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The new numerical solutions of conformable time fractional generalized Burgers equation with proportional delay

Oransal gecikmeli uyumlu zaman kesirli mertebeden genelleştirilmiş Burgers denkleminin yeni sayısal çözümleri

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Abstract

The conformable time-fractional partial differential equations with proportional delay are studied using two new methods: the conformable fractional q-homotopy analysis transform method and the conformable Shehu homotopy perturbation method. The numerical solutions to this equation are graphed. Numerical simulations show that the proposed techniques are effective and trustworthy.

Keywords: Conformable q-homotopy analysis transform method, Conformable time-fractional generalized Burgers equation, Proportional delay.

Öz

Oransal gecikmeli uyumlu zaman-kesirli kısmi diferansiyel denklemler, iki yeni yöntem olan uyumlu kesirli q-homotopi analizi dönüşüm yöntemi ve uyumlu Shehu homotopi pertürbasyon yöntemi kullanılarak incelenir. Bu denklemin sayısal çözümleri grafiklerle gösterilmiştir. Sayısal simülasyonlar, önerilen tekniklerin etkili ve güvenilir olduğunu göstermektedir.

***Anahtar kelimeler**: Uyumlu q-homotopi analiz dönüşümü metodu, Uyumlu kesirli mertebeden genelleştirilmiş Burgers denklemi, Oransal gecikme.*

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1. Introduction

Fractional calculus (FC) extends integer order calculus to arbitrary order. It was discussed in an early communication between the eminent mathematicians Leibniz and L'Hospital around 1695. Because of its ability to provide an exact description for numerous sorts of non-linear events, numerous authors have begun to investigate fractional calculus in recent years. Fractional order differential equations are a type of differential equation that has non-local and genetic material property consequences. Many prominent academics investigated and defined the concept of fractional calculus, developing revolutionary definitions that provided the groundwork for fractional calculus (Liouville, 1832; Riemann, 1896; Caputo, 1969; Miller&Ross, 1993; Podlubny, 1999; Baleanu et al. 2012; Povstenko, 2015). Fractional partial differential equations are now commonly used in the creation of nonlinear models and the study of dynamical systems. Many phenomena, including chaos theory (Baleanu et al., 2017) financial models (Sweilam et al., 2017), a noisy environment (Liu et al., 2015), optics (Esen et al., 2018), and others, have been associated with fractional-order calculus theory (Veeresha et al., 2019a; Caponetto et al., 2010; Prakash et al., 2019; Veeresha et al., 2019b; Atangana et al., 2022; Shahzad et al., 2023; Iqbal et al., 2023; Iyanda et al., 2023, Hasan et al., 2023; Liaqat et al., 2023). The solutions of fractional differential equations are crucial in describing the properties of natural nonlinear systems. We apply a number of analytical and numerical techniques to obtain exact solutions to fractional differential equations characterizing nonlinear processes.

The investigation deals with the numerical solution of conformable time-fractional partial differential equations with proportional delay defined by

$$\begin{cases} D_t^{\alpha} w(x,t) = \psi\left(x, w(\rho_0 x, \sigma_0 t), \frac{\partial w(\rho_1 x, \sigma_1 t)}{\partial x}, \dots, \frac{\partial^m w(\rho_m x, \sigma_m t)}{\partial x^m}\right), \\ \qquad w^{(k)}(x,0) = \varphi_k(x). \end{cases} \tag{1}$$

where $\rho_i, \sigma_i \in (0,1)$ for all $i \in N$, φ_k is initial value, ψ differential operator and D_t^{α} is conformable time-fractional operator.

There are few publications about time fractional partial differential equatiions with proportional delay in the literature. These include the Chebyshev pseudospectral method (Zubik-Kawal, 2000), the homotopy analysis method (Alkan, 2022), the spectral collocation and waveform relaxation methods (Jackiewicz&Zubik-Kawal, 2006), and the iterated pseudospectral method (Mead&Zubik-Kawal, 2005). Abazari & Ganji (2011) were able to find approximate solutions to PDEs utilizing RDTM. These solutions involved proportional delay. Abazari & Ganji (2014) employed DTM to obtain analytical solutions to nonlinear integro-differential equations with proportional delay. These answers were obtained by solving the equations using DTM. Tanthanuch (2012) was successful in solving the non-homogeneous inviscid Burgers equation with proportional delay by employing a method known as group analysis. Analytical solutions to TFPDE with proportionate delay were found by using the homotopy perturbation approach by Sakar et al. (2016) and Biazar & Ghanbari (2012). Chen & Wang (2010) used the variational iteration method to solve a neutral functional-differential problem with proportional delays. Singh & Kumar (2017) accomplished their goal of finding an alternate approximation solution to the initial valued autonomous system of TFPDE with proportional delay by employing an additional variational iteration approach, abbreviated as AVIM. The fundamental objective of this research is to make two novel methodological suggestions: the conformable q-homotopy analysis transform method (Cq-HATM) and the conformable Shehu homotopy perturbation method (CSHPM).

