SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE
e-ISSN: 2147-835X
Dergi sayfası: http://dergipark.gov.tr/saufenbilder

Geliş/Received
11-04-2017
Kabul/Accepted
22-08-2017

Doi
10.16984/saufenbilder.305632

On the solution of a nonlinear Volterra integral equation with delay

Aynur Şahin[*1] , Zeynep Kalkan[2], Hakan Arısoy[3]

ABSTRACT

In this paper, we show that the iterative sequence which is a simplified form of the iteration method introduced by Ullah and Arshad (SpringerPlus, (2016)5:1616), is convergent strongly to the solution of a nonlinear Volterra integral equation with delay in a complete metric space. Furthermore, we prove a data dependence result for the solution of this integral equation.

Keywords: Volterra integral equations, fixed point, data dependence, iteration methods

Gecikmeli lineer olmayan bir Volterra integral denkleminin çözümü

ÖZ

Bu makalede, Ullah ve Arshad (SpringerPlus (2016)5:1616) tarafından tanımlanan iterasyon metodunun basitleştirilmiş hali olan bir iteratif dizisinin gecikmeli lineer olmayan bir Volterra integral denkleminin çözümüne kuvvetli yakınsadığı gösterilmiştir. Dahası bu integral denklemin çözümü için bir veri bağımlılığı sonucu ispatlanmıştır.

Anahtar Kelimeler: Volterra integral denklemleri, sabit nokta, veri bağımlılığı, iterasyon metotları

[*] Sorumlu Yazar / Corresponding Author
[1] Department of Mathematics, Sakarya University, Sakarya, 54050, Turkey-ayuce@sakarya.edu.tr
[2] Department of Mathematics, Sakarya University, Sakarya, 54050, Turkey-zeynepyildiz28@gmail.com
[3] Department of Mathematics, Sakarya University, Sakarya, 54050, Turkey-hakanarisoy34@hotmail.com

1. INTRODUCTION AND PRELIMINARIES

Let $I=[a,b]$ be a fixed finite interval such that $a<b$. We consider the following metric on $C(I)$, the space of the complex-valued continuous functions on I,

$$d(u,v)=\sup_{x\in I}\frac{|u(x)-v(x)|}{\varphi(x)}$$

where $\varphi: I\to(0,\infty)$ is a non-decreasing continuous function. It is clear that $(C(I),d)$ is a complete metric space (cf., e.g., [1]).

In 2013, Castro and Guerra [2] proved the existence and uniqueness of the solution of a nonlinear Volterra integral equation with delay as follows.

Theorem 1. (see [2, Theorem 2.1]) Let us consider continuous given functions $\mu: I\times I\to[0,\infty)$ and $\eta: I\times I\to[0,\infty)$. Moreover, assume that $g\in C(I)$, $f: I\times I\times\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is a continuous function, $\alpha: I\to I$ is a continuous delay function fulfilling $\alpha(x)\le x$ for all $x\in I$ and $\psi: C(I)\to C(I)$ is a bounded function in the sense that there exists a constant $K>0$ such that

$$d(\psi(h_1),\psi(h_2))\le K\,d(h_1,h_2).$$

In addition, suppose that there are constants $\beta,\gamma\in[0,1)$ such that $\int_a^x\mu(x,t)\varphi(t)dt\le\beta\varphi(x)$ and $\int_a^x\eta(x,t)\varphi(t)dt\le\gamma\varphi(x)$, and that

$$\left|f\left(x,t,u(t),u(\alpha(t))\right)-f\left(x,t,v(t),v(\alpha(t))\right)\right|$$
$$\le\mu(x,t)|u(t)-v(t)|+\eta(x,t)\left|u(\alpha(t))-v(\alpha(t))\right|$$

for all $x,t\in I$, $u,v\in\mathbb{C}(I)$. If $K(\beta+\gamma)<1$, then there is a unique solution $y_0\in C(I)$ of the nonlinear Volterra integral equation

$$y(x)=g(x)+\psi\left(\int_a^x f\left(x,t,y(t),y(\alpha(t))\right)dt\right)\quad(1)$$

for all $x\in I$.

Very recently, Ullah and Arshad [3] introduced the following iteration method in a Banach space:

$$\begin{cases} x_{n+1}^{(1)}=Tx_n^{(2)},\\ x_n^{(2)}=T\left((1-\alpha_n^{(1)})x_n^{(3)}+\alpha_n^{(1)}Tx_n^{(3)}\right),\\ x_n^{(3)}=T\left((1-\alpha_n^{(2)})x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right),\quad n\in\mathbb{N}\end{cases}\quad(2)$$

where $\{\alpha_n^{(1)}\}$ and $\{\alpha_n^{(2)}\}$ are real sequences in $[0,1]$.

Putting $\alpha_n^{(1)}=1$ for all $n\in\mathbb{N}$ in (2), Ertürk et al. [4] studied an iteration method as follows:

$$\begin{cases} x_{n+1}^{(1)}=Tx_n^{(2)},\\ x_n^{(2)}=T(Tx_n^{(3)}),\\ x_n^{(3)}=T\left((1-\alpha_n^{(2)})x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right),\quad n\in\mathbb{N}.\end{cases}$$
(3)

(3)

They showed that the iteration method (3) is faster than all Picard [5], Mann [6], Ishikawa [7], Noor [8], S [9], Normal-S [10], CR [11], Picard-S [12], Modified-SP [13], Thakur et al. [14], Vatan two-step [15], Abbas and Nazır [16], S* [17] and Ullah and Arshad [3] iteration methods.

In this paper, we prove the strong convergence and data dependence theorems for the nonlinear Volterra integral equation with delay (1) by using the iteration method (3). Also we present an example to support our results.

