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# Some Properties of CR-submanifolds of an S-manifold with a Semi-Symmetric Metric Connection

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**Abstract**

We define a semi-symmetric metric connection in an S-manifold and study CR-submanifolds of an S-manifold with a semi-symmetric metric connection. Moreover, we also obtain integrability and parallel conditions of the distributions on CR-submanifolds. Finally, we give some results of the sectional curvatures of CR-submanifolds of an S-space form with a semi-symmetric metric connection.
**Keywords**—CR-submanifold, S-manifold, S-space form, Semi-symmetric metric connection, Distributions.

## 1 Introduction

Many authors have studied the geometry of submanifolds of Kaehlerian and Sasakian manifolds. In this manner, the notion of a *CR*-submanifold of Kaehler manifold was introduced by Bejancu in [4]. Later, *CR*-submanifold of Sasakian manifolds were studied by Kobayaski in [17]. For manifolds with an *f*-structure, Blair has initiated the study of *S*-manifolds, which reduce, in particular cases, to Sasakian manifolds. Mihai [18] and Ornea [19] have investigated *CR*-submanifold of S-manifolds. Also, Algahemi studied *CR*-submanifold of an *S*-manifold in [3]. For CR-submanifolds see also: ([11], [12], [20]). In [10], Cabrerizo et al. are studied curvature of submanifolds of an *S*-space form. They are investigated some properties of invariant and anti-invarinat submanifolds of an *S*-space forms with constant sectional curvature.

Let $\nabla$ be a linear connection in an $n$-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of $\nabla$ are given respectively by [5].

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection $\nabla$ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection $\nabla$ is a metric connection if there is a Riemannian metric $g$ in M such that $\nabla g = 0$ otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection. In [16], Friedmann and Schouten introduced the idea of a semi-symetric linear connection. A linear connection $\nabla$ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y$$

where $\eta$ is a 1-form. In [23], Yano studied some properties of semi-symmetric metric connections. The semi-symmetric connection is important in Riemannian manifolds having also physical applications. The purpose of the present paper is to study CR-submanifolds of an *S*-manifold endowed with a semi-symetric metric connection.

The paper is organized as follows: In Section 2, we give a brief description of *S*-manifolds. In Section 3, we give some properties of CR-submanifolds of *S*-manifolds and find necessary conditions for the induced connection on CR-submanifolds of an *S*-manifold with a semi-symmetric metric connection to be also a semi-symmetric metric connection. In Section 4, we obtain some basic lemmas of CR-submanifold of an *S*-manifold with a semi-symmetric metric connection. In Section 5, we investigate the integrability conditions of $D$ and $D^{\perp}$ distributions of CR-submanifolds of an *S*-manifold with a semi-symmetric

metric connection. In Section 6, we study the geometry of foliations of CR-submanifolds of an *S*-manifold with a semi-symmetric metric connection. Finally, in the last section, we give CR-submanifolds of *S*-space forms with a semi-symmetric metric connection. Some results of the sectional curvatures of CR-submanifolds of *S*-space forms are studied.

**2 S-manifolds**

Let $(\widetilde{M}, g)$ be a (2n+s)-dimensional Riemannian manifold. Then, it is said to be a metric *f*-manifold if there exist on $(\widetilde{M}, g)$ an *f*-structure *f*, that is a tensor field *f* of type (1,1) satisfying $f^3 + f = 0$ (see [22]), of rank *2n* and *s* local vector fields $\xi_1, \ldots, \xi_s$ (called structure vector fields) such that, if $\eta^1, \ldots, \eta^s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$ then

$$f\xi_\alpha = 0, \eta^\alpha \circ f = 0, f^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha \qquad (2.1)$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y) \qquad (2.2)$$

for any $X, Y \in \Gamma(T\widetilde{M})$ and $\alpha = 1, \ldots, s$ The *f*-structure *f* is normal if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\,\eta^\alpha = 0,$$

where $[f, f]$is the Nijenhuis tensor fields of *f*. Let *F* be the fundamental 2-form defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \Gamma(T\widetilde{M})$. Then $\widetilde{M}$ is said to be an *S*-manifold if the *f*-structure is normal and

$$\eta^1 \wedge \ldots \wedge \eta^s \wedge (d\eta^\alpha)^n \neq 0, \quad F = d\eta^\alpha$$

for any $\alpha = 1, \ldots, s$ In this case, the structure vector fields are Killing vector fields. When *s=1* *S*-manifolds are Sasakian manifolds.

