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A Boundary Value Problem with Retarded Argument Containing an Eigenparameter in the Transmission Condition

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Abstract

In this work, a discontinuous boundary value problem with retarded argument is studied. At the discontinuity point there is a transmission condition that contains a parameter. Asymptotic properties of eigenvalues and corresponding eigenfunctions of the boundary value problem are studied.

Keywords: Retarded argument, transmission condition, eigenvalue, eigenfunction, boundary value problem

Geçiş koşulunda Özdeğer Parametresi İçeren Geç Kalan Argümanlı Bir Sınır Değer Problemi

Öz

Bu çalışmada, süreksizlik içeren geç kalan argümanlı bir sınır değer problemi ele alınmıştır. Süreksizlik noktasında parametre içeren bir geçiş koşulu vardır. Ele alınan sınır değer probleminin özdeğer ve özfonksiyonlarının asimptotik özellikleri incelenecektir.

Anahtar Kelimeler: Geç kalan argüman, geçiş koşulu, özdeğer, özfonksiyon, sınır değer problemi

INTRODUCTION

Differential equations with retarded argument becomes the popular branches of functional differential equations. After the development of control systems in engineering retarded equations become important. Before that scientists were aware of this type of delays in the control systems but there was not enough theory about this subject. Because of that this type of affects were ignored in the models. Delays have an important role to explain complex models mathematically and it also has important affects.

Standart problem in this area was given in 1956 by Norkin (Norkin,1956). And in 1958 Norkin studied the same equation with more general boundary conditions(Norkin, 1958). Most of the work done in this area is seperated into two parts as continuous and discontinuous problems. Continuous problems have either standart boundary conditions or eigenparameters at the boundary conditions (Norkin, 1972; Bayramoğlu, Köklü, Baykal 2002; Koparan 2019) Discontinuous problems varries as having eigenparameter at the boundary condition and having transmission conditions at the discontinuity points (Şen, Bayramov 2011; Şen, Bayramov 2011b; Yang 2012; Aydin Akgun, Bayramov, Bayramoğlu 2013; Şen, Seo, Araci 2013, Hira 20017). There is only one work done about the problem that has eigenparameter at the transmission condition (Şen, Bayramov 2013). Applications of differential equations with retarded argument is given in the book by Kolmanovskii, and Myshkis (Kolmanovskii and Myshkis, 1999).

In this work, below differential equation with retarded argument will be studied. Here the problem has discontinuity and at the discontinuity point transmission conditions have eigenparameter.

$$y''(x) + \lambda^2 y(x) + q(x) y\big(x - \Delta(x)\big) = 0, \qquad x \in \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right] \tag{1}$$

$$y(0) + \alpha y'(0) = 0, \tag{2}$$

$$y(\pi) = 0, \tag{3}$$

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and the transmission conditions

$$y\left(\frac{\pi}{2}+0\right)=\frac{\delta}{\lambda}y\left(\frac{\pi}{2}-0\right), \quad (4)$$

$$y'\left(\frac{\pi}{2}+0\right)=\frac{\delta}{\lambda}y'\left(\frac{\pi}{2}-0\right), \quad (5)$$

where $q(x)$ and $\Delta(x)\geq 0$ are real valued functions continuous in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and have finite left and right limits at each side of $\frac{\pi}{2}$, if $x\in\left[0,\frac{\pi}{2}\right)$ then $x-\Delta x\geq 0$, if $x\in\left(\frac{\pi}{2},\pi\right]$ then $x-\Delta x\geq\frac{\pi}{2}$, λ is an eigenparameter and $\alpha,\delta\neq 0$ are arbitrary real numbers.

Consider the solution of equation (1) on $\left[0,\frac{\pi}{2}\right)$ such that it satisfies the initial conditions

$$\omega_1(0,\lambda)=\alpha, \qquad \omega'_1(0,\lambda)=-1. \quad (6)$$

These initial conditions (6) define a unique solution of equation (1) on $\left[0,\frac{\pi}{2}\right)$ (Norkin,1972). By transmission conditions, solution of equation (1) on $\left(\frac{\pi}{2},\pi\right]$ can be defined in terms of $\omega_1(x,\lambda)$. $\omega_2(x,\lambda)$ can be written as

$$\omega_2\left(\frac{\pi}{2},\lambda\right)=\frac{\delta}{\lambda}\omega_1\left(\frac{\pi}{2},\lambda\right), \quad (7)$$

$$\omega'_2\left(\frac{\pi}{2},\lambda\right)=\frac{\delta}{\lambda}\omega'_1\left(\frac{\pi}{2},\lambda\right). \quad (8)$$

These initial contions (7) define a unique solution of equation (1) on $\left(\frac{\pi}{2},\pi\right]$.