We end this section with the following lemma which will be needed in the sequel.

Lemma 1. (see [18]) Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative real sequence and there exists $n_0\in\mathbb{N}$ such that for all $n\ge n_0$ satisfying the following condition:

$$a_{n+1}\le(1-\mu_n)a_n+\mu_n\eta_n$$

where $\mu_n\in(0,1)$ such that $\sum_{n=1}^{\infty}\mu_n=\infty$ and $\eta_n\ge0$. Then the following inequality holds:

$$0\le\limsup_{n\to\infty}a_n\le\limsup_{n\to\infty}\eta_n.$$

2. MAIN RESULTS

We first give the strong convergence theorem of the iterative sequence $\{x_n^{(1)}\}$ defined by (3) for the nonlinear Volterra integral equation (1) in the complete metric space $(C(I),d)$.

Theorem 2. Let $\{\alpha_n^{(2)}\}$ be a real sequence in $[0,1]$ satisfying $\sum_{n=1}^{\infty}\alpha_n^{(2)}=\infty$. Under the assumptions of Theorem 1, the equation (1) has a unique solution, say y_0, in $C(I)$ and the iteration method (3) is convergent strongly to y_0.

Proof. Let $\{x_n^{(1)}\}$ be an iterative sequence generated by the iteration method (3) for the operator $T: C(I)\to C(I)$ defined by

$$T(u)(x) = g(x) + \psi\left(\int_a^x f\left(x, t, u(t), u\left(\alpha(t)\right)\right) dt\right).$$

We will show that $x_n^{(1)} \to y_0$ as $n \to \infty$. From (1), (3) and the assumptions of Theorem 1, we obtain

$$d\left(x_n^{(3)}, y_0\right) = \sup_{x\in I} \frac{\left|x_n^{(3)}(x) - y_0(x)\right|}{\varphi(x)}$$

$$= \sup_{x\in I} \frac{\left|T\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)} T x_n^{(1)}\right)(x) - T y_0(x)\right|}{\varphi(x)}$$

$$= \sup_{x\in I} \frac{\left|\psi\left(\int_a^x f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(t),\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))\right)dt\right)-\psi\left(\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$

$$\leq K\sup_{x\in I} \frac{\left|\int_a^x f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(t),\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))\right)dt-\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right|}{\varphi(x)}$$

$$\leq K\sup_{x\in I} \frac{\int_a^x\left|f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(t),\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))\right)-f\left(x,t,y_0(t),y_0(\alpha(t))\right)\right|dt}{\varphi(x)}$$

$$\leq K\sup_{x\in I} \frac{\int_x^\alpha\left(\mu(x,t)\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(t)-y_0(t)\right|+\eta(x,t)\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))-y_0(\alpha(t))\right|\right)dt}{\varphi(x)}$$

$$= K\sup_{x\in I} \frac{\int_a^x \mu(x,t)\varphi(t)\frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(t)-y_0(t)\right|}{\varphi(t)}dt+\int_a^x \eta(x,t)\varphi(\alpha(t))\frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))-y_0(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}$$

$$\leq K\left[\begin{array}{l}\sup_{t\in I} \frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(t)-y_0(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x \mu(x,t)\varphi(t)dt}{\varphi(x)}\\ +\sup_{t\in I}\frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))-y_0(\alpha(t))\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x \eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\end{array}\right]$$

$$\leq K\left[d\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)},y_0\right).\beta+d\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)},y_0\right).\gamma\right]$$

$$=K(\beta+\gamma)d\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)}+\alpha_n^{(2)}Tx_n^{(1)},y_0\right)$$

$$\leq K(\beta+\gamma)\left(\left(1-\alpha_n^{(2)}\right)d\left(x_n^{(1)},y_0\right)+\alpha_n^{(2)}d\left(Tx_n^{(1)},y_0\right)\right) \quad (4)$$

and

$$d\left(Tx_n^{(1)},y_0\right)=\sup_{x\in I}\frac{\left|Tx_n^{(1)}(x)-Ty_0(x)\right|}{\varphi(x)}=\sup_{x\in I}\frac{\left|\psi\left(\int_a^x f\left(x,t,x_n^{(1)}(t),x_n^{(1)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$

$$\leq K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,x_n^{(1)}(t),x_n^{(1)}(\alpha(t))\right)dt-\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right|}{\varphi(x)}$$

$$\leq K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,x_n^{(1)}(t),x_n^{(1)}(\alpha(t))\right)dt-f\left(x,t,y_0(t),y_0(\alpha(t))\right)\right|dt}{\varphi(x)}$$

$$\leq K\sup_{x\in I}\frac{\int_x^\alpha\left(\mu(x,t)\left|x_n^{(1)}(t)-y_0(t)\right|+\eta(x,t)\left|x_n^{(1)}(\alpha(t))-y_0(\alpha(t))\right|\right)dt}{\varphi(x)}$$

$$= K\sup_{x\in I}\frac{\int_a^x \mu(x,t)\varphi(t)\frac{\left|x_n^{(1)}(t)-y_0(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|x_n^{(1)}(\alpha(t))-y_0(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}$$

$$\leq K\left[\sup_{t\in I}\frac{\left|x_n^{(1)}(t)-y_0(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}+\sup_{t\in I}\frac{\left|x_n^{(1)}(\alpha(t))-y_0(t)\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\right]$$

$$\leq K\left[d\left(x_n^{(1)},y_o\right).\beta+d\left(x_n^{(1)},y_0\right).\gamma\right]$$

$$= K(\beta+\gamma)d\left(x_n^{(1)},y_0\right). \quad (5)$$

Combining (4) and (5), we have

$$d\left(x_n^{(3)},y_0\right)\leq K(\beta+\gamma)\left(1-\alpha_n^{(2)}\left(1-K(\beta+\gamma)\right)\right)d\left(x_n^{(1)},y_0\right).$$