2. Preliminaries

Now let's give the definitions to be used in the study.

Definition 2.1. Let a function $f: [0,\infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f order α is described by (Khalil et al., 2014; Abdeljawad, 2015; Ala et al., 2020; Gözütok et al., 2019)

$$T_{\alpha}(f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \tag{2}$$

for all $x > 0, \alpha \in (0,1]$.

Theorem 2.1. Let $\alpha \in (0,1]$ and f, g be α −differentiable at a point $x > 0$. Then (Khalil et al., 2014; Abdeljawad, 2015; Gözütok and Gözütok, 2017)

(i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$,
(3)

(ii) $T_\alpha(x^p) = px^{p-1}$, for all $p \in \mathbb{R}$, (4)

(iii) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$, (5)

(iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$, (6)

(v) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$. (7)

Definition 2.2. Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. Then, the conformable fractional Shehu transform (CFST) of order α of f is defined by (Benattia & Belghaba, 2021)

$$ {}_cS_\alpha[f(t)] = V_\alpha(s;u) = \int_0^\infty exp\left(\frac{-st^\alpha}{u\alpha}\right) f(t)t^{\alpha-1}dt. \tag{8}$$

Definition 2.3 Let $0 < \alpha \leq 1$, $f: [0,\infty) \to \mathbb{R}$ be real valued function. The conformable Shehu transform for the conformable fractional-order derivative of the function $f(t)$ is described by (Benattia & Belghaba, 2021)

$$V_\alpha[T_\alpha f(t)](v) = \frac{s}{u} V_\alpha(s;u) - f(0). \tag{9}$$

3. New numerical methods

We will introduce new methods.

3.1. Conformable q-homotopy analysis transform method

Consider the conformable time-fractional order nonlinear partial differential equation (CTFNPDE) with proportional delay to explain the fundamental idea of Cq-HATM:

$$ {}_tT_\alpha w(x,t) + Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) = f(x,t), t > 0, n-1 < \alpha \leq n, \tag{10}$$

where M is a linear operator, N is a nonlinear operator, $f(x,t)$ is a source term, $\rho_i, \sigma_i \in (0,1)$ and ${}_tT_\alpha$ is a conformable fractional derivative of order α.

Applying the conformable Laplace transform to Eq. (10) and utilizing the initial condition, then, we have

$$s\mathcal{L}_\alpha[w(x,t)] - w(x,0) + \mathcal{L}_\alpha[Mw(\rho_i x, \sigma_i t)] + \mathcal{L}_\alpha[Nw(\rho_i x, \sigma_i t)] = \mathcal{L}_\alpha[f(x,t)]. \tag{11}$$

Rearranging the last equation, then we get

$$\mathcal{L}_\alpha[w(x,t)] - \frac{1}{s}w(x,0) + \frac{1}{s}\mathcal{L}_\alpha[Mw(\rho_i x, \sigma_i t)] + \frac{1}{s}\mathcal{L}_\alpha[Nw(\rho_i x, \sigma_i t)] - \frac{1}{s}\mathcal{L}_\alpha[f(x,t)] = 0. \tag{12}$$

With the help of HAM, we can describe the nonlinear operator for real function $\varphi(x,t;q)$ as follows:

$$N[\varphi(x,t;q)] = \mathcal{L}_\alpha[\varphi(x,t;q)] - \frac{1}{s}\varphi(x,t;q)\,(0^+) + \frac{1}{s}(\mathcal{L}_\alpha[M\varphi(\rho_i x, \sigma_i t;q)]$$

$$+\mathcal{L}_\alpha[N\varphi(\rho_i x, \sigma_i t;q)] - \mathcal{L}_\alpha[f(x,t)]), \tag{13}$$

where $q\epsilon\left[0,\frac{1}{n}\right]$.

