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Research Article

Estimates of the norms of some cosine and sine series

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ABSTRACT. In the work, we estimate the $\mathbb{L}^1$ norms of some special cosine and sine series used in studying fractional integrals.

Keywords: Fourier series, Dirichlet kernel, cosine and sine sums.

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1. INTRODUCTION

Let $\mathbb{L}^1$ be the (class) of all 2π-periodic, Lebesgue integrable functions f on $\mathbb{R}$ such that

$$\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\ f(x)\ |\ dx < \infty.$$

For $0 < \gamma < 1$, in this work, we study properties of the series

$$\varphi_\gamma(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^\gamma} \qquad \text{and} \qquad \psi_\gamma(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^\gamma} \tag{1.1}$$

in $\mathbb{L}^1$.

Cosine series of the form

$$f(x) = \sum_{n=1}^{\infty} \mu_n \cos(nx) \tag{1.2}$$

have been studied by several authors (see [1], [14], [12] and [11]). In particular, necessary and sufficient conditions for the convergence in $\mathbb{L}^1$ of the partial sums of the series (1.2) are known (see [7], [8] and [3] and the references therein).

Here we are interested in the series given in (1.1), because of their applications in studying fractional integrals (see [5, p. 422] and [6], where the complex case was considered).

In this work, we look for estimates of the $\mathbb{L}^1$ norms of the functions in (1.1). We restrict the analysis to the case $0 < \gamma < 1$, because it follows from a result proved by Young in [13] (see also [4]) that, for $\gamma \geq 1$, $1 + \varphi_\gamma(x) \geq 0$. Moreover there exists a number α_0 such that, for $0 < \gamma < \alpha_0$, the series $\varphi_\gamma(x)$ is not uniformly bounded below (see [9] or [14, p. 191]).

Here we proof that, if $0 < \gamma < 1$, then

$$\|\varphi_\gamma\|_1 \leq 2 - \frac{1}{2^\gamma} \qquad \text{and} \qquad \|\psi_\gamma\|_1 \leq 2^{1+\gamma}\Big(1 + \frac{1}{\gamma}\Big).$$

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2. Notations and known results

Recall that, for $0 <| x |\leq \pi$ and $n \in \mathbb{N}$, the Dirichlet kernel is given by

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin((2n+1)x/2)}{\sin(x/2)}, \quad D_0(x) = 1,$$

while the Fejér kernel is defined by

$$F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) = \frac{1}{n+1}\Big(\frac{\sin((n+1)x/2)}{\sin(x/2)}\Big)^2$$

$$= 1 + 2\sum_{k=1}^{n}\Big(1 - \frac{k}{n+1}\Big)\cos(kx) \tag{2.3}$$

(see [5, p. 42-43]).

The associated conjugate Dirichlet kernel is defined by (see [5, p. 48] or [14, p. 49])

$$\widetilde{D}_n(x) = 2\sum_{k=1}^{n} \sin(kx) = \frac{\cos(x/2) - \cos((2n+1)x/2)}{\sin(x/2)} \tag{2.4}$$

and the conjugate Fejér kernel is given by (see [14, p. 91])

$$\widetilde{F}_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} \widetilde{D}_k(x) = \frac{1}{\tan(x/2)} - \frac{1}{2(n+1)}\frac{\sin((n+1)x)}{\sin^2(x/2)}.$$

Recall that, for $n \geq 2$ (see [10, p. 151]),

$$\|D_n\|_1 \leq 2 + \ln n. \tag{2.5}$$

3. Auxiliary results

As usually, for a given sequence $\{c_k\}$, we denote $\Delta c_k = c_k - c_{k+1}$ and $\Delta^2 c_k = c_k - 2c_{k+1} + c_{k+2}$.

The first identity in the next lemma is well known, but the second and third ones will help us to simplify some computations.

Lemma 3.1. *Let $\{c_k\}_{k=0}^{\infty}$ and $\{d_k\}_{k=0}^{\infty}$ be two numerical sequences. Set $E_k = \sum_{j=0}^{k} d_j$. For each $n \in \mathbb{N}$, $n > 1$,*

$$\sum_{k=0}^{n} c_k d_k = c_n E_n + \Delta c_{n-1}\sum_{k=0}^{n-1} E_k + \sum_{k=0}^{n-2}\Delta^2 c_k \sum_{j=0}^{k} E_j,$$

$$\sum_{k=0}^{n-2}(k+1)\Delta^2 c_k = c_0 - c_n - n\Delta c_{n-1}$$

and

$$\sum_{k=n-1}^{n+m-2}(k+1)\Delta^2 c_k = c_n - c_{n+m} + n\Delta c_{n-1} - (n+m)\Delta c_{n+m-1}.$$

