcal{T}z)\} \tag{7}$$

for all $x,\xi,z\in\Im$ where $0<\kappa\leq 1$, then $\mathcal{T}$ has a unique fixed point in $\Im$ and $\mathcal{T}$ is tvs-G-cone continuous on $\Im$.

Proof.
For any $p \in \mathrm{int}(P)$ in Y, we consider the function ζ_p. Then, by Theorem 3.4, $M_{\mathcal{G}} = \zeta_p o\mathcal{G}$ is a G-metric on $\Im$. Since $\mathcal{G}$ is a tvs-G-cone complete metric on $\Im$, thus $(\Im, M_{\mathcal{G}})$ is also G-complete. If, for all $x,\xi,z\in\Im$, $\mathcal{T}$ satisfies Condition 6, then applying Lemma 2.1, if

$$\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \preceq \kappa \max\{\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x),\ \mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ \mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)$$

then

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \zeta_p(\kappa \max\{\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x),\ \mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ \mathcal{G}(z,\mathcal{T}z,\mathcal{T}z)\}$$

Thus,

$$\zeta_p(\mathcal{G}(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z)) \leq \kappa \max\{\zeta_p(\mathcal{G}(x,\mathcal{T}x,\mathcal{T}x)),\ \zeta_p(\mathcal{G}(\xi,\mathcal{T}\xi,\mathcal{T}\xi)),\ \zeta_p(\mathcal{G}(z,\mathcal{T}z,\mathcal{T}z))\}$$

Hence,

$$(\zeta_p o\mathcal{G})(\mathcal{T}x,\mathcal{T}\xi,\mathcal{T}z) \leq \kappa \max\{(\zeta_p o\mathcal{G})(x,\mathcal{T}x,\mathcal{T}x),\ (\zeta_p o\mathcal{G})(\xi,\mathcal{T}\xi,\mathcal{T}\xi),\ (\zeta_p o\mathcal{G})(z,\mathcal{T}z,\mathcal{T}z)\}$$

Therefore,

$$M_{\mathcal{G}}(\mathcal{T}x, \mathcal{T}\xi, \mathcal{T}z)) \leq \kappa \max\{M_{\mathcal{G}}(x, \mathcal{T}x, \mathcal{T}x),\ M_{\mathcal{G}}(\xi, \mathcal{T}\xi, \mathcal{T}\xi),\ M_{\mathcal{G}}(z, \mathcal{T}z, \mathcal{T}z)\}$$

This shows that $\mathcal{T}$ satisfies Condition 2.19 [20] in the complete G-metric space $(\Im, M_{\mathcal{G}})$. Thus, Theorem 2.3 [20] ensures that $\mathcal{T}$ has a unique fixed point in $(\Im, M_{\mathcal{G}})$. Therefore, $\mathcal{T}$ has a unique fixed point in $(\Im, \mathcal{G})$. Moreover, Theorem 2.3 [20] shows that $\mathcal{T}$ is continuous in $(\Im, M_{\mathcal{G}})$. Since, by the Theorem 3.7, continuity of $\mathcal{T}$ in $(\Im, M_{\mathcal{G}})$ implies the continuity of $\mathcal{T}$ in $(\Im, \mathcal{G})$, thus $\mathcal{T}$ is tvs-G-cone continuous. We can prove the theorem similarly, if $\mathcal{T}$ satisfies Condition 7 in $(\Im, \mathcal{G})$. $\square$

4. Conclusion

In this paper, we investigated the relationship between the vector-valued version and scalar-valued version of fixed point theorems of generalized cone-metric spaces and G-metric spaces. We showed a correspondence between G-metric and tvs-G cone metric with the help of a scalarization function defined on a locally convex Hausdorff topological vector space. If we take a real Banach space E instead of locally convex Hausdorff space X and P is the cone in E as defined in [8]. Then, all the results for X hold for G-cone metric spaces. Hence, these theorems extended some results of G-cone metric spaces and proved a correspondence between any G-cone metric space and the G-metric space. The remarkable point is that all of these are possible only because of the non-empty interior of P. Like Theorems 3.10 and 3.12, the equivalence between the non-negative scalar-valued version and vector-valued versions of these fixed point theorems can be proved easily.