Also, we get

$$d\left(x_n^{(2)},y_0\right)=\sup_{x\in I}\frac{\left|T\left(Tx_n^{(3)}\right)(x)-Ty_0(x)\right|}{\varphi(x)}=\sup_{x\in I}\frac{\left|\psi\left(\int_a^x f\left(x,t,Tx_n^{(3)}(t),Tx_n^{(3)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$

$$\leq K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,Tx_n^{(3)}(t),Tx_n^{(3)}(\alpha(t))\right)dt-\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right|}{\varphi(x)}$$

$$\leq K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,Tx_n^{(3)}(t),Tx_n^{(3)}(\alpha(t))\right)-f\left(x,t,y_0(t),y_0(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\int_x^a\left(\mu(x,t)\left|Tx_n^{(3)}(t)-y_0(t)\right|+\eta(x,t)\left|Tx_n^{(3)}(\alpha(t))-y_0(\alpha(t))\right|\right)dt}{\varphi(x)}$$
$$= K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|Tx_n^{(3)}(t)-y_0(t)\right|dt}{\varphi(t)}+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|Tx_n^{(3)}(\alpha(t))-y_0(\alpha(t))\right|}{\varphi(\alpha(t))}d\,t}{\varphi(x)}$$
$$\leq K\left[\sup_{t\in I}\frac{\left|Tx_n^{(3)}(t)-y_0(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}+\sup_{t\in I}\frac{\left|Tx_n^{(3)}(\alpha(t))-y_0(t)\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\right]$$
$$\leq K(\beta+\gamma)d\left(Tx_n^{(3)},y_0\right)$$

and

$$d\left(Tx_n^{(3)},y_0\right)=\sup_{x\in I}\frac{\left|Tx_n^{(3)}(x)-Ty_0(x)\right|}{\varphi(x)}=\sup_{x\in I}\frac{\left|\psi\left(\int_a^x f\left(x,t,x_n^{(3)}(t),x_n^{(3)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,x_n^{(3)}(t),x_n^{(3)}(\alpha(t))\right)dt-\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right|}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,x_n^{(3)}(t),x_n^{(3)}(\alpha(t))\right)dt-f\left(x,t,y_0(t),y_0(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\int_x^a\left(\mu(x,t)\left|x_n^{(3)}(t)-y_0(t)\right|+\eta(x,t)\left|x_n^{(3)}(\alpha(t))-y_0(\alpha(t))\right|\right)dt}{\varphi(x)}$$
$$= K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|x_n^{(3)}(t)-y_0(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|x_n^{(3)}(\alpha(t))-y_0(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}$$
$$\leq K\left[\sup_{t\in I}\frac{\left|x_n^{(3)}(t)-y_0(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}+\sup_{t\in I}\frac{\left|x_n^{(3)}(\alpha(t))-y_0(t)\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\right]$$
$$\leq K(\beta+\gamma)d(x_n^3,y_0).$$

Then, we have

$$d\left(x_n^{(2)},y_0\right)\leq[K(\beta+\gamma)]^3\left(1-\alpha_n^{(2)}\left(1-K(\beta+\gamma)\right)\right)d\left(x_n^{(1)},y_0\right). \tag{6}$$

Similarly, we obtain

$$d(x_{n+1}^{(1)},y_0)=\sup_{x\in I}\frac{\left|Tx_n^{(2)}(x)-Ty_0(x)\right|}{\varphi(x)}=\sup_{x\in I}\frac{\left|\psi\left(\int_a^x f\left(x,t,x_n^{(2)}(t),x_n^{(2)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,x_n^{(2)}(t),x_n^{(2)}(\alpha(t))\right)dt-\int_a^x f\left(x,t,y_0(t),y_0(\alpha(t))\right)dt\right|}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,x_n^{(2)}(t),x_n^{(2)}(\alpha(t))\right)dt-f\left(x,t,y_0(t),y_0(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$\leq K\sup_{x\in I}\frac{\int_x^a\left(\mu(x,t)\left|x_n^{(2)}(t)-y_0(t)\right|+\eta(x,t)\left|x_n^{(2)}(\alpha(t))-y_0(\alpha(t))\right|\right)dt}{\varphi(x)}$$
$$= K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|x_n^{(2)}(t)-y_0(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|x_n^{(2)}(\alpha(t))-y_0(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}$$
$$\leq K\left[\sup_{t\in I}\frac{\left|x_n^{(2)}(t)-y_0(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}+\sup_{t\in I}\frac{\left|x_n^{(2)}(\alpha(t))-y_0(t)\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\right]$$
$$\leq K(\beta+\gamma)d(x_n^{(2)},y_0). \tag{7}$$

From (6) and (7), we get

$$d\left(x_{n+1}^{(1)},y_0\right)\leq[K(\beta+\gamma)]^4\left(1-\alpha_n^{(2)}\left(1-K(\beta+\gamma)\right)\right)d\left(x_n^{(1)},y_0\right).$$

Since $K(\beta+\gamma)<1$, then we have

$$d\left(x_{n+1}^{(1)},y_0\right)\leq\left(1-\alpha_n^{(2)}\left(1-K(\beta+\gamma)\right)\right)d\left(x_n^{(1)},y_0\right).$$

Thus, by induction, we get

$$d\left(x_{n+1}^{(1)}, y_0\right) \leq d\left(x_0^{(1)}, y_0\right) \prod_{k=0}^{n}\left[1-\alpha_k^{(2)}\left(1-K(\beta+\gamma)\right)\right]. \quad (8)$$

Since $\alpha_k^{(2)} \in [0,1]$ for all $k \in \mathbb{N}$ and $K\,(\beta+\gamma) < 1$, then we obtain

$$0 \leq \alpha_k^{(2)}\left(1-K(\beta+\gamma)\right) \leq 1.$$

Having regard to the fact that $1-x \leq e^{-x}$ for all $x \in [0,1]$, we can write (8) as

$$d(x_{n+1}^{(1)}, y_0) \leq d(x_0^{(1)}, y_0) e^{-[1-K(\beta+\gamma)]\sum_{k=0}^{n}\alpha_k^{(2)}}$$

which yields $\lim_{n\to\infty} d(x_n^{(1)}, y_0) = 0$. This completes the proof.