We construct a homotopy as follows:

$$(1-nq)\mathcal{L}_\alpha[\varphi(x,t;q)-w_0(x,t)]=hqH(x,t)N[\varphi(\rho_i x,\sigma_i t;q)], \tag{14}$$

where, $h\neq 0$ is an auxiliary parameter and $\mathcal{L}_\alpha$ represents conformable Laplace transform. For $q=0$ and $q=\frac{1}{n}$, the results of Eq. (14) are as follows:

$$\varphi(x,t;0)=w_0(x,t),\varphi\left(x,t;\frac{1}{n}\right)=w(x,t), \tag{15}$$

Thus, by amplifying q from 0 to $\frac{1}{n}$, then the solution $\varphi(x,t;q)$ converges from $w_0(x,t)$ to the solution $w(x,t)$.

Using the Taylor theorem around q and then expanding $\varphi(x,t;q)$, we get

$$\varphi(x,t;q)=w_0(x,t)+\sum_{i=1}^{\infty}w_m(x,t)q^m, \tag{16}$$

where

$$w_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;q)}{\partial q^m}|_{q=0}. \tag{17}$$

Eq. (16) converges at $q=\frac{1}{n}$ for the appropriate $w_0(x,t)$, n and h. Then, we have

$$w(x,t)=w_0(x,t)+\sum_{m=1}^{\infty}w_m(x,t)\left(\frac{1}{n}\right)^m. \tag{18}$$

If we differentiate the zeroth order deformation Eq. (14) m-times with respect to q and we divide by $m!$, respectively, then for $q=0$, we acquire

$$\mathcal{L}_\alpha[w_m(x,t)-k_m w_{m-1}(x,t)]=hH(x,t)\mathcal{R}_m(\vec{w}_{m-1}), \tag{19}$$

where the vectors are described by

$$\vec{w}_m=\{w_0(x,t),w_1(x,t),\dots,w_m(x,t)\}. \tag{20}$$

Applying the inverse conformable Laplace transform to Eq. (20), we get

$$w_m(x,t)=k_m w_{m-1}(x,t)+h\mathcal{L}_\alpha{}^{-1}[H(x,t)\mathcal{R}_m(\vec{w}_{m-1})], \tag{21}$$

where

$$\mathcal{R}_m(\vec{w}_{m-1})=\mathcal{L}_\alpha[w_{m-1}(x,t)]-\left(1-\frac{k_m}{n}\right)\frac{1}{s}w_0(x,t)+\frac{1}{s}\mathcal{L}_\alpha[Mw_{m-1}(\rho_i x,\sigma_i t)$$

$$+H_{m-1}(x,t)-f(x,t)], \tag{22}$$

and

$$k_m=\begin{cases}0, & m\leq 1,\\ n, & m>1.\end{cases} \tag{23}$$

Here, H_m is homotopy polynomial and presented by

$$H_m = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}|_{q=0} \text{ and } \varphi(x,t;q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \cdots. \quad (24)$$

Using Eqs. (21) - (22), we get

$$w_m(x,t) = (k_m + h)w_{m-1}(x,t) - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}w_0(x,t) + \mathrm{h}\mathcal{L}_\alpha{}^{-1}\left[\left(\frac{1}{s}\mathcal{L}_\alpha[Mw_{m-1}(\rho_i x, \sigma_i t)\right.\right.$$

$$\left.\left.+H_{m-1}(x,t) - f(x,t)]\right)\right]. \quad (25)$$

When q-HATM is used, the series solution is given by

$$w(x,t) = \sum_{i=0}^{\infty} w_m(x,t). \quad (26)$$

3.2. Conformable Shehu homotopy perturbation method

We analyze the CTFNPDE with proportional delay:

$${}_tT_\alpha w(x,t) + Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) = f(x,t), t > 0, n-1 < \alpha \leq n, \quad (27)$$

with initial condition

$$w(x,0) = a(x), \quad (28)$$

where M is a linear operator, N is a nonlinear operator, $f(x,t)$ is a source term, $\rho_i, \sigma_i \in (0,1)$ and ${}_tT_\alpha$ is a conformable fractional derivative of order α.