Now, we can define the function $\omega(x,\lambda)$ on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ such that

$$\omega(x,\lambda)=\begin{cases}\omega_1(x,\lambda), & x\in[0\frac{\pi}{2})\\ \omega_2(x,\lambda), & x\in\left(\frac{\pi}{2}\pi\right]\end{cases}. \quad (9)$$

This function is a solution of equation (1) on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, that satisfies left boundary condition (2) and both transmission conditions (4) and (5).

Lemma 1: Let $\omega(x,\lambda)$ be a solution of equation (1) and $\lambda>0$. Then $\omega_1(x,\lambda)$ and $\omega_2(x,\lambda)$ have the following forms:

$$\omega_1(x,\lambda)=\alpha cos\lambda x-\frac{sin\lambda x}{\lambda}+$$
$$+\frac{1}{\lambda}\int_0^x q(\tau)sin\,\lambda(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau, \quad (10)$$

$$\omega_2(x,\lambda)=\frac{\delta}{\lambda}\omega_1\left(\frac{\pi}{2},\lambda\right)cos\lambda\left(x-\frac{\pi}{2}\right)+$$
$$+\frac{\delta}{\lambda^2}\omega'_1\left(\frac{\pi}{2},\lambda\right)sin\lambda(x-\frac{\pi}{2})+$$
$$+\frac{1}{\lambda}\int_{\frac{\pi}{2}}^x q(\tau)sin\,\lambda(x-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau. \quad (11)$$

Proof: By differentiating $\omega_1(x,\lambda)$ and $\omega_2(x,\lambda)$ twice and applying integration by parts we see that (10) and (11) satisfy equation (1).

Theorem 1: The eigenvalue problem (1)-(5) have only simple eigenvalues.

Proof: Let $\tilde{\lambda}$ be an eigenvalue of the problem (1)-(5) and

$$\tilde{u}(x,\tilde{\lambda})=\begin{cases}\tilde{u}_1(x,\tilde{\lambda}), & x\in[0\frac{\pi}{2})\\ \tilde{u}_2(x,\tilde{\lambda}), & x\in\left(\frac{\pi}{2}\pi\right]\end{cases}$$

be a corresponding eigenfunction. Then from (2) and (6) it follows that the Wronskian

$$W[\tilde{u}_1(0,\tilde{\lambda}),\tilde{\omega}_1(0,\tilde{\lambda})]=\begin{vmatrix}\widetilde{u_1}(0,\tilde{\lambda}) & \alpha\\ \tilde{u}'_1(0,\tilde{\lambda}) & -1\end{vmatrix}=0.$$

Then it means that $\tilde{u}_1(x,\tilde{\lambda})$ and $\tilde{\omega}_1(0,\tilde{\lambda})$ are linearly dependent on $[0\frac{\pi}{2})$. Similarly it can be proved that $\tilde{u}_2(x,\tilde{\lambda})$ and $\tilde{\omega}_2(0,\tilde{\lambda})$ are linearly dependent. Consequently, it follows that $\omega(x,\tilde{\lambda})$ is an eigenfunction for the boundary value problem (1)-(5) and all the eigenfunctions to this problem corresponding to $\tilde{\lambda}$ are linearly dependent. Hence eigenvalues of the problem are simple.

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MATERIAL AND METHODS

The function $\omega(x,\lambda)$ defined by (10) is a solution of equation (1) satisfying left boundary condition (2) and transmission conditions (4) and (5). Writing $\omega(x,\lambda)$ into (3), characteristic equation is obtained.

$$F(\lambda)=\omega(\pi,\lambda)=\omega_2(\pi,\lambda)=0. \tag{12}$$

Theorem 1 guarantees that the set of eigenvalues of the boundary value problem (1)-(5) coincides with the set of real roots of equation (12).