A direct application of Theorem 2 for the particular case $\psi(g) = \lambda g$, for some parameter λ, yields the following corollary for corresponding linear Volterra integral equations.

Corollary 1. Let $\left\{\alpha_n^{(2)}\right\}$ be the same as in Theorem 2, and let $\mu: I \times I \to [0,\infty)$ and $\eta: I \times I \to [0,\infty)$ be continuous functions. Moreover, assume that $g \in C(I)$, $f: I \times I \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function and $\alpha: I \to I$ is a continuous delay function fulfilling $\alpha(x) \leq x$ for all $x \epsilon I$. In addition, suppose that there are constants $\beta, \gamma \epsilon [0,1)$ such that $\int_a^x \mu(x,t)\varphi(t)dt \leq \beta\varphi(x)$ and $\int_a^x \eta(x,t)\varphi(t)dt \leq \gamma\varphi(x)$, and that

$$\left|f\left(x,t,u(t),u(\alpha(t))\right) - f\left(x,t,v(t),v(\alpha(t))\right)\right| \leq \mu(x,t)|u(t)-v(t)| + \eta(x,t)\left|u(\alpha(t))-v(\alpha(t))\right|$$

for all $x,t \in I$, $u,v \in \mathbb{C}(I)$. If $|\lambda|(\beta+\gamma) < 1$, then there is a unique solution $y_0 \in C(I)$ of the linear Volterra integral equation

$$y(x) = g(x) + \lambda \int_a^x f\left(x,t,y(t),y(\alpha(t))\right)dt$$

and the iterative sequence $\left\{x_n^{(1)}\right\}$ defined by (3) is convergent strongly to y_0.

Example 1. Let us consider the function $f(x,t,y(t),y(\alpha(t))) = \frac{1}{x}\left(y(t)+y(\alpha(t))\right)$ with the delay function $\alpha(t) = t$ for $t \in [1,10]$. Let be $g(x) = \frac{1}{2x} + \frac{x}{2}$ and $\lambda = \frac{1}{2}$. Moreover, let us take $\alpha_n^{(2)} = \frac{1}{n}$ for all $n \in \mathbb{N}$ and $\varphi: [1,10] \to (0,\infty)$ such that $\varphi(x) = x^2$. We can see that all the considered functions are under the conditions of Corollary 1. Namely:

- $g: [1,10] \to \mathbb{C}$ is continuous;
- $\alpha: [1,10] \to [1,10]$ is continuous and such that $\alpha(x) \leq x$ for all $x \epsilon [1,10]$;
- $f: [1,10] \times [1,10] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous and such that $\left|f\left(x,t,u(t),u(\alpha(t))\right) - f\left(x,t,v(t),v(\alpha(t))\right)\right| = \left|\frac{1}{x}\left(u(t)+u(\alpha(t))\right) - \frac{1}{x}\left(v(t)+v(\alpha(t))\right)\right| \leq \frac{1}{x}|u(t)-v(t)| + \frac{1}{x}\left|u(\alpha(t))-v(\alpha(t))\right|$;
- From previous item, we can see that the functions $\mu: [1,10] \times [1,10] \to [0,\infty)$ and $\eta: [1,10] \times [1,10] \to [0,\infty)$ are such that $\mu(x,t) = \eta(x,t) = \frac{1}{x}$. The functions μ and η are continuous;
- Now, we compute β and γ. Since $\mu(x,t) = \eta(x,t) = \frac{1}{x}$, then we have
$$\int_1^x \mu(x,t)\varphi(t)dt = \int_1^x \eta(x,t)\varphi(t)dt = \int_1^x \frac{1}{x}t^2 dt = \frac{1}{x}\left(\frac{x^3}{3} - \frac{1}{3}\right) = \frac{1}{3}x^2\left(1-\frac{1}{x^3}\right) \leq \frac{1}{3}\varphi(x)$$
for all $x,t \in [1,10\,]$. Thus, we may take $\beta = \gamma = \frac{1}{3} \epsilon [0,1)$;
- $|\lambda|(\beta+\gamma) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{1}{3} < 1$.

In addition, the exact solution of the equation

$$y(x) = \frac{1}{2x} + \frac{x}{2} + \frac{1}{2}\int_1^x \frac{1}{x}\left(y(t)+y(\alpha(t))\right)dt$$

for all $x \in [1,10\,]$, is the function $y_0(x) = x$. Indeed,

$$\frac{1}{2x} + \frac{x}{2} + \frac{1}{2}\int_1^x \frac{1}{x}(t+t)dt = \frac{1}{2x} + \frac{x}{2} + \frac{1}{x}\int_1^x tdt = \frac{1}{2x} + \frac{x}{2} + \frac{1}{x}\left(\frac{x^2}{2} - \frac{1}{2}\right) = x.$$

We now prove the data dependence theorem of the solution for the nonlinear Volterra integral equation (1) with the help of the iteration method (3). In this theorem we shall use the following notations:

$$T(u)(x) = g(x) + \psi\left(\int_a^x f\left(x,t,u(t),u(\alpha(t))\right)dt\right) \quad (9)$$

and

$$\breve{T}(u)(x) = \breve{g}(x) + \psi\left(\int_a^x \breve{f}\left(x,t,u(t),u(\alpha(t))\right)dt\right) \quad (10)$$

where $g, \breve{g} \in C(I)$ and $f, \breve{f}: I \times I \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions.