Applying the conformable fractional Shehu transform to Eq. (27) and using the initial condition, then we get

$$\frac{s}{u}\ {}_cS_\alpha[w(x,t)] - \sum_{m=0}^{k-1} w(x,0) + {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) - f(x,t)] = 0. \quad (29)$$

Eq. (29) is simplified, then we have

$${}_cS_\alpha[w(x,t)] - \frac{u}{s}a(x) + \frac{u}{s}\ {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) - f(x,t)] = 0. \quad (30)$$

When Eq. (30) is rearranged, it is obtained as

$${}_cS_\alpha[w(x,t)] = \frac{u}{s}a(x) - \frac{u}{s}\ {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t) - f(x,t)]. \quad (31)$$

When the inverse conformable fractional Shehu transform is implemented to both sides of Eq. (31), we have

$$w(x,t) = A(x,t) - \left({}_cS_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_cS_\alpha[Mw(\rho_i x, \sigma_i t) + Nw(\rho_i x, \sigma_i t)]\right\}, \quad (32)$$

where the term $A(x,t)$ emerges from the in-homogeneous term and initial conditions.

Applying the homotopy perturbation method yields

$$w(x,t) = \sum_{n=0}^{\infty} p^n w_n(x,t). \quad (33)$$

Now, let the nonlinear term be represented as

$$Nw(x,t) = \sum_{n=0}^{\infty} p^n H_n(w), \quad (34)$$

where $H_n(w)$ is defined by the form

$$H_n(w_0, w_1, \dots, w_n) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i w_i\right)\right]_{p=0}, n = 0,1,2, \dots \tag{35}$$

Substituting the Eqs. (33)-(34) into Eq. (32), it is obtained as

$$\sum_{n=0}^{\infty} p^n w_n(x,t) = A(x,t) - p\left\{\left({}_{c}S_\alpha\right)^{-1}\left[\frac{u}{s}\ {}_{c}S_\alpha\{M\sum_{n=0}^{\infty} p^n w_n(\rho_i x, \sigma_i t) + \sum_{n=0}^{\infty} p^n H_n(w)\}\right]\right\}. \tag{36}$$

Eq. (36) is the combination of the conformable fractional Shehu transform and the homotopy perturbation method. The coefficients of the same power terms of p is compared, then we have the following iterations.

$$p^0: w_0(x,t) = A(x,t), \tag{37}$$

$$p^1: w_1(x,t) = -\left({}_{c}S_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_{c}S_\alpha[M w_0(\rho_i x, \sigma_i t) + H_0(w)]\right\}, \tag{38}$$

$$p^2: w_2(x,t) = -\left({}_{c}S_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_{c}S_\alpha[M w_1(\rho_i x, \sigma_i t) + H_1(w)]\right\}, \tag{39}$$

$$p^3: w_3(x,t) = -\left({}_{c}S_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_{c}S_\alpha[M w_2(\rho_i x, \sigma_i t) + H_2(w)]\right\}, \tag{40}$$

$\vdots$

Thus, the series solution of the equation is obtained in the form

$$w(x,t) = \lim_{p\to 1} \sum_{m=0}^{\infty} p^m w_m(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots \tag{41}$$

4. Application

Consider the conformable time-fractional Burgers equation with proportional delay (Sakar et al., 2016; Singh & Kumar, 2017)

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} - \frac{1}{8}\frac{\partial w(x,t)}{\partial x} - w(x,t), \tag{42}$$

where $x, t \in [0,1], 0 < \alpha \leq 1$, subject to initial condition

$$w(x,0) = x^2. \tag{43}$$

Case (i) Cq-HATM solution

Implementing the conformable Laplace transform to Eq. (42) and using Eq. (43), then we get

$$\mathcal{L}_\alpha[w(x,t)] = \frac{1}{s}w(x,0) + \frac{1}{s}\mathcal{L}_\alpha\left[\frac{\partial^2 w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} - \frac{1}{8}\frac{\partial w(x,t)}{\partial x} - w(x,t)\right], \tag{44}$$

We define the nonlinear operators by using Eq. (44), as

$$N[\varphi(x,t;q)] = \mathcal{L}_\alpha[\varphi(x,t;q)] - \frac{1}{s}x^2 - \frac{1}{s}\mathcal{L}_\alpha\left[\frac{\partial^2 w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} - \frac{1}{8}\frac{\partial w(x,t)}{\partial x} - w(x,t)\right]. \tag{45}$$