Proof. The first identity is obtained by applying twice the Abel transform

$$\sum_{k=0}^{n} c_k d_k = c_n \sum_{k=0}^{n} d_k + \sum_{k=0}^{n-1}(c_k - c_{k+1})\sum_{j=0}^{k} d_j. \tag{3.6}$$

That is

$$co\sum_{k=0}^{n} c_k d_k = c_n E_n + \sum_{k=0}^{n-1}(c_k - c_{k+1})E_k$$
$$= c_n E_n + (c_{n-1} - c_n)\sum_{k=0}^{n-1} E_k + \sum_{k=0}^{n-2}(c_k - 2c_{k+1} + c_{k+2})\sum_{j=0}^{k} E_j.$$

In particular, if

$$d_k = \begin{cases} 1, & k = 0, \\ 0, & k \geq 1, \end{cases} \tag{3.7}$$

one has $E_k = 1$ ($k \geq 0$). Hence

$$c_0 = c_n + n(c_{n-1} - c_n) + \sum_{k=0}^{n-2}(k+1)(c_k - 2c_{k+1} + c_{k+2})$$

and

$$0 = \sum_{k=0}^{n+m} c_k d_k - \sum_{k=0}^{n} c_k d_k = c_{n+m} - c_n + (n+m)\Delta c_{n+m-1}$$
$$- n\Delta c_{n-1} + \sum_{k=n-1}^{n+m-2}(k+1)\Delta^2 c_k.$$

□

Lemma 3.2. *If $\gamma > 0$ and $c_k = k^{-\gamma}$ for $k \in \mathbb{N}$, then*

$$0 < c_k - c_{k+1} < \frac{\gamma}{k^{1+\gamma}} \quad \textit{and} \quad 0 < c_k - 2c_{k+1} + c_{k+2} < \frac{\gamma(1+\gamma)}{k^{2+\gamma}}.$$

Proof. Set $f_\gamma(x) = x^{-\gamma}$. If $x \geq 1$, then

$$f_\gamma(x) - f_\gamma(x+1) = -\int_x^{x+1} f'_\gamma(y)dy = \int_x^{x+1} \frac{\gamma}{y^{1+\gamma}}dy < \frac{\gamma}{x^{1+\gamma}}$$

and

$$f_\gamma(x) - 2f_\gamma(x+1) + f_\gamma(x+2) = \gamma\int_x^{x+1}\Big(f_{1+\gamma}(y) - f_{1+\gamma}(y+1)\Big)dy$$
$$= \gamma(1+\gamma)\int_x^{x+1}\int_y^{y+1}\frac{dz}{z^{2+\gamma}}dy < \frac{\gamma(1+\gamma)}{x^{2+\gamma}}.$$

□

Proposition 3.1. *If $0 < \gamma < 1$ and $n \geq 2$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=1}^{n}\frac{\cos(kx)}{k^\gamma}\Big|dx \leq 2 - \frac{1}{2^\gamma} + \frac{1+\ln n}{2n^\gamma}.$$

Moreover, if $m \in \mathbb{N}$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=n+1}^{n+m}\frac{\cos(kx)}{k^\gamma}\Big|dx \leq \frac{1+\ln(n+m)}{(n+m)^\gamma} + \frac{(3+\ln n)}{n^\gamma} + \frac{2n}{(n-1)^{1+\gamma}}. \tag{3.8}$$

Proof. Set $a_k = 1/k^\gamma$ for $k \in \mathbb{N}$, $a_0 = 2 - 1/2^\gamma$ and

$$L_n(x) = 2 - \frac{1}{2^\gamma} + 2\sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma}. \tag{3.9}$$

Notice that

$$a_0 - 2a_1 + a_2 = 0$$

and $\Delta^2 a_k \geq 0$ for $k \geq 0$ (see Lemma 3.2).