Theorem 3. Let f, g, α and ψ be the same as in Theorem 1. Let $\left\{x_n^{(1)}\right\}$ and $\left\{\breve{x}_n^{(1)}\right\}$ be two iterative sequences defined by (3) and

$$\begin{cases} \breve{x}_{n+1}^{(1)} = \breve{T}\breve{x}_n^{(2)}, \\ \breve{x}_n^{(2)} = \breve{T}\left(\breve{T}\breve{x}_n^{(3)}\right), \\ \breve{x}_n^{(3)} = \breve{T}\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right), \quad n \in \mathbb{N}, \end{cases} \tag{11}$$

respectively, where $\left\{\alpha_n^{(2)}\right\}$ is a real sequence in $[0,1]$ satisfying (i) $\frac{1}{2} \leq \alpha_n^{(2)}$ for all $n \in \mathbb{N}$. Also, we suppose that there exist non-negative constants ε_1 and ε_2 such that (ii) $|g(x) - \breve{g}(x)| \leq \varepsilon_1$ and $\left|f(x,t,u(t),u(\alpha(t))) - \breve{f}(x,t,u(t),u(\alpha(t)))\right| \leq \frac{\varepsilon_2}{b-a}$ for all $x, t \in I$, $u \in C(I)$. If y_0 and $\breve{y}_0$ are solutions of corresponding equations (9) and (10), respectively, then we obtain

$$d(y_0, \breve{y}_o) \leq \frac{9M(\varepsilon_1 + K\varepsilon_2)}{1 - K(\beta+\gamma)}$$

where $M = \sup_{x\in I} \frac{1}{\varphi(x)}$.

Proof. From (3), (9), (10), (11) and hypothesis in Theorem 1 and (ii), we obtain

$$d(x_n^{(3)}, \breve{x}_n^{(3)}) = \sup_{x\in I} \frac{\left|x_n^{(3)}(x) - \breve{x}_n^{(3)}(x)\right|}{\varphi(x)} = \sup_{x\in I} \frac{\left|T\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(x) - \breve{T}\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(x)\right|}{\varphi(x)}$$

$$\leq \sup_{x\in I} \frac{|g(x) - \breve{g}(x)| + \left|\begin{array}{l} \psi\left(\int_a^x f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))\right)dt\right) \\ -\psi\left(\int_a^x \breve{f}\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right)dt\right) \end{array}\right|}{\varphi(x)}$$

$$\leq \sup_{x\in I} \frac{\varepsilon_1}{\varphi(x)} + K\sup_{x\in I} \frac{\left|\begin{array}{l} \int_a^x f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))\right)dt \\ -\int_a^x \breve{f}\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right)dt \end{array}\right|}{\varphi(x)}$$

$$\leq \varepsilon_1.M + K\sup_{x\in I} \frac{\int_a^x \left|\begin{array}{l} f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t))\right) \\ -f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right) \end{array}\right| dt}{\varphi(x)}$$

$$+K\sup_{x\in I} \frac{\int_a^x \left|\begin{array}{l} f\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right) \\ -\breve{f}\left(x,t,\left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t), \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right) \end{array}\right| dt}{\varphi(x)}$$

$$\leq \varepsilon_1.M + K\sup_{x\in I} \frac{\int_a^x \left(\begin{array}{l} \mu(x,t)\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(t) - \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t)\right| \\ +\eta(x,t)\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t)) - \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right| \end{array}\right) dt}{\varphi(x)}$$
$$+K\sup_{x\in I} \int_a^x \frac{\varepsilon_2}{b-a}.\frac{1}{\varphi(x)}dt$$

$$\leq \varepsilon_1.M + K\sup_{x\in I} \frac{\begin{array}{l} \int_a^x \mu(x,t)\varphi(t)\frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(t) - \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t)\right|}{\varphi(t)}dt \\ +\int_a^x \eta(x,t)\varphi(\alpha(t))\frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t)) - \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right|}{\varphi(\alpha(t))}dt \end{array}}{\varphi(x)} + K\varepsilon_2.M$$

$$\leq M(\varepsilon_1 + K\varepsilon_2)$$
$$+K\left[\begin{array}{l} \sup_{t\in I} \frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(t) - \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(t)\right|}{\varphi(t)} \sup_{x\in I} \frac{\int_a^x \mu(x,t)\varphi(t)dt}{\varphi(x)} \\ +\sup_{t\in I} \frac{\left|\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}\right)(\alpha(t)) - \left(\left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)(\alpha(t))\right|}{\varphi(\alpha(t))} \sup_{x\in I} \frac{\int_a^x \eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)} \end{array}\right]$$

$$\leq M(\varepsilon_1 + K\varepsilon_2) + K(\beta+\gamma)d\left(\left(1-\alpha_n^{(2)}\right)x_n^{(1)} + \alpha_n^{(2)}Tx_n^{(1)}, \left(1-\alpha_n^{(2)}\right)\breve{x}_n^{(1)} + \alpha_n^{(2)}\breve{T}\breve{x}_n^{(1)}\right)$$