By applying the proposed algorithm, the $m-th$ order deformation equations are defined by

$$\mathcal{L}_\alpha[w_m(x,t) - k_m w_{m-1}(x,t)] = h\mathcal{R}_m[\vec{w}_{m-1}], \tag{46}$$

where

$$\mathcal{R}_m[\vec{w}_{m-1}] = \mathcal{L}_\alpha[\vec{w}_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\frac{1}{s}x^2 - \frac{1}{s}\mathcal{L}_\alpha\left[\sum_{r=0}^{m-1}\frac{\partial^2 w_r\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_r\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} - \frac{1}{8}\frac{\partial w_{m-1}(x,t)}{\partial x}\right.$$

$$\left. - w_{m-1}(x,t)\right]. \quad (47)$$

On applying inverse conformable Laplace transform to Eq. (46), then we have

$$w_m(x,t) = k_m w_{m-1}(x,t) + h\mathcal{L}_\alpha^{-1}\{\mathcal{R}_m[\vec{w}_{m-1}]\}. \quad (48)$$

By the use of initial condition, then we get

$$w_0(x,t) = x^2. \quad (49)$$

To find the value of $w_1(x,t)$, putting $m = 1$ in Eq. (48), then we obtain

$$w_1(x,t) = hx^2\frac{t^\alpha}{\alpha}. \quad (50)$$

In the same way, if we put $m = 2$ in Eq. (48), we can obtain the value of $w_2(x,t)$

$$w_2(x,t) = (n+h)\left(hx^2\frac{t^\alpha}{\alpha}\right) - h^2\left(\frac{x}{2.2^\alpha} - \frac{x}{4} - x^2\right)\frac{t^{2\alpha}}{2\alpha^2}. \quad (51)$$

In this way, the other terms can be found. So, the Cq-HATM solution of the equaiton is given by

$$w(x,t) = w_0(x,t) + \sum_{m=1}^{\infty} w_m(x,t)\left(\frac{1}{n}\right)^m. \quad (52)$$

If we put $\alpha = 1, n = 1,\ h = -1$ in Eq. (52), then the obtained results $\sum_{m=1}^{M} w_m(x,t)\left(\frac{1}{n}\right)^m$ converges to the exact solution $w(x,t) = x^2e^{-t}$ of the equation when $M \to \infty$.

Case (ii) CSHPM solution

Applying the conformable Shehu transform to Eq. (42) and using Eq. (43), then we get

$$_cS_\alpha[w(x,t)] = \frac{u}{s}x^2 + \frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^2 w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} - \frac{1}{8}\frac{\partial w(x,t)}{\partial x} - w(x,t)\right]. \quad (53)$$

Applying the inverse conformable Shehu transform to Eq (53), then we obtain

$$w(x,t) = x^2 + \left({}_cS_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_cS_\alpha\left[\frac{\partial^2 w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} - \frac{1}{8}\frac{\partial w(x,t)}{\partial x} - w(x,t)\right]\right\}. \quad (54)$$

Now HPM is applied, then we have

$$\sum_{m=0}^{\infty} p^m w_m(x,t) = x^2 + p\left[\left({}_cS_\alpha\right)^{-1}\left\{\frac{u}{s}\ {}_cS_\alpha\left[\sum_{m=0}^{\infty} p^m H_m(w)\right.\right.\right.$$

$$\left.\left.\left. - \frac{1}{8}\sum_{m=0}^{\infty} p^m\frac{\partial w_m(x,t)}{\partial x} - \sum_{m=0}^{\infty} p^m w_m(x,t)\right]\right\}\right] \quad (55)$$

We get to the first few components of $H_m(w)$ by

$$H_0(w) = \frac{\partial^2 w_0\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_0\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x}, \quad (56)$$

$$H_1(w) = \frac{\partial^2 w_0\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_1\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} + \frac{\partial^2 w_1\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_0\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x}, \tag{57}$$

$$H_2(w) = \frac{\partial^2 w_0\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_2\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} + \frac{\partial^2 w_1\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_1\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x} + \frac{\partial^2 w_2\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x^2}\frac{\partial w_0\left(\frac{x}{2},\frac{t}{2}\right)}{\partial x}, \tag{58}$$