The Riemannian connection $\widetilde{\nabla}$ of an *S*-manifold satisfying ([7])

$$(\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} \{g(fX, fY)\xi_\alpha + \eta^\alpha(Y)f^2X\} \qquad (2.3)$$

and

$$\widetilde{\nabla}_X \xi_\alpha = -fX \qquad (2.4)$$

for any $X, Y \in \Gamma(T\widetilde{M})$ and $\alpha = 1, \ldots, s$.

**3 *CR*-Submanifold of *S*-Manifolds**

**Definition 3.1** An (2m+s)-dimensional Riemannian submanifold *M* of S-manifold $\widetilde{M}$ is called a CR-submanifold if $\xi_1, \ldots, \xi_s$ is tangent to M and there exists on *M* two differentiable distributions *D* and $D^\perp$ on *M* satisfying:

1. $TM = D \oplus D^\perp \oplus sp\{\xi_1, \ldots, \xi_s\}$

2. The distribution *D* is invariant under f that is $fD_x = D_x$ for any $x \in M$

3. The distribution $D^\perp$ is anti-invariant under *f*, that is, $fD^\perp \subseteq T_x^\perp M$ for any $x \in M$ where $T_xM$ and $T_x^\perp M$ are the tangent space of *M* at *x*.

We denote by *2p* and *q* the real dimensions of $D_x$ and $D_x^\perp$ respectively, for any $x \in M$. Then if *p=0* we have an anti-invariant submanifold tangent to $\xi_1, \ldots, \xi_s$ and if *q=0* we have an invariant submanifold.

Now, we give the following example.

**Example 3.1** In what follows, $(\mathbb{R}^{2n+s}, f, \eta, \xi, g)$ will denote the manifold $\mathbb{R}^{2n+s}$ with its usual *S*-structure given by

$$\eta^\alpha = \frac{1}{2}(dz_\alpha - \sum_{i=1}^{n} y_i\, dx_i), \;\; \xi_\alpha = 2\frac{\partial}{\partial z_\alpha}$$

$$f\left(\sum_{i=1}^{n}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s} z_\alpha \frac{\partial}{\partial z_\alpha}\right) = \sum_{i=1}^{n}(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s}\sum_{i=1}^{n} Y_i\, y_i \frac{\partial}{\partial z_\alpha}$$

$$g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}(\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i),$$

$(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_\alpha)$ denoting the Cartesian coordinates on $\mathbb{R}^{2n+s}$. The consider a submanifold of $\mathbb{R}^{10}$ defined by

$$M = X(u, vk, l, t_1, t_2) = 2(u, k, 0,0, v, 0, l, 0, t_1, t_2)$$

Then local frame of TM

$$e_1 = 2\frac{\partial}{\partial x_1}, e_2 = 2\frac{\partial}{\partial y_1}, e_3 = 2\frac{\partial}{\partial x_2},$$

$$e_4 = 2\frac{\partial}{\partial y_3}, e_5 = 2\frac{\partial}{\partial z_1} = \xi_1, e_6 = 2\frac{\partial}{\partial z_2} = \xi_2$$

and

$$e_1^* = \frac{\partial}{\partial x_3}, e_2^* = \frac{\partial}{\partial y_2}$$

from a basis of $T^\perp M$. We determine $D_1 = sp\{e_1, e_2\}$ and $D_2 = sp\{e_3, e_4\}$. Then $D_1$, $D_2$ are invariant and anti-invariant distribution, respectively. Thus $TM = D_1 \oplus D_2 \oplus sp\{\xi_1, \xi_2\}$ is a CR-submanifold of $\mathbb{R}^{10}$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ with respect to the induced metric *g*. Then Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla^*_X Y + h(X,Y) \quad (3.1)$$
$$\widetilde{\nabla}_X N = \nabla^{*\perp}_X N - A_N X \quad (3.2)$$