$$\leq K(\beta+\gamma)\left(\left(1-\alpha_n^{(2)}\right)d(x_n^{(1)},\breve{x}_n^{(1)})+\alpha_n^{(2)}d(Tx_n^{(1)},\breve{T}\breve{x}_n^{(1)})\right)+M(\varepsilon_1+K\varepsilon_2) \tag{12}$$

and

$$d(Tx_n^{(1)},\breve{T}\breve{x}_n^{(1)})=\sup_{x\in I}\frac{\left|Tx_n^{(1)}(x)-\breve{T}\breve{x}_n^{(1)}(x)\right|}{\varphi(x)}$$

$$\leq \sup_{x\in I}\frac{|g(x)-\breve{g}(x)|+\left|\psi\left(\int_a^x f\left(x,t,x_n^{(1)}(t),x_n^{(1)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x \breve{f}\left(x,t,\breve{x}_n^{(1)}(t),\breve{x}_n^{(1)}(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$

$$\leq \sup_{x\in I}\frac{\varepsilon_1}{\varphi(x)}+K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,x_n^{(1)}(t),x_n^{(1)}(\alpha(t))\right)dt-\int_a^x \breve{f}\left(x,t,\breve{x}_n^{(1)}(t),\breve{x}_n^{(1)}(\alpha(t))\right)dt\right|}{\varphi(x)}$$

$$\leq \varepsilon_1 M+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,x_n^{(1)}(t)x_n^{(1)},(\alpha(t))\right)-f\left(x,t,\breve{x}_n^{(1)}(t),\breve{x}_n^{(1)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,\breve{x}_n^{(1)}(t),\breve{x}_n^{(1)}(\alpha(t))\right)-\breve{f}\left(x,t,\breve{x}_n^{(1)}(t),\breve{x}_n^{(1)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$

$$\leq \varepsilon_1.M\ +K\sup_{x\in I}\ \frac{\int_a^x\left(\mu(x,t)\left|x_n^{(1)}(t)-\breve{x}_n^{(1)}(t)\right|+\eta(x,t)\left|x_n^{(1)}(\alpha(t))-\breve{x}_n^{(1)}(\alpha(t))\right|\right)dt}{\varphi(x)}+K\sup_{x\in I}\int_a^x\frac{\varepsilon_2}{b-a}.\frac{1}{\varphi(x)}dt$$

$$\leq \varepsilon_1.M+\ K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|x_n^{(1)}(t)-\breve{x}_n^{(1)}(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|x_n^{(1)}(\alpha(t))-\breve{x}_n^{(1)}(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}+K\varepsilon_2.M$$

$$\leq M(\varepsilon_1+K\varepsilon_2)+K\begin{bmatrix}\sup_{t\in I}\frac{\left|x_n^{(1)}(t)(t)-\breve{x}_n^{(1)}(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}\\+\sup_{t\in I}\frac{\left|x_n^{(1)}(t)(\alpha(t))-\breve{x}_n^{(1)}(\alpha(t))\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\end{bmatrix}$$

$$\leq M(\varepsilon_1+K\varepsilon_2)+K(\beta+\gamma)d\left(x_n^{(1)},\breve{x}_n^{(1)}\right). \tag{13}$$

Combining (12) and (13), we get

$$d(x_n^{(3)},\breve{x}_n^{(3)})\leq K(\beta+\gamma)\left(1-\alpha_n^{(2)}(1-K(\beta+\gamma)\right)d\left(x_n^{(1)},\breve{x}_n^{(1)}\right)+K(\beta+\gamma)\alpha_n^{(2)}M(\varepsilon_1+K\varepsilon_2)+M(\varepsilon_1+K\varepsilon_2). \tag{14}$$

Also, we have

$$d(x_n^{(2)},\breve{x}_n^{(2)})=\sup_{x\in I}\frac{\left|T(Tx_n^{(3)})(x)-\breve{T}(\breve{T}\breve{x}_n^{(3)})(x)\right|}{\varphi(x)}$$

$$\leq \sup_{x\in I}\frac{|g(x)-\breve{g}(x)|+\left|\psi\left(\int_a^x f\left(x,t,Tx_n^{(3)}(t),Tx_n^{(3)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x \breve{f}\left(x,t,\breve{T}\breve{x}_n^{(3)}(t),\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$

$$\leq \sup_{x\in I}\frac{\varepsilon_1}{\varphi(x)}+K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,Tx_n^{(3)}(t),Tx_n^{(3)}(\alpha(t))\right)dt-\int_a^x \breve{f}\left(x,t,\breve{T}\breve{x}_n^{(3)}(t),\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right)dt\right|}{\varphi(x)}$$

$$\leq \varepsilon_1 M+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,Tx_n^{(3)}(t),Tx_n^{(3)},(\alpha(t))\right)-f\left(x,t,\breve{T}\breve{x}_n^{(3)}(t),\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,\breve{T}\breve{x}_n^{(3)}(t),\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right)-\breve{f}\left(x,t,\breve{T}\breve{x}_n^{(3)}(t),\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$

$$\leq \varepsilon_1.M\ +K\sup_{x\in I}\ \frac{\int_a^x\left(\mu(x,t)\left|Tx_n^{(3)}(t)-\breve{T}\breve{x}_n^{(3)}(t)\right|+\eta(x,t)\left|Tx_n^{(3)}(\alpha(t))-\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right|\right)dt}{\varphi(x)}+K\sup_{x\in I}\int_a^x\frac{\varepsilon_2}{b-a}.\frac{1}{\varphi(x)}dt$$

$$\leq \varepsilon_1.M+\ K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|Tx_n^{(3)}(t)-\breve{T}\breve{x}_n^{(3)}(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|Tx_n^{(3)}(\alpha(t))-\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}+K\varepsilon_2.M$$