$$\vdots \tag{59}$$

Comparing the coefficients of the same powers of p, then we have

$$p^0: w_0(x,t) = x^2, H_0(w) = \frac{x}{4}, \tag{60}$$

$$p^1: w_1(x,t) = -x^2\frac{t^\alpha}{\alpha}, H_1(w) = \frac{-xt^\alpha}{2\alpha 2^\alpha}, \tag{61}$$

$$p^2: w_2(x,t) = -h^2\left(\frac{x}{2.2^\alpha} - \frac{x}{4} - x^2\right)\frac{t^{2\alpha}}{2\alpha^2}, H_2(w) = -\frac{h^2}{2}\left(\frac{1}{42^\propto} - \frac{1}{8} - \frac{x}{2}\right)\frac{t^{2\alpha}}{.2^{2\alpha+1}\alpha^2} + \frac{xt^{2\alpha}}{.2^{2\alpha+2}\alpha^2} + \frac{h^2}{2}\frac{xt^{2\alpha}}{.2^{2\alpha+3}\alpha^2}, \tag{62}$$

As a result, the solution to Eq. (42) for CSHPM is given by

$$w(x,t) = x^2 - x^2\frac{t^\alpha}{\alpha} - h^2\left(\frac{x}{2.2^\alpha} - \frac{x}{4} - x^2\right)\frac{t^{2\alpha}}{2\alpha^2}. \tag{63}$$

Figure 1 shows the graphs of Cq-HATM, exact solution and absolute error.

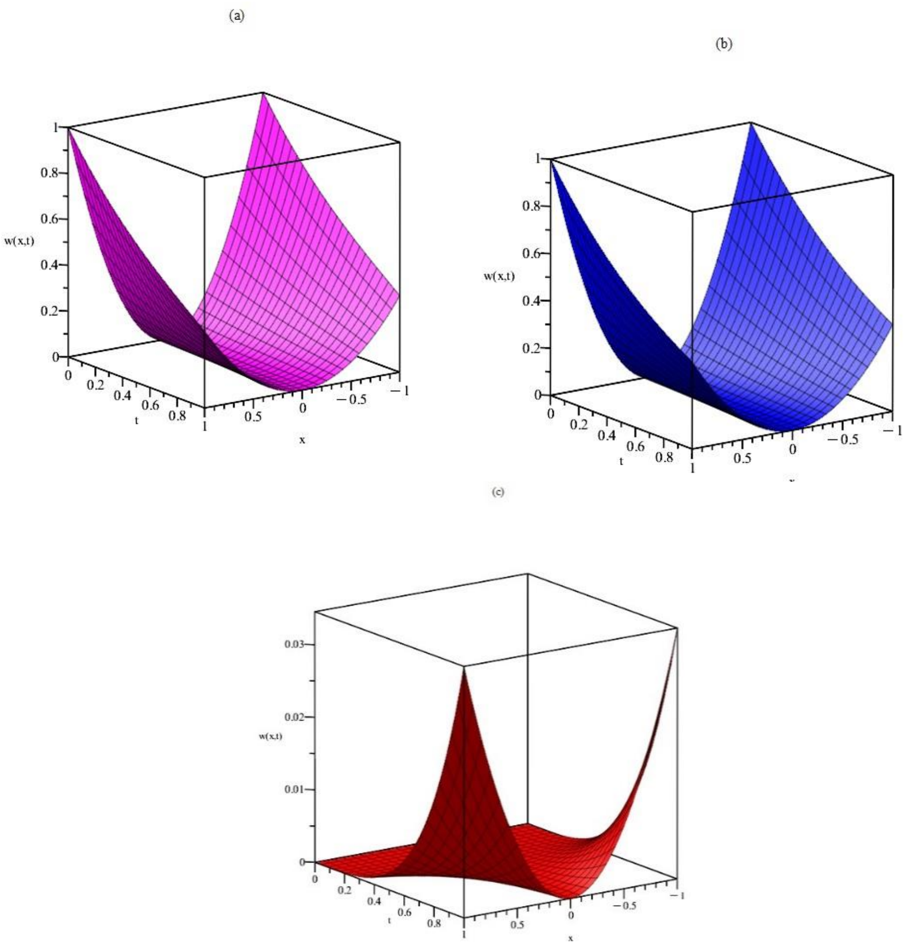

Figure 1. (a) Nature of Cq-HATM solution **(b)** Nature of exact solution **(c)** Nature of absolute error=$\left|w_{exact} - w_{Cq-HATM}\right|$ at $\mathrm{h} = -1, \mathrm{n} = 1, \alpha = 1$.