Taking into account Lemma 3.1 (with $d_0 = 1$ and $d_k = 2\cos(kx)$ for $k \geq 1$) and the definition of the Dirichlet and Fejér kernels, we obtain

$$\begin{aligned} L_n(x) &= a_n D_n(x) + \Delta a_{n-1} \sum_{k=0}^{n-1} D_k(x) + \sum_{k=0}^{n-2} \Delta^2 a_k \sum_{j=0}^{k} D_j(x) \\ &= a_n D_n(x) + n\Delta a_{n-1} F_{n-1}(x) + \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k F_k(x). \end{aligned}$$

Hence

$$2\sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma} = -a_0 + a_n D_n(x) + n\Delta a_{n-1} F_{n-1}(x) + \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k F_k(x).$$

Recall that (see (2.3)) $F_k(x) \geq 0$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_k(x)dx = 1.$$

Taking into account (2.5) and Lemma 3.1, for $n \geq 2$ and $0 < \gamma < 1$, one has

$$\begin{aligned} \frac{2}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma}\right| dx &\leq a_0 + a_n(2 + \ln n) + n\Delta a_{n-1} + \sum_{k=0}^{n-2} (k+1)\Delta^2 a_k \\ &= a_0 + a_n(2 + \ln n) + n\Delta a_{n-1} + a_0 - a_n - n\Delta a_{n-1} \\ &= 2a_0 + a_n(1 + \ln n). \end{aligned}$$

Moreover

$$\begin{aligned} L_{n+m}(x) - L_n(x) &= a_{n+m} D_{n+m}(x) + (n+m)\Delta a_{n+m-1} F_{n+m-1}(x) \\ &\quad - a_n D_n(x) - n\Delta a_{n-1} F_{n-1}(x) + \sum_{k=n-1}^{n+m-2} (k+1)\Delta^2 a_k F_k(x). \end{aligned} \tag{3.10}$$

Therefore

$$\begin{aligned}
&\frac{2}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=1}^{n+m}\frac{\cos(kx)}{k^\gamma}-\sum_{k=1}^{n}\frac{\cos(kx)}{k^\gamma}\Big|dx\\
=&\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|L_{n+m}(x)-L_n(x)\Big|dx\\
\leq& a_{n+m}(2+\ln(n+m))+(n+m)\Delta a_{n+m-1}+a_n(2+\ln n)+n\Delta a_{n-1}\\
+&\sum_{k=n-1}^{n+m-2}(k+1)\Delta^2 a_k\\
=&a_{n+m}(2+\ln(n+m))+(n+m)\Delta a_{n+m-1}+a_n(2+\ln n)+n\Delta a_{n-1}\\
+&a_n-a_{n+m}+n\Delta a_{n-1}-(n+m)\Delta a_{n+m-1}\\
=&a_{n+m}(1+\ln(n+m))+a_n(3+\ln n)+2n\Delta a_{n-1}\\
\leq&\frac{1+\ln(n+m)}{(n+m)^\gamma}+\frac{(3+\ln n)}{n^\gamma}+\frac{2n}{(n-1)^{1+\gamma}}.
\end{aligned}$$

□

Remark 3.1. *We know that (see* [5, p. 50 and 43]*), if* $0<\delta<\pi$ *and* $n\in\mathbb{N}$*, then*

$$\sup_{\delta\leq|x|\leq\pi}|D_n(x)|\leq\frac{1}{\sin(\delta/2)}\quad\text{and}\quad\sup_{\delta\leq|x|\leq\pi}F_n(x)\leq\frac{1}{(n+1)\sin^2(\delta/2)}.$$

Therefore, it follows from (3.10) *that* $\{L_n\}$ *is a Cauchy sequence in the uniform norm in* $[-\pi,-\delta)\cup(\delta,\pi]$*. Hence* $\{L_n\}$ *converges uniformly to a continuous function in this fixed interval. Since* $\delta\in(0,\pi)$ *is arbitrary, it implies continuity in the open interval. In particular*

$$\varphi_\gamma(x)=-\frac{a_0}{2}+\frac{1}{2}\sum_{k=0}^{\infty}(k+1)\Delta^2 a_k F_k(x).$$

We have not found good estimates for the $\mathbb{L}^1$ norm of the conjugate of the Dirichlet kernel in the existing literature, that is the reason why we include the following lemma.