Lemma 2:

1. Let $\lambda\geq 2q_1$. Then $\omega_1(x,\lambda)$ given by equation (10) satisfy the following inequality:

$$|\omega_1(x,\lambda)|\leq\frac{1}{q_1}\sqrt{4q_1^2\alpha^2+1}. \tag{13}$$

2. Let $\lambda\geq max\{2q_1,2q_2\}$. Then $\omega_2(x,\lambda)$ given by equation (10) satisfies the following inequality:

$$|\omega_2(x,\lambda)|\leq\frac{4|\delta|}{\lambda q_1}\sqrt{4q_1^2\alpha^2+1}. \tag{14}$$

Proof: Let $B_{1\lambda}=max_{\left[0,\frac{\pi}{2}\right]}|\omega_1(x,\lambda)|$. Then from (10), it follows that for every $\lambda>0$, the following inequality holds:

$$B_{1\lambda}\leq\sqrt{\alpha^2+\frac{1}{\lambda^2}}+\frac{1}{\lambda}q_1B_{1\lambda}. \tag{15}$$

For $\lambda\geq 2q_1$, (13) is obtained. Differentiating (10) with respect to x,

$$\omega_1{}'(x,\lambda)=-\alpha\lambda sin\lambda x-\cos\lambda x-$$
$$-\int_0^x q(\tau)cos\,\lambda(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \tag{16}$$

is obtained. From (13) and (16), for $\lambda\geq 2q_1$, we have $|\omega_1'(x,\lambda)|\leq\sqrt{\alpha^2\lambda^2+1}+\sqrt{4q_1^2\alpha^2+1}$.

Hence for $\lambda\geq 2q_1$, and $x\in\left[0\frac{\pi}{2}\right)$,

$$\frac{|\omega_1{}'(x,\lambda)|}{\lambda}\leq\frac{1}{q_1}\sqrt{4q_1^2\alpha^2+1}. \tag{17}$$

Let $B_{2\lambda}=max_{\left[\frac{\pi}{2},\pi\right]}|\omega_2(x,\lambda)|$, then from (10), (13) and (17), for $\lambda\geq 2q_1$, the following inequality holds:

$$B_{2\lambda}\leq\frac{2|\delta|}{\lambda q_1}\sqrt{4q_1^2\alpha^2+1}+\frac{1}{\lambda}q_2B_{2\lambda}.$$

Hence for $\lambda\geq max\{2q_1,2q_2\}$, (14) is obtained.

Theorem 2: Eigenvalues of the problem (1)-(5) form an infinite set of positive real numbers.

Proof: Differentiating (10) with respect to x,

$$\omega_2'(x,\lambda)=-\delta\omega_1\left(\frac{\pi}{2},\lambda\right)sin\lambda\left(x-\frac{\pi}{2}\right)+$$
$$\frac{\delta}{\lambda}\omega_1'\left(\frac{\pi}{2},\lambda\right)cos\lambda\left(x-\frac{\pi}{2}\right)+$$
$$+\frac{1}{\lambda}\int_{\frac{\pi}{2}}^{x}q(\tau)cos\,\lambda(x-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau. \tag{18}$$

Writing (10), (11), (16) and (18) into the characteristic equation (12), we obtain,

$$\frac{\alpha\delta}{\lambda}cos\lambda\pi-\frac{\delta}{\lambda^2}sin\lambda\pi+$$
$$+\frac{\delta}{\lambda^2}\int_0^{\frac{\pi}{2}}q(\tau)sin\,\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau$$
$$+\frac{1}{\lambda}\int_{\frac{\pi}{2}}^{\pi}q(\tau)sin\,\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau=0. \tag{19}$$

For $\alpha\neq 0$ and sufficiently large λ, by (13) and (14) equation (19) may be written as

$$\lambda cos\lambda\pi+\,O(1)=0. \tag{20}$$

Clearly there are infinitely many real numbers λ that satisfy (20).

Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we will study the asymptotic behavior of eigenvalues and eigenfunctions. From now on we will assume λ is sufficienly large. On $\left[0\frac{\pi}{2}\right)$, from (9) and (13)

$$\omega_1(x,\lambda)=O(1). \tag{21}$$

On $\left(\frac{\pi}{2}\pi\right]$, from (10) and (14)

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$$\omega_2(x,\lambda) = O\left(\frac{1}{\lambda}\right). \tag{22}$$

Derivatives of of $\omega_1(x,\lambda)$ and $\omega_2(x,\lambda)$ with respect to λ on $x \in [0\frac{\pi}{2})$, and $x \in \left(\frac{\pi}{2}\pi\right]$ respectively, exist and are continuous for $|\lambda| < \infty$ by Norkin (Norkin, 1972).

Lemma 3:

$$\omega'_{1\lambda}(x,\lambda) = O(1), for\ x \in \left[0,\frac{\pi}{2}\right], \tag{23}$$

$$\omega'_{2\lambda}(x,\lambda) = O\left(\frac{1}{\lambda}\right), \text{ for } x \in \left[\pi,\frac{\pi}{2}\right], \tag{24}$$

Proof: Differentiating (9) with respect to λ and by (21),

$$\omega'_{1\lambda}(x,\lambda) = \frac{1}{\lambda}\int_0^x q(\tau)sin\,\lambda(x-\tau)\omega'_{1\lambda}(\tau - \Delta(\tau),\lambda)d\tau + K(x,\lambda),\ (|K(x,\lambda)| \le K_0). \tag{25}$$

Let $C_{1\lambda} = max_{\left[0,\frac{\pi}{2}\right]}|\omega'_{1\lambda}(x,\lambda)|$. The existence of $C_{1\lambda}$ follows from the continuity of derivative for $x \in \left[0,\frac{\pi}{2}\right]$. From (25)

$$C_{1\lambda} = \frac{1}{\lambda}q_1 C_{1\lambda} + K_0$$

is obtained. So for $\lambda \ge 2q_1$, $C_{1\lambda} \le 2K_0$. Therefore (23) is proved. Similarly (24) may be proved.

Theorem 3: Let n be a sufficiently large natural number. Then there is only one eigevalue of the problem (1)-(5) around $n+\frac{1}{2}$.

Proof: First consider the $O(1)$ term in equation (20)

$$-\frac{\lambda}{\alpha}sin\lambda\pi + \\ +\frac{1}{\alpha}\int_0^{\frac{\pi}{2}} q(\tau)sin\,\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau + \\ +\frac{\lambda}{\alpha}\int_{\frac{\pi}{2}}^{\pi} q(\tau)sin\,\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau$$

From (21)-(24), this expression has bounded derivative with respect to λ. It is clear that for λ big enough roots of equation (20) are located close to $n+\frac{1}{2}$. We need to show that there is only one solution of (20) around each $n+\frac{1}{2}$.

Consider the function $F(\lambda) = \lambda cos\lambda\pi + O(1)$. Its derivative $F'(\lambda) = \cos\lambda\pi - \lambda\pi\sin\lambda\pi + O(1)$ is equal to zero for λ close to $n+\frac{1}{2}$ for sufficiently large n. Consequently by Rolle's Theorem proof is completed.

From equation (20) eigenvalues of the boundary value problem (1)-(5) are obtained as:

$$\lambda_n = n + \frac{1}{2} + O\left(\frac{1}{n}\right), \tag{26}$$

From equation (10), (16) and (21)

$$\omega_1(x,\lambda) = \alpha cos\lambda x + O\left(\frac{1}{\lambda}\right), \tag{27}$$

$$\omega'_1(x,\lambda) = -\alpha\lambda\sin\lambda x + O(1), \tag{28}$$

From equation (11), (22), (27) and (28),

$$\omega_2(x,\lambda) = \frac{\alpha\delta}{\lambda}cos\,\lambda\,x + O\left(\frac{1}{\lambda^2}\right). \tag{29}$$

Hence eigenfunctions $u_n(x)$ have the following asymptotic representation.

$$u_n(x) = \begin{cases} \alpha\ cos\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n}\right),\ x \in \left[0,\frac{\pi}{2}\right) \\ \frac{\delta}{n}\alpha\ cos\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n^2}\right), x \in \left(\frac{\pi}{2},\pi\right] \end{cases}$$

RESULTS AND DISCUSSION (Main title)

Under additional assumptions we can improve these formulas for the eigenvalues and the eigenfunctions.