for any $X,Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$. $\nabla^{*\perp}$ is the connection in the normal bundle, $h$ is the second fundamental from of $\widetilde{M}$ and $A_N$ is the Weingarten endomorphism associated with *N*. The second fundamental form *h* and the shape operator *A* related by

$$g(h(X,Y),N) = g(A_N X, Y). \quad (3.3)$$

Let $M$ be CR-submanifold of $\widetilde{M}$. $M$ is said to be totally geodesic if $h(X,Y) = 0$ for any $X,Y \in \Gamma(TM)$.

We denote by $\tilde{R}$ and *R* the curvature tensor fields associated with $\widetilde{\nabla}$ and $\nabla$ respectively. The Gauss equation is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g\big(h(X,Z),h(Y,W)\big) - g(h(X,W),h(Y,Z))$$

for all $X,Y,Z,W \in \Gamma(TM)$.

The projection morphisms of TM to $D$ and $D^\perp$ are denoted by $P$ and $Q$ respectively. For any $X,Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, we have

$$X = PX + QX + \textstyle\sum_{\alpha=1}^{s} \eta^\alpha(X)\xi_\alpha, 1 \le \alpha \le s \quad (3.4)$$
$$fN = BN + CN \quad (3.5)$$

where *BN* (resp. *CN*) denotes the tangential (resp. normal) component of *fN*.

Now, we define a connection $\overline{\nabla}$ as

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \textstyle\sum_{\alpha=1}^{s}\{\eta^\alpha(Y)X - g(X,Y)\,\xi_\alpha\}.$$

Then, $\overline{\nabla}$ is lineer connection.
Let $\overline{\mathrm{T}}$ be the torsion tensor of $\overline{\nabla}$. Then, for all $X,Y \in \Gamma(T\widetilde{M})$

$$\overline{\mathrm{T}}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = \textstyle\sum_{\alpha=1}^{s}\{\eta^\alpha(Y)X - \eta^\alpha(X)Y\} \quad (3.6)$$

Then $\overline{\nabla}$ is semi symmetric. Morever we get,

$$(\overline{\nabla}_X g)(Y,Z) = X[g(Y,Z)] - g(\overline{\nabla}_X Y, Z) - g(Y, \overline{\nabla}_X Z).$$

In view of (3.6) and the above equation, we give the following theorem.

**Theorem 3.1** *Let $\widetilde{\nabla}$ be the Riemannian connection on an S-manifold $\widetilde{M}$. Then the linear connection which is defined as*

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \textstyle\sum_{\alpha=1}^{s}\{\eta^\alpha(Y)X - g(X,Y)\,\xi_\alpha\} \quad (3.7)$$

is a semi-symmetric metric connection on $\widetilde{M}$.

**Theorem 3.2** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$. Then*

$$(\overline{\nabla}_X f)Y = \textstyle\sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\} \quad (3.8)$$

for all $X,Y \in \Gamma(TM)$.

*Proof*. By the use of (3.7), we get

$$(\overline{\nabla}_X f)Y = (\widetilde{\nabla}_X f)Y - \textstyle\sum_{\alpha=1}^{s}\{g(X,fY)\xi_\alpha - \eta^\alpha(Y)X\}$$

for all $X,Y \in \Gamma(TM)$. Now using (2.3), we obtain (3.8).

As an immediate consequence of Theorem 3.2 we have the following result.

**Corollary 3.1** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection $\overline{\nabla}$. Then*

$$\overline{\nabla}_X \xi_\alpha = -fX - f^2 X \quad (3.9)$$

for all $X \in \Gamma(TM)$.