The graphs of Cq-HATM, exact solution, and absolute error are depicted in Figure 2

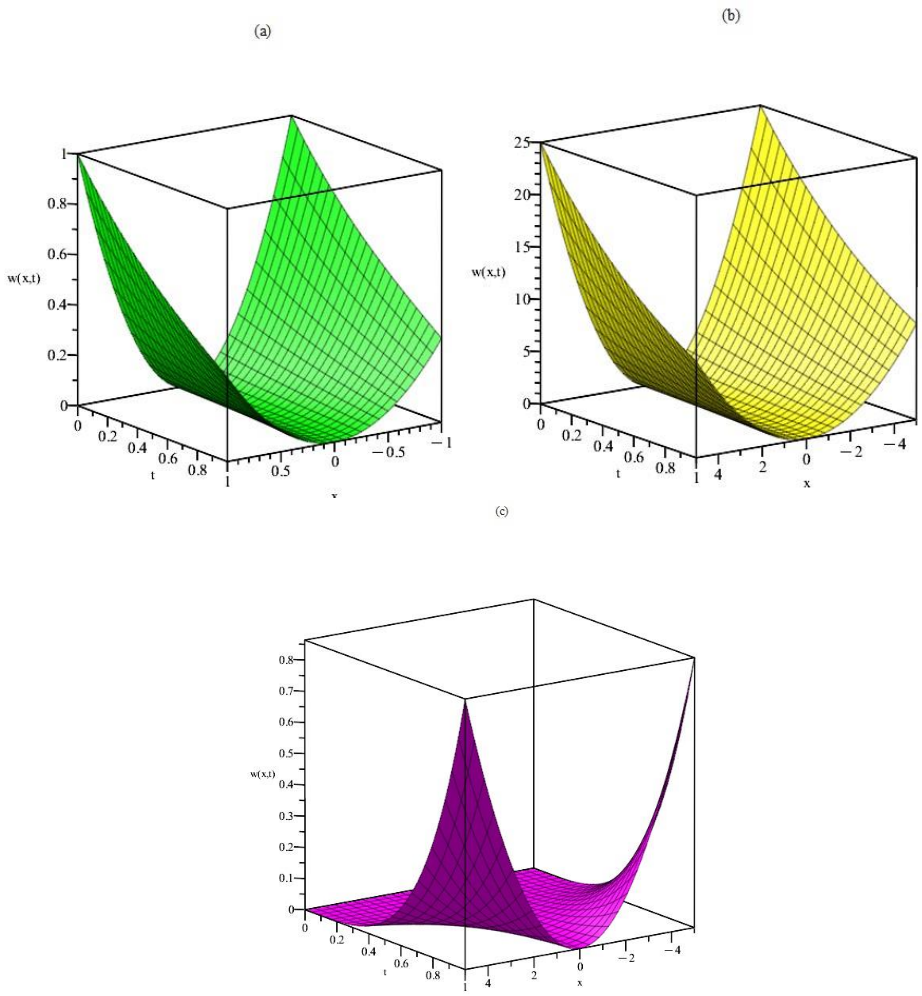

Figure 2. (a) Nature of CSHPM solution **(b)** Nature of exact solution **(c)** Nature of absolute error=$|w_{exact} - w_{CSHPM}|$ at $\alpha = 1$.

Figure 3 depicts comparison plots of Cq-HATM, CSHPM, and exact solutions for distinct α values.