Improved Asymptotic Representations for the Eigenvalues and Eigenfunctions

Theorem 4: Assume that $q'(x)$ and $\Delta''(x)$ exist, are bounded on $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$ and have left and right limits at $\frac{\pi}{2}$, on $\left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$, $\Delta'(x) \le 1$, $\Delta(0) = 0$ and $\lim_{x\to\frac{\pi}{2}\mp 0}\Delta(x) = 0$. Then for $n\to\infty$, eigenvalues of the problem (1)-(5) are as follows:

$$\lambda_n = n + \frac{1}{2} + \frac{\alpha B(\pi,n,\Delta(\tau))-1}{\alpha n\pi} + O\left(\frac{1}{n^2}\right) \tag{30}$$

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Proof: It is clear that from (27) and (29)

$$\omega_1(\tau-\Delta(\tau),\lambda) = \alpha\cos\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda}\right) \quad (31)$$
$$\omega_2(\tau-\Delta(\tau),\lambda) = \frac{\alpha\delta}{\lambda}\cos\lambda(\tau-\Delta(\tau)) + O\left(\frac{1}{\lambda^2}\right) \quad (32)$$

Writing these expressions into the equation (19) we obtain

$$\lambda\alpha\delta\cos\lambda\pi - \delta\sin\lambda\pi +$$
$$+\alpha\delta\int_0^\pi q(\tau)sin\,\lambda\pi\frac{1}{2}[\cos\lambda\Delta(\tau) + \cos\lambda(2\tau-\Delta(\tau))]d\tau$$
$$-\delta\int_0^\pi q(\tau)sin\,\lambda\tau\frac{1}{2}[sin\,\lambda\Delta(\tau) + sin\,\lambda(2\tau-\Delta(\tau))]d\tau + O\left(\frac{1}{\lambda}\right) = 0. \quad (33)$$

Defining

$$A(x,\lambda,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)sin\lambda\,\Delta(\tau)d\tau,$$
$$B(x,\lambda,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\cos\lambda\,\Delta(\tau)d\tau, \quad (34)$$

and the Lemma III.3.3 in (Norkin,1972),

$$\int_0^x q(\tau)\cos\lambda(2\tau-\Delta(\tau))\,d\tau = O\left(\frac{1}{\lambda}\right),$$
$$\int_0^x q(\tau)\,sin\,\lambda\left(2\tau-\Delta(\tau)\right)d\tau = O\left(\frac{1}{\lambda}\right). \quad (35)$$

Using (34) and (35), equation (33) can be written as

$$\lambda\alpha\delta\cos\lambda\pi - \delta\sin\lambda\pi +$$
$$\alpha\delta\sin\lambda\pi B(\pi,\lambda,\Delta(\tau)) - \alpha\delta\cos\lambda\pi\,A(\pi,\lambda,\Delta(\tau)) + O\left(\frac{1}{\lambda}\right) = 0\,.$$

Hence

$$\cot\lambda\pi = \frac{1-\alpha B(\pi,\lambda,\Delta(\tau))}{\alpha\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

Writing $\lambda=\lambda_n=n+\frac{1}{2}+\delta_n$ into above equation

$$\delta_n = \frac{\alpha B(\pi,n,\Delta(\tau))-1}{\alpha n\pi} + O\left(\frac{1}{n^2}\right)$$

and

$$\lambda_n = n + \frac{1}{2} + \frac{\alpha B(\pi,n,\Delta(\tau))-1}{\alpha n\pi} + O\left(\frac{1}{n^2}\right)$$

are obtained.