**Theorem 3.3** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection $\overline{\nabla}$. Then, M is trans Sasakian manifold of type (1,1) with s=1.*

We denote by same symbol g both metrics on $\widetilde{M}$ and *M*. Let $\overline{\nabla}$ be the semi-symmetric metric connection on $\widetilde{M}$ and $\nabla$ be the induced connection on *M*. Then,

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y) \quad (3.10)$$

where *m* is a tensor field on CR-submanifold *M*. Using (3.1) and (3.4) we have,

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) + \textstyle\sum_{\alpha=1}^{s}\eta^\alpha(Y)X \,. \quad (3.11)$$

Comparing tangential and normal components from both the sides in (3.11), we get

$$m(X,Y) = h(X,Y)$$

and

$$\nabla_X Y = \nabla^*_X Y + \textstyle\sum_{\alpha=1}^{s}\eta^\alpha(Y)X \,. \quad (3.12)$$

Thus $\nabla$ is also a semi-symmetric metric connection. For $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$ from (3.2) and (3.12), we have

$$\nabla_X N = \nabla^*_X N + \textstyle\sum_{\alpha=1}^{s}\eta^\alpha(N)X = -A_N X + \sum_{\alpha=1}^{s}\eta^\alpha(N)X.$$

Now, Gauss and Weingarten formulas for a *CR*-submanifolds of a *S*-manifold with a semi-symmetric metric connection is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (3.13)$$

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N + \sum_{\alpha=1}^{s}\eta^\alpha(N)X \quad (3.14)$$

for all $X,Y \in \Gamma(TM)$, $N \in \Gamma(T^\perp M)$, *h* second fundamental form of *M* and $A_N$ is the Weingarten endomorphism associated with *N*.

**Theorem 3.4** *The connection induced on CR-submanifolds*

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*of an S-manifold with a semi-symmetric metric connection is also a semi-symetric metric connection.*

## 4 Some Basic Lemmas

**Lemma 4.1** *If M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then,*

$$P\nabla_X fPY - PA_{fQY}X - fP\nabla_X Y = -\sum_{\alpha=1}^{s}\{\eta^\alpha(Y)PX + \eta^\alpha(Y)fPX\}, \quad (4.1)$$

$$Q\nabla_X fPY - QA_{fQY}X - Bh(X,Y) = -\sum_{\alpha=1}^{s}\eta^\alpha(Y)QX, \quad (4.2)$$

$$h(X,fPY) - fQ\nabla_X Y + \nabla_X^{\perp} fQY = -\sum_{\alpha=1}^{s}\eta^\alpha(Y)fQX + Ch(X,Y) \quad (4.3)$$

$$\sum_{\alpha=1}^{s}\{\eta^\alpha(\nabla_X fPY)\xi_\alpha - \eta^\alpha(A_{fQY}X)\xi_\alpha\} = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha + \eta^\alpha(\nabla_X Y)\xi_\alpha - \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha\} \quad (4.4)$$

for all $X,Y \in \Gamma(TM)$.

*Proof.* By direct differentiating covariantly, we have

$$\overline{\nabla}_X fY = (\overline{\nabla}_X f)Y + f\overline{\nabla}_X Y.$$

By virtue of (3.4), (3.8), (3.13) and (3.14), we get

$$\nabla_X fPY + h(X,fPY) + (-A_{fQY}X + \nabla_X^{\perp} fQY) = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha + \eta^\alpha(Y)X - \eta^\alpha(Y)fX\} + f\nabla_X Y + fh(X,Y).$$

Then, from (3.4), we have

$$P\nabla_X fPY + Q\nabla_X fPY + h(X,fPY) - PA_{fQY}X - QA_{fQY}X + \nabla_X^{\perp} fQY = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha - \eta^\alpha(Y)PX - \eta^\alpha(Y)QX - \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha + \eta^\alpha(\nabla_X Y)\xi_\alpha - \eta^\alpha(Y)fPX - \eta^\alpha(Y)fQX\} + fP\nabla_X Y + fQ\nabla_X Y + Bh(X,Y) + Ch(X,Y.)$$

Comparing tangential, vertical and normal components in above equation, we get desired results.