Lemma 3.3. *For each* $n\in\mathbb{N}$*, one has*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\widetilde{D}_n(t)\Big|\,dt\;\leq\;2+2\ln n.$$

Proof. It is known that

$$\frac{2x}{\pi}\leq\sin x,\qquad 0<x\leq\pi/2,$$

and

$$\frac{|\sin(nx)|}{|\sin x|}=\frac{\sin(nx)}{\sin x}\leq n,\qquad 0<x\leq\pi/(2n).$$

For instance, similar inequalities appeared in [10, p. 151]. Since the second one is less known, we include a proof. Since the function $\cos x$ decreases in the interval $(0,\pi/2]$, for $0<x\leq\pi/(2n)$, $\cos(nx)\leq\cos x$. If $g(x)=\sin(nx)-n\sin x$, then $g'(x)=n(\cos(nx)-\cos x)<0$. Hence $g(x)$ decreases in $[0,\pi/(2n)$. But $g(0)=0$. Therefore

$$0\leq\sin(nx)\leq n\sin(x),\qquad 0<x\leq\pi/(2n).$$

Since $\widetilde{D}_n$ is an odd function, taking into account the trigonometric identity

$$\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right),$$

one has

$$\begin{aligned}
\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left|\widetilde{D}_n(t)\right|dt &= \frac{1}{\pi}\int\limits_{0}^{\pi}\left|\widetilde{D}_n(t)\right|dt = \frac{1}{\pi}\int\limits_{0}^{\pi}\left|\frac{\cos(t/2)-\cos((2n+1)t/2)}{\sin(t/2)}\right|dt \\
&= \frac{2}{\pi}\int\limits_{0}^{\pi/2}\left|\frac{\cos(s)-\cos((2n+1)s)}{\sin s}\right|ds = \frac{4}{\pi}\int\limits_{0}^{\pi/2}\left|\frac{\sin((n+1)s)\sin(ns)}{\sin s}\right|ds \\
&\le \frac{4}{\pi}\int\limits_{0}^{\pi/2}\left|\frac{\sin(ns)}{\sin s}\right|ds \le \frac{4}{\pi}\int\limits_{0}^{\pi/(2n)} n dt + \frac{4}{\pi}\int\limits_{\pi/(2n)}^{\pi/2}\frac{\pi}{2t}dt \\
&= 2 + 2\Big(\ln\frac{\pi}{2} - \ln\frac{\pi}{2n}\Big) = 2 + 2\ln n.
\end{aligned}$$

□

Lemma 3.4. *If* $0<\gamma<1$ *and* $n>3$, *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=1}^{n}\frac{\sin(kx)}{k^\gamma}\right|dx \le 2^{1+\gamma}\Big(1+\frac{1}{\gamma}\Big) + \frac{(1+\ln n)}{n^\gamma}.$$

Moreover, if $m\in\mathbb{N}$, *then*

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left|\sum_{k=n+1}^{n+m}\frac{\sin(kt)}{k^\gamma}\right|dt \le \frac{1+\ln(n+m)}{(n+m)^\gamma} + (2+2^{1+\gamma})\frac{1+\ln n}{n^\gamma} + \frac{2^{1+\gamma}}{\gamma n^\gamma}.$$

Proof. We use the notations of Lemma 3.1 by setting $c_0 = 0$, $d_0 = 1$, and $c_k = 1/k^\gamma$ and $d_k = d_k(x) = 2\sin(kx)$, for $k\ge 1$. With these notations

$$\sum_{j=1}^{k} d_j(x) = 1 + \widetilde{D}_k(x), \qquad k\ge 1.$$

If we set

$$M_n(x) = 2\sum_{k=1}^{n} c_k\sin(kx) = \sum_{k=0}^{n} c_k d_k(x),$$

it follows from (3.6) that

$$\begin{aligned} M_n(x) &= c_n \sum_{k=0}^{n} d_k(x) + \sum_{k=0}^{n-1} (c_k - c_{k+1}) \sum_{j=0}^{k} d_j(x) \\ &= c_n\Big(1+\widetilde{D}_n(x)\Big) - c_1 + \sum_{k=1}^{n-1}(c_k - c_{k+1})\Big(1+\widetilde{D}_k(x)\Big) \\ &= c_n\Big(1+\widetilde{D}_n(x)\Big) - c_1 + \sum_{k=1}^{n-1}(c_k - c_{k+1}) + \sum_{k=1}^{n-1}(c_k - c_{k+1})\widetilde{D}_k(x) \\ &= c_n\widetilde{D}_n(x) + \sum_{k=1}^{n-1}(c_k - c_{k+1})\widetilde{D}_k(x). \end{aligned}$$

Taking into account Lemmas 3.2 and 3.3, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |M_n(t)|\,dt \le 2c_n(1+\ln n) + 2\gamma \sum_{k=1}^{n-1} \frac{(1+\ln k)}{k^{1+\gamma}}.$$