$$\leq M(\varepsilon_1+K\varepsilon_2)+K\begin{bmatrix}\sup_{t\in I}\frac{\left|Tx_n^{(3)}(t)-\breve{T}\breve{x}_n^{(3)}(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}\\+\sup_{t\in I}\frac{\left|Tx_n^{(3)}(\alpha(t))-\breve{T}\breve{x}_n^{(3)}(\alpha(t))\right|}{\varphi(\alpha(t))}\sup_{x\in I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\end{bmatrix}$$

$$\leq M(\varepsilon_1+K\varepsilon_2)+K(\beta+\gamma)d\left(Tx_n^{(3)},\breve{T}\breve{x}_n^{(3)}\right)$$

and

$$d\left(Tx_n^{(3)},\breve{T}\breve{x}_n^{(3)}\right)=\sup_{x\in I}\frac{\left|Tx_n^{(3)}(x)-\breve{T}\breve{x}_n^{(3)}(x)\right|}{\varphi(x)}$$
$$\leq \sup_{x\in I}\frac{|g(x)-\breve{g}(x)|+\left|\psi\left(\int_a^x f\left(x,t,x_n^{(3)}(t),x_n^{(3)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x \breve{f}\left(x,t,\breve{x}_n^{(3)}(t),\breve{x}_n^{(3)}(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$
$$\leq \sup_{x\in I}\frac{\varepsilon_1}{\varphi(x)}+K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,x_n^{(3)}(t),x_n^{(3)}(\alpha(t))\right)dt-\int_a^x \breve{f}\left(x,t,\breve{x}_n^{(3)}(t),\breve{x}_n^{(3)}(\alpha(t))\right)dt\right|}{\varphi(x)}$$
$$\leq \varepsilon_1 M+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,x_n^{(3)}(t),x_n^{(3)},(\alpha(t))\right)-f\left(x,t,\breve{x}_n^{(3)}(t),\breve{x}_n^{(3)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,\breve{x}_n^{(3)}(t),\breve{x}_n^{(3)}(\alpha(t))\right)-\breve{f}\left(x,t,\breve{x}_n^{(3)}(t),\breve{x}_n^{(3)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$\leq \varepsilon_1.M\ +K\sup_{x\in I}\ \frac{\int_a^x\left(\mu(x,t)\left|x_n^{(3)}(t)-\breve{x}_n^{(3)}(t)\right|+\eta(x,t)\left|x_n^{(3)}(\alpha(t))-\breve{x}_n^{(3)}(\alpha(t))\right|\right)dt}{\varphi(x)}+K\sup_{x\in I}\int_a^x\frac{\varepsilon_2}{b-a}.\frac{1}{\varphi(x)}dt$$
$$\leq \varepsilon_1.M+\ K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|x_n^{(3)}(t)-\breve{x}_n^{(3)}(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|x_n^{(3)}(\alpha(t))-\breve{x}_n^{(3)}(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}+K\varepsilon_2.M$$
$$\leq M(\varepsilon_1+K\varepsilon_2)+K\begin{bmatrix}\sup_{t\in I}\frac{\left|x_n^{(3)}(t)-\breve{x}_n^{(3)}(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}\\+\sup_{t\in I}\frac{\left|x_n^{(3)}(\alpha(t))-\breve{x}_n^{(3)}(\alpha(t))\right|}{\varphi(\alpha(t))}\sup_{x\epsilon I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\end{bmatrix}$$
$$\leq M(\varepsilon_1+K\varepsilon_2)+K(\beta+\gamma)d\left(x_n^{(3)},\breve{x}_n^{(3)}\right).$$

Then, we have

$$d(x_n^{(2)},\breve{x}_n^{(2)})\leq[K(\beta+\gamma)]^2d\left(x_n^{(3)},\breve{x}_n^{(3)}\right)+K(\beta+\gamma)M(\varepsilon_1+K\varepsilon_2)+M(\varepsilon_1+K\varepsilon_2). \tag{15}$$

Substituting (14) in (15), we get

$$d\left(x_n^{(2)},\breve{x}_n^{(2)}\right)\leq[K(\beta+\gamma)]^3\left(1-\alpha_n^{(2)}(1-K(\beta+\gamma)\right)d\left(x_n^{(1)},\breve{x}_n^{(1)}\right)+[K(\beta+\gamma)]^3\alpha_n^{(2)}M(\varepsilon_1+K\varepsilon_2)$$
$$+[K(\beta+\gamma)]^2M(\varepsilon_1+K\varepsilon_2)+K(\beta+\gamma)M(\varepsilon_1+K\varepsilon_2)+M(\varepsilon_1+K\varepsilon_2). \tag{16}$$