**Lemma 4.2** *If M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then,*

$$-A_{fY}X - fP\nabla_X Y - Bh(X,Y) = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX \quad (4.5)$$

$$\nabla_X^{\perp} fY = fQ\nabla_X Y + Ch(X,Y) \quad (4.6)$$

for all $X,Y \in \Gamma(D^{\perp} \oplus sp\{\xi_1,\dots,\xi_s\})$.

*Proof.* By the use of (3.8) and $fY \in \Gamma(T^{\perp}M)$, then for all $X,Y \in \Gamma(D^{\perp} \oplus sp\{\xi_1,\dots,\xi_s\})$ we get

$$(\overline{\nabla}_X f)Y = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\}.$$

From the above equation, we have

$$\overline{\nabla}_X fY - f\overline{\nabla}_X Y = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\}.$$

Now using (3.13) and (3.14) in the above equation, we have

$$-A_{fY}X + \nabla_X^{\perp} fY - f\nabla_X Y - fh(X,Y) = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\}$$

or

$$-A_{fY}X + \nabla_X^{\perp} fY - fP\nabla_X Y - fQ\nabla_X Y - Bh(X,Y) - Ch(X,Y) = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\}$$

for all $X,Y \in \Gamma(D^{\perp} \oplus sp\{\xi_1,\dots,\xi_s\})$ Now, comparing tangential, vertical and normal components in the above equation, we get desired results.

**Lemma 4.3** *If M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then,*

$$\nabla_X fY - fP\nabla_X Y = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha\} - Bh(X,Y) \quad (4.7)$$

$$h(X,fY) = fQ\nabla_X Y + Ch(X,Y) \quad (4.8)$$

for all $X,Y \in \Gamma(D)$.

*Proof.* From (3.8), we have

$$\overline{\nabla}_X fY - f\overline{\nabla}_X Y = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha\}$$

for all $X,Y \in \Gamma(D)$. Now using (2.2), we get

$$\nabla_X fY + h(X,fY) - fP\nabla_X Y - fQ\nabla_X Y - Bh(X,Y) - Ch(X,Y) = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - g(X,fY)\xi_\alpha\}.$$

In the above equation, comparing tangential, vertical and normal components, we get (4.7) and (4.8).

## 5 Integrability Conditions of Distributions

**Theorem 5.1** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then the distribution $D \oplus D^{\perp}$ is not integrable.*

*Proof.* For any $X,Y \in \Gamma(D \oplus D^{\perp})$, we have

$$g([X,Y],\xi_\alpha) = g(Y,\widetilde{\nabla}_X \xi_\alpha) + g(X,\widetilde{\nabla}_Y \xi_\alpha).$$

Using (3.9) and (3.13), we get

$$g([X,Y],\xi_\alpha) = -g(Y,\overline{\nabla}_X \xi_\alpha - X - \eta^\alpha(X)\xi_\alpha) + g(X,\overline{\nabla}_Y \xi_\alpha - Y - \eta^\alpha(Y)\xi_\alpha) = g(Y,fX + f^2X) + g(X,fY + f^2Y)$$

This completes the proof.

**Theorem 5.2** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. The distribution $D \oplus sp\{\xi_1,\dots,\xi_s\}$ is integrable if and only if*

$$h(X,fY) = h(Y,fX)$$

for all $X,Y \in \Gamma(D \oplus sp\{\xi_1,\dots,\xi_s\})$.

*Proof.* By using of (3.21), we have

$$h(X,fY) - h(Y,fX) = fQ[X,Y]. \quad (5.1)$$

Let $D \oplus sp\{\xi_1, ..., \xi_s\}$ be integrable. Then $Q[X,Y] = 0$. From (5.1), we have

$$h(X, fY) = h(Y, fX) \quad (5.2)$$

Vice verse, $h(X, fY) = h(Y, fX)$ or $fQ[X,Y] = 0$. This completes the proof.

As an immediate consequence of Theorem 5.2 we have the following result.

**Corollary 5.1** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. The distribution* $D \oplus sp\{\xi_1, ..., \xi_s\}$ *is integrable if and only if*

$$A_N fX = -f A_N X$$

for all $X \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$.