Shortly, new generalized metric spaces are expected to be introduced, and studies on fixed point theory are expected to continue. We hope that the results of this paper will be helpful to researchers in this field for further research. Researchers may study the equivalence between the vector-valued and scalar-valued versions of fixed point results in new generalized metric spaces, getting inspired by the relations provided herein between the tvs-G cone metric spaces and G-metric spaces.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Acknowledgement

The first author is thankful to University Grant Commission (UGC), New Delhi, India, for awarding her senior research fellowship, Grant number: 1221/ (CSIRNETJUNE2019). We are thankful to the Department of Mathematics, Siksha-Bavana, Visva-Bharati.

References

[1] S Gahler, *2-Metrische Raume Und Ihre Topologische Struktur*, Mathematische Nachrichten 26 (1-4) (1963) 115–148.

[2] S. Czerwik, *Contraction Mappings in b-Metric Spaces*, Acta Mathematica et Informatica Universitatis Ostraviensis 1 (1) (1993) 5–11.

[3] S. Czerwik, *Nonlinear Set-Valued Contraction Mappings in b-Metric Spaces*, Atti del Seminario Matematico e Fisico dell'Universita di Modena 46 (1998) 263–276.

[4] W. Kirk, N. Shahzad, Fixed Point Theory in Distance Spaces, Springer, Cham, 2014.

[5] B. C. Dhage, *Generalized Metric Spaces Mappings with Fixed Point*, Bulletin of Calcutta Mathematical Society 84 (1992) 329–336.

[6] Z. Mustafa, B. Sims, *A New Approach to Generalized Metric Spaces*, Journal of Nonlinear Convex Analysis 7 (2) (2006) 289–297.

[7] S. Sedghi, N. Shobe, A. Aliouche, *A Generalization of Fixed Point Theorems in S-Metric Spaces*, Matematoqki Vesnik 64 (3) (2012) 258–266.

[8] H. L. Guang, Z. Xian, *Cone Metric Space and Fixed Point Theorems of Contractive Mapping*, Journal of Mathematical Analysis and Applications 322 (2) (2007) 1468–1476.

[9] A. Das, T. Bag, *A Study on Parametric S-metric Spaces*, Communications in Mathematics and Applications 13 (3) (2022) 921–933.

[10] A. Das, T. Bag, *A Generalization to Parametric Metric Spaces*, International Journal of Nonlinear Analysis and Application 14 (1) (2023), 229–244.

[11] G. Rano, T. Bag, S.K. Samanta, *Fuzzy Metric Space and Generating Space of Quasi-Metric Family*, Annals of Fuzzy Mathematics and Informatics 11 (2016) 183–195.

[12] I. Beg, M. Abbas, T. Nazir, *Generalized Cone Metric Spaces*, The Journal of Nonlinear Sciences and Its Applications 3 (1) (2010) 21–31.

[13] G. Y. Chen, X. X. Huang, X. Q. Yang, Vector Optimization: Set-Valued and Variational Analysis, Springer Berlin, Heidelberg, 2005.

[14] W. S. Du, *On Some Nonlinear Problems Induced by an Abstract Maximal Element Principle*, Journal of Mathematical Analysis and Applications 347 (2) (2008) 391–399.

[15] C. Gerth, P. Weidner, *Nonconvex Separation Theorems and Some Applications in Vector Optimization*, Journal of Optimization Theory and Applications 67 (2) (1990) 297–320.

[16] A. Göpfert, C. Tammer, C. Z. Linescu, *On the Vectorial Ekeland's Variational Principle and Minimal Points in Product Spaces*, Nonlinear Analysis: Theory, Methods and Applications 39 (7) (2000) 909–922.

[17] A. Göpfert, C. Tammer, H. Riahi, C. Z. Linescu, Variational Methods in Partially Ordered Spaces, Springer, New York, 2003.

[18] W. Du, *A Note on Cone Metric Fixed Point Theory and Its Equivalence*, Nonlinear Analysis: Theory, Methods and Applications 72 (5) (2010) 2259–2261.

[19] I. J. Lin, C. M. Chen, C. H. Chen, T. Y. Cheng, *A Note on tvs-G-Cone Metric Fixed Point Theory*, Journal of Applied Mathematics 2012 (2012) Article ID 407071 10 pages.

[20] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces*, Fixed Point Theory and Applications 2008 (2008) Article Number 189870 12 pages.