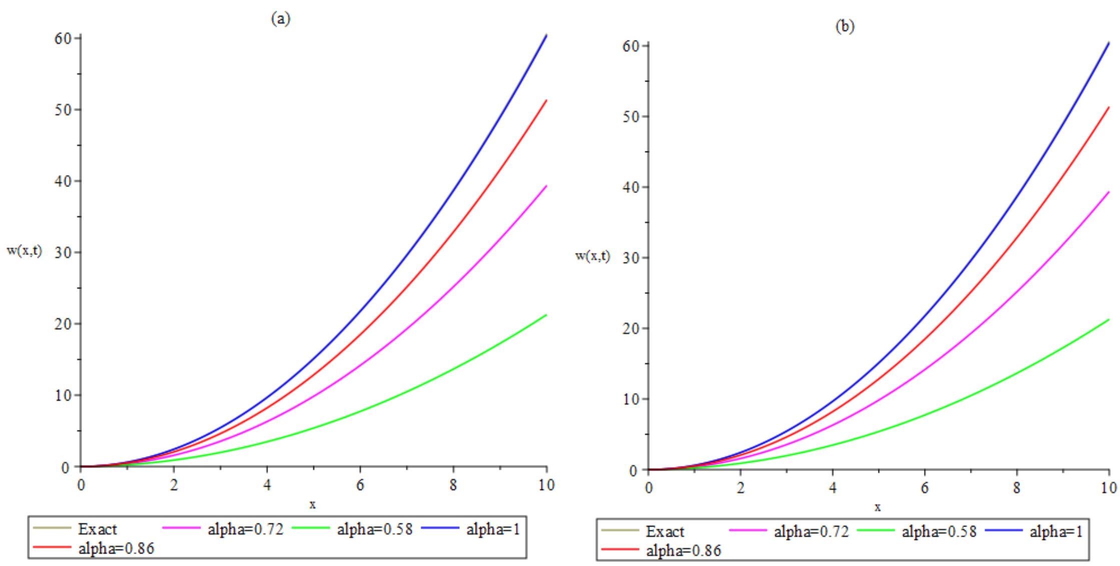

Figure 3. The comparison of the Cq-HATM solutions and exact solution **(b)** The comparison of the CSHPM solutions and exact solution at $\mathrm{h} = -1, \mathrm{n} = 1, t = 0.5$ with different α.

Table 1. Comparison of absolute error between Cq-HATM, CSHPM, and FVIM (Singh & Kumar, 2017) for Eq. (42) with $\alpha = 1$.

x		t			
		0.025	0.050	0.075	0.1
Cq-HATM	0.25	1.0×10^{-9}	1.6×10^{-8}	8.1×10^{-8}	2.5×10^{-7}
CSHPM		1.0×10^{-9}	1.6×10^{-8}	8.1×10^{-8}	2.5×10^{-7}
FVIM		5.8×10^{-5}	2.3×10^{-4}	5.3×10^{-4}	9.4×10^{-4}
Cq-HATM	0.50	4.0×10^{-9}	6.4×10^{-6}	3.2×10^{-7}	1.0×10^{-6}
CSHPM		4.0×10^{-9}	6.4×10^{-6}	3.2×10^{-7}	1.0×10^{-6}
FVIM		2.3×10^{-4}	9.4×10^{-4}	2.1×10^{-3}	3.7×10^{-3}
Cq-HATM	0.75	9.1×10^{-9}	1.4×10^{-7}	7.3×10^{-7}	2.2×10^{-6}
CSHPM		9.1×10^{-9}	1.4×10^{-7}	7.3×10^{-7}	2.2×10^{-6}
FVIM		5.2×10^{-4}	2.1×10^{-3}	4.7×10^{-3}	8.5×10^{-3}

5. Results and discussion

Table 1 evaluates the absolute error comparison between Cq-HATM, CSHPM, and FVIM for Eq. (42) with $\alpha = 1$ for the conformable time-fractional generalized Burgers equation (CTFGBE) with proportional delay. The 3D graphs of Cq-HATM, exact solution, and absolute error are depicted in Figure 1. Figure 2 depicts 3D graphs of Cq-HATM, exact solution, and absolute error. Figure 3 depicts a comparison of Cq-HATM, CSHPM, and exact solutions in 2D plots for various α values. It was observed that the proposed methods outlined in Table 1 yielded the same and even better outcomes than FVIM.

6. Conclusion

Conformable time-fractional partial differential equations with proportional delay are analyzed with Cq-HATM and CSHPM in this paper. In addition, graphs of the solutions to this equation for various values of have been generated using the MAPLE program. The general structure of the surface graphs generated by the Maple software for Equation (42) is observed to vary. It is possible to conclude that the recently proposed methods for solving nonlinear conforming time-fractional partial differential equations with proportional delay are both advantageous and effective.

Authors contribution

The first author plotted the graphs using the Maple program. The second author obtained the solutions and composed the article. The article was edited by the third author.

Declaration of ethical code

The authors of this article declares that the materials and methods used in this study do not require ethical committee approval and/or legal-specific permission.

Conflicts of interest

The authors declare that there is no conflict of interest.

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