**Theorem 5.3** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. The distribution* $D^\perp \oplus sp\{\xi_1, ..., \xi_s\}$ *is integrable if and only if*

$$A_{fX}Y - A_{fY}X = \sum_{\alpha=1}^{s}\{\eta^\alpha(X)Y - \eta^\alpha(Y)X + \eta^\alpha(X)fY - \eta^\alpha(Y)fX \quad (5.3)$$

for all $X, Y \in \Gamma(D^\perp \oplus sp\{\xi_1, ..., \xi_s\})$.

*Proof.* If $X, Y \in \Gamma(D^\perp \oplus sp\{\xi_1, ..., \xi_s\})$, then from (4.4)

$$-A_{fY}X - fP\nabla_X Y - Bh(X,Y) = \sum_{\alpha=1}^{s}\{g(X,Y)\xi_\alpha - \eta^\alpha(Y)X - \eta^\alpha(Y)fX\} \quad (5.4)$$

Now interchanging $X$ and $Y$, subtracting the equations, we have

$$-A_{fY}X + A_{fX}Y - fP[X,Y] = \sum_{\alpha=1}^{s}\{-\eta^\alpha(Y)X + \eta^\alpha(X)Y - \eta^\alpha(Y)fX + \eta^\alpha(X)fY\} \quad (5.5)$$

From (5.5), we obtain

$$-A_{fY}X + A_{fX}Y - fP[X,Y] = \sum_{\alpha=1}^{s}\{-\eta^\alpha(Y)X + \eta^\alpha(X)Y\}$$

Now, let $D^\perp \oplus sp\{\xi_1, ..., \xi_s\}$ be integrable. For all $X, Y \in \Gamma(D^\perp \oplus sp\{\xi_1, ..., \xi_s\})$, $[X,Y] = 0$. Then

$$A_{fX}Y - A_{fY}X = \sum_{\alpha=1}^{s}\{\eta^\alpha(X)Y - \eta^\alpha(Y)X + \eta^\alpha(X)fY - \eta^\alpha(Y)fX\}.$$

By using (5.5), $fP[X,Y] = 0$ then $[X,Y] = 0$.

**Corollary 5.2** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then the distribution* $D^\perp$ *is integrable if and only if*

$$A_{fY}X = A_{fX}Y \quad (5.6)$$

for all $X, Y \in \Gamma(D^\perp)$.

## 6 Parallel Distributions

**Definition 6.1** *The horizontal (resp.vertical) distribution on D (resp.* $D^\perp$*) is said to be parallel with respect to the connection $\nabla$ on M if*
$\nabla_X Y \in D\ (resp. \nabla_Z W \in D^\perp)\ for\ any\ X, Y \in \Gamma(D)\ (resp.\ Z, W \in \Gamma(D^\perp))$.

Now, we have the following Theorem:

**Theorem 6.1** *Let M be a $\xi_\alpha$ −horizontal CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then, the horizontal distribution D is parallel if and only if*

$$h(X, fY) = h(Y, fX) = fh(X,Y) \quad (6.1)$$

for all $X, Y \in \Gamma(D)$.

*Proof.* Since every parallel distribution is involutive then the first equality follows immediately. Now since $D$ is parallel, we have

$$\nabla_X fY \in D, \forall X, Y \in \Gamma(D).$$

From (4.2), we have

$$Bh(X,Y) = 0, \qquad \forall X, Y \in \Gamma(D) \quad (6.2)$$

and from (4.3), if $\xi_\alpha \in \Gamma(D)$, then $D$ is parallel if and only if

$$h(X, fY) = Ch(X,Y).$$

But we have,

$$fh(X,Y) = Bh(X,Y) + Ch(X,Y),$$

and from (4.7), $fh(X,Y) = Ch(X,Y)$ which completes the proof.

**Lemma 6.1** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then the distribution* $D^\perp$ *is parallel if and only if*

$$-A_{fW}Z = \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha + Bh(Z,W) \quad (6.3)$$

for all $Z, W \in \Gamma(D^\perp)$.