Similarly, we obtain

$$d(x_{n+1}^{(1)},\breve{x}_{n+1}^{(1)})=\sup_{x\in I}\frac{\left|Tx_n^{(2)}(x)-\breve{T}\breve{x}_n^{(2)}(x)\right|}{\varphi(x)}$$
$$\leq \sup_{x\in I}\frac{|g(x)-\breve{g}(x)|+\left|\psi\left(\int_a^x f\left(x,t,x_n^{(2)}(t),x_n^{(2)}(\alpha(t))\right)dt\right)-\psi\left(\int_a^x \breve{f}\left(x,t,\breve{x}_n^{(2)}(t),\breve{x}_n^{(2)}(\alpha(t))\right)dt\right)\right|}{\varphi(x)}$$
$$\leq \sup_{x\in I}\frac{\varepsilon_1}{\varphi(x)}+K\sup_{x\in I}\frac{\left|\int_a^x f\left(x,t,x_n^{(2)}(t),x_n^{(2)}(\alpha(t))\right)dt-\int_a^x \breve{f}\left(x,t,\breve{x}_n^{(2)}(t),\breve{x}_n^{(2)}(\alpha(t))\right)dt\right|}{\varphi(x)}$$
$$\leq \varepsilon_1.M+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,x_n^{(2)}(t)x_n^{(2)},(\alpha(t))\right)-f\left(x,t,\breve{x}_n^{(2)}(t),\breve{x}_n^{(2)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$+K\sup_{x\in I}\frac{\int_a^x\left|f\left(x,t,\breve{x}_n^{(2)}(t),\breve{x}_n^{(2)}(\alpha(t))\right)-\breve{f}\left(x,t,\breve{x}_n^{(2)}(t),\breve{x}_n^{(2)}(\alpha(t))\right)\right|dt}{\varphi(x)}$$
$$\leq \varepsilon_1.M\ +K\sup_{x\in I}\ \frac{\int_a^x\left(\mu(x,t)\left|x_n^{(2)}(t)-\breve{x}_n^{(2)}(t)\right|+\eta(x,t)\left|x_n^{(2)}(\alpha(t))-\breve{x}_n^{(2)}(\alpha(t))\right|\right)dt}{\varphi(x)}+K\sup_{x\in I}\int_a^x\frac{\varepsilon_2}{b-a}.\frac{1}{\varphi(x)}dt$$
$$\leq \varepsilon_1.M+\ K\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)\frac{\left|x_n^{(2)}(t)-\breve{x}_n^{(2)}(t)\right|}{\varphi(t)}dt+\int_a^x\eta(x,t)\varphi(\alpha(t))\frac{\left|x_n^{(2)}(\alpha(t))-\breve{x}_n^{(2)}(\alpha(t))\right|}{\varphi(\alpha(t))}dt}{\varphi(x)}+K\varepsilon_2.M$$
$$\leq M(\varepsilon_1+K\varepsilon_2)+K\begin{bmatrix}\sup_{t\in I}\frac{\left|x_n^{(2)}(t)-\breve{x}_n^{(2)}(t)\right|}{\varphi(t)}\sup_{x\in I}\frac{\int_a^x\mu(x,t)\varphi(t)dt}{\varphi(x)}\\+\sup_{t\in I}\frac{\left|x_n^{(2)}(\alpha(t))-\breve{x}_n^{(2)}(\alpha(t))\right|}{\varphi(\alpha(t))}\sup_{x\epsilon I}\frac{\int_a^x\eta(x,t)\varphi(\alpha(t))dt}{\varphi(x)}\end{bmatrix}$$
$$\leq M(\varepsilon_1+K\varepsilon_2)+K(\beta+\gamma)d\left(x_n^{(2)},\breve{x}_n^{(2)}\right). \tag{17}$$

Combining (16) and (17), we get

$$d\left(x_{n+1}^{(1)},\breve{x}_{n+1}^{(1)}\right)\leq[K(\beta+\gamma)]^4\left(1-\alpha_n^{(2)}(1-K(\beta+\gamma)\right)d\left(x_n^{(1)},\breve{x}_n^{(1)}\right)+[K(\beta+\gamma)]^4\alpha_n^{(2)}M(\varepsilon_1+K\varepsilon_2)$$
$$+[K(\beta+\gamma)]^3M(\varepsilon_1+K\varepsilon_2)+[K(\beta+\gamma)]^2M(\varepsilon_1+K\varepsilon_2)+K(\beta+\gamma)M(\varepsilon_1+K\varepsilon_2)+M(\varepsilon_1+K\varepsilon_2).$$

Since $K\,(\beta+\gamma)<1$, then we have

$$\begin{aligned} d\left(x_{n+1}^{(1)},\check{x}_{n+1}^{(1)}\right) &\leq \left(1-\alpha_n^{(2)}(1-K(\beta+\gamma))\right) d\left(x_n^{(1)},\check{x}_n^{(1)}\right)+\alpha_n^{(2)}M(\varepsilon_1+K\varepsilon_2)+4M(\varepsilon_1+K\varepsilon_2) \\ &= \left(1-\alpha_n^{(2)}(1-K(\beta+\gamma))\right) d\left(x_n^{(1)},\check{x}_n^{(1)}\right)+4M\varepsilon_1+\alpha_n^{(2)}M\varepsilon_1+4MK\varepsilon_2+\alpha_n^{(2)}MK\varepsilon_2. \end{aligned} \tag{18}$$

Using assumption (i), we obtain

$$\frac{1}{\alpha_n^{(2)}} \leq 2 \quad \text{for all } n\in\mathbb{N}.$$

Hence, from (18), we get

$$d\left(x_{n+1}^{(1)},\check{x}_{n+1}^{(1)}\right) \leq \left(1-\alpha_n^{(2)}(1-K(\beta+\gamma))\right) d\left(x_n^{(1)},\check{x}_n^{(1)}\right)+\alpha_n^{(2)}(1-K(\beta+\gamma))\frac{9M(\varepsilon_1+K\varepsilon_2)}{1-K(\beta+\gamma)}. \tag{19}$$

It is clear that the inequality (19) satisfies all conditions in Lemma 1, and hence it follows that

$$d(y_0,\check{y}_0) \leq \frac{9M(\varepsilon_1+K\varepsilon_2)}{1-K(\beta+\gamma)}.$$

Remark 1. The results of Theorems 2 and 3 can be proved similarly for the iteration method (2).

ACKNOWLEDGMENTS

The authors would like to thank the editor and referees their careful reading and valuable comments and suggestions which led to the present form of the paper. This paper was supported by Sakarya University Scientific Research Projects Coordination Unit. (Project Number: 2017-02-00-003).

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