Theorem 5: Under the same hypothesis of Theorem 4 eigenfunctions of the boundary value problem (1)-(5) are as follows:

$$u_{1n}(x) = \alpha\left(1-\frac{A(x,n,\Delta(\tau))}{n}\right)\cos\left(n+\frac{1}{2}\right)x - \frac{(\alpha B(\pi,n,\Delta(\tau))-1)x}{n\pi}\sin\left(n+\frac{1}{2}\right)x + \left(\frac{\alpha B(x,n,\Delta(\tau)-1}{n}\right)\sin\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n^2}\right), \quad (36)$$

$$u_{2n}(x) = \frac{\alpha\delta}{n}\left(1-\frac{A(x,n,\Delta(\tau)1}{n}\right)\cos\left(n+\frac{1}{2}\right)x - \frac{\delta(\alpha B(\pi,n,\Delta(\tau))-1)x}{n^2\pi}\sin\left(n+\frac{1}{2}\right)x + \frac{\delta(\alpha B(x,n,\Delta(\tau))-1)}{n^2}\sin\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n^3}\right). \quad (37)$$

Proof: From (10) and (31) we obtain

$$\omega_1(x,\lambda) = \alpha cos\lambda x - \frac{sin\lambda x}{\lambda} + \frac{\alpha}{\lambda}\int_0^x q(\tau)sin\,\lambda(x-\tau)\cos\lambda(\tau-\Delta(\tau))\,d\tau + O\left(\frac{1}{\lambda^2}\right).$$

Using (34) and (35),

$$\omega_1(x,\lambda) = \alpha\left(1-\frac{A(x,\lambda,\Delta(\tau))}{\lambda}\right)cos\lambda x + \left(\frac{\alpha B(x,\lambda,\Delta(\tau))-1}{\lambda}\right)\sin\lambda x + O\left(\frac{1}{\lambda^2}\right). \quad (38)$$

Replacing λ with λ_n and using (30),

$$u_{1n}(x) = \alpha\left(1-\frac{A(x,n,\Delta(\tau))}{n}\right)\cos\left(n+\frac{1}{2}\right)x - \frac{(\alpha B(\pi,n,\Delta(\tau))-1)x}{n\pi}\sin\left(n+\frac{1}{2}\right)x + \left(\frac{\alpha B(x,n,\Delta(\tau)-1}{n}\right)\sin\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n^2}\right).$$

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Writing (10), (16), (31) and (32) into the equation (11), we have

$$\omega_2(x,\lambda) = \frac{\alpha\delta}{\lambda} cos\lambda x - \frac{sin\lambda x}{\lambda^2} + \frac{\alpha\delta}{\lambda^2}\int_0^x q(\tau) sin\,\lambda(x-\tau)\cos\lambda(\tau - \Delta(\tau))\, d\tau + +O\left(\frac{1}{\lambda^3}\right).$$

Using (34) and (35),

$$\omega_2(x,\lambda) = \frac{\alpha\delta}{\lambda}\left(1 - \frac{A(x,\lambda,\Delta(\tau))}{\lambda}\right) cos\lambda x + +\delta\left(\frac{\alpha B(x,\lambda,\Delta(\tau))-1}{\lambda^2}\right)\sin\lambda x + O\left(\frac{1}{\lambda^3}\right). \quad (39)$$

Replacing λ with λ_n and using (30),

$$u_{2n}(x) = \frac{\alpha\delta}{n}\left(1 - \frac{A(x,n,\Delta(\tau)1}{n}\right)\cos\left(n+\frac{1}{2}\right)x - \frac{\delta(\alpha B(\pi,n,\Delta(\tau)) - 1)x}{n^2\pi}\sin\left(n+\frac{1}{2}\right)x + \frac{\delta(\alpha B(x,n,\Delta(\tau)) - 1)}{n^2}\sin\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n^3}\right)$$

is obtained.

CONCLUSION

Eigenvalues and eigenfunctions of the (1)-(5) boundary value problem have the following asymptotic representation:

$$\lambda_n = n + \frac{1}{2} + \frac{\alpha B(\pi,n,\Delta(\tau))-1}{\alpha n\pi} + O\left(\frac{1}{n^2}\right)$$

$$u_n(x) = \begin{cases} u_{1n}(x), & x \in \left[0,\frac{\pi}{2}\right) \\ u_{2n}(x), & x \in \left(\frac{\pi}{2},\pi\right] \end{cases}$$

here $u_{1n}(x)$ and $u_{2n}(x)$ are given by the equation (36) and (37) respectively.

CONFLICT OF INTEREST

The authors have no conflicts of interest to declare.

RESEARCH AND PUBLICATION ETHICS STATEMENT

The authors declare that this study complies with research and publication ethics.

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