In order to estimate the sum in the previous inequality, we include some computations. By integration by part, we obtain

$$\begin{aligned} \gamma \sum_{k=1}^{n-1} \frac{(1+\ln k)}{k^{1+\gamma}} &\le 2^{1+\gamma}\gamma \sum_{k=1}^{n-1} \frac{(1+\ln k)}{(k+1)^{1+\gamma}} \le 2^{1+\gamma}\gamma \sum_{k=1}^{n-1} \int_k^{k+1} \frac{(1+\ln x)}{x^{1+\gamma}}dx \\ &= 2^{1+\gamma}\gamma \int_1^n \frac{(1+\ln x)}{x^{1+\gamma}}dx = 2^{1+\gamma}\Big(1 - \frac{1+\ln n}{n^\gamma} + \int_1^n \frac{1}{x^{1+\gamma}}dx\Big) \\ &= 2^{1+\gamma}\Big(1 - \frac{1+\ln n}{n^\gamma} + \frac{1}{\gamma}\Big(1-\frac{1}{n^\gamma}\Big)\Big) \le 2^{1+\gamma}\Big(1+\frac{1}{\gamma}\Big). \end{aligned}$$

We conclude that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{k=1}^{n} \frac{\sin(kt)}{k^\gamma}\right| dt = \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid M_n(t) \mid dt \le c_n(1+\ln n) + 2^{1+\gamma}\Big(1+\frac{1}{\gamma}\Big).$$

Moreover,

$$\begin{aligned} &\frac{2}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{k=1}^{n+m} \frac{\sin(kt)}{k^\gamma} - \sum_{k=1}^{n} \frac{\sin(kt)}{k^\gamma}\right| dt \\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| c_{n+m}\widetilde{D}_{n+m}(x) - c_n\widetilde{D}_n(x) + \sum_{k=n}^{n+m-1} (c_k - c_{k+1})\widetilde{D}_k(x)\right| dx \\ \le& 2c_{n+m}(1+\ln(n+m)) + 2c_n(1+\ln n) + 2\gamma \sum_{k=n}^{n+m-1} \frac{(1+\ln k)}{k^{1+\gamma}} \\ \le& 2c_{n+m}(1+\ln(n+m)) + 2c_n(1+\ln n) + 2^{2+\gamma}\frac{(1+\ln n)}{n^\gamma} + \frac{2^{2+\gamma}}{\gamma n^\gamma}. \end{aligned}$$

□

Remark 3.2. *It is known that (see* [14, p. 92]*)*

$$\widetilde{F}_n(t)\,\operatorname{sign} t \geq 0, \qquad t \in (-\pi, \pi).$$

Hence $\widetilde{F}_n(x)$ is not a positive operator and different Cesàro means of $\widetilde{D}_n(x)$ share this properties. That is the reason why we use $\widetilde{D}_n(x)$ instead of $\widetilde{F}_n(x)$.

4. Main results

Theorem 4.1. *If $0 < \gamma < 1$, then $\varphi_\gamma, \psi_\gamma \in \mathbb{L}^1$,*

$$\|\varphi_\gamma\|_1 \leq 2 - \frac{1}{2^\gamma} \qquad \text{and} \qquad \|\psi_\gamma\|_1 \leq 2^{1+\gamma}\Big(1 + \frac{1}{\gamma}\Big).$$

Proof. If a series converges to a function $f \in \mathbb{L}^1$, then the series is the Fourier series of f (see [5, p. 51]).

If

$$H_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k^\gamma},$$

equation (3.8) can be rewriten as

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \Big|H_{n+m}(x) - H_n(x))\Big| dx \leq \frac{1+\ln(n+m)}{(n+m)^\gamma} + \frac{(3+\ln n)}{n^\gamma} + \frac{2n}{(n-1)^{1+\gamma}}.$$

Hence $\{H_n\}$ is a Cauchy sequence in $\mathbb{L}^1$. Therefore there exists a function $F \in \mathbb{L}^1$ such that $\|F - H_n\|_1 \to 0$ as $n \to \infty$. But $F(x) = \varphi_\gamma(x)$ a.e. . Since the series is continuous for $0 <\mid x \mid \leq \pi$, we have equality for $x \neq 0$.

Taking into account Proposition 3.1 (see also [2, p. 50]) and (3.9) with L_n defined as in (3.9), one has

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mid \varphi_\gamma(t) \mid dt = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mid H_n(t) \mid dt \leq \lim_{n\to\infty} \Big(2 - \frac{1}{2^\gamma} + \frac{1+\ln n}{2n^\gamma}\Big)$$
$$= 2 - \frac{1}{2^\gamma}.$$

The assertions for ψ_γ follow analogously. □

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