*Proof.* Using (4.4), we have,

$$-A_{fW}Z - fP\nabla_Z W = \sum_{\alpha=1}^{s} g(Z,W)\xi_\alpha + Bh(Z,W),$$
$\forall Z, W \in \Gamma(D^\perp)$

Hence

$$\nabla_Z W \in \Gamma(D^\perp)\ if and\ only\ if\ P\nabla_Z W = 0.$$

Since $P\nabla_Z W = 0$ we get (6.3).

**Lemma 6.2** *Let M be CR-submanifolds of an S-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then the distribution* $D^\perp$ *is parallel is parallel if and only if*

$$A_{fW}Z \in \Gamma(D^\perp) \quad (6.4)$$

for all $Z, W \in \Gamma(D^\perp)$

*Proof.* Using Gauss and Weingarten formulas in (3.8), we have
$(\nabla_Z f)W = \sum_{\alpha=1}^{s}\{g(fZ,fW)\xi_\alpha + \eta^\alpha(W)(f^2Z - fZ)\}$
for $Z,W \in \Gamma(D^\perp)$ By using (3.13) and (3.14), we get
$\nabla_Z fW - f\nabla_Z W = \sum_{\alpha=1}^{s}\{g(fZ,fW)\xi_\alpha + \eta^\alpha(W)(f^2Z - fZ)\}$

or
$$-A_{fW}Z + \nabla_Z^\perp fW - f\nabla_Z W - fh(Z,W) = \sum_{\alpha=1}^{s}\{g(fZ,fW)\xi_\alpha + \eta^\alpha(W)(f^2Z - fZ)\}$$

Now taking inner product with $Y \in \Gamma(D)$ in above equation, we have
$$-g(A_{fW}Z,Y) + g(\nabla_Z^\perp fW,Y) - g(f\nabla_Z W,Y) - g(fh(Z,W),Y) = \sum_{\alpha=1}^{s}\{g(fZ,fW)g(\xi_\alpha,Y) + \eta^\alpha(W)g(f^2Z,Y) - \eta^\alpha(W)g(fZ,Y)\}$$

This implies that
$$g(A_{fW}Z,Y) = 0 \; if \; and \; only \; if \; A_{fW}Z \in \Gamma(D^\perp).$$

Therefore, we get
$$\nabla_Z W \in D^\perp \text{ if and only if } A_{fW}Z \in D^\perp.$$

This completes the proof.

## 7 CR-Submanifolds of an S-Space form with a semi symmetric metric connection

In [1], Akyol et al introduced constant φ sectional curvature $R$ with a semi symmetric metric connection. Let $M$ be *CR*-submanifolds of an *S*-manifold $\widetilde{M}$ with a semi-symmetric metric connection. Then a *CR*-submanifold $M$ has constant φ sectional curvature $c$ if and only if the Riemannian curvature tensor $\bar{R}$ satisfied
$$\bar{R}(X,Y,Z,W) = 2\sum_{i,j=1}^{2m+s}\{g(X,W)\eta^i(Y)\eta^j(Z) + g(Y,W)\eta^i(X)\eta^j(Z) + g(Y,Z)\eta^i(X)\eta^j(W) - g(X,Z)\eta^i(Y)\eta^j(W)\} + \sum_{i,j=1}^{2m+s}\{\eta^i(X)\eta^k(Y)\eta^j(Z)\eta^k(W) - \eta^k(W)\eta^i(Y)\eta^j(Z)\eta^k(W) + \eta^k(X)\eta^i(Y)\eta^k(Z)\eta^j(W) - \eta^k(X)\eta^i(Y)\eta^k(W)\eta^i(Z)\}$$
$$+\frac{c+3s}{4}\{g(\varphi X,\varphi W)g(\varphi Y,\varphi Z) - g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}$$
$$+\frac{c-s}{4}\{g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W)\} + s\{g(\varphi Z,X)g(Y,W) - g(X,W)g(\varphi Z,Y) + g(Y,\varphi Z)g(\varphi X,W) + g(X,Z)g(Y,W) - g(Y,Z)g(X,W) - g(X,Z)g(\varphi Y,W)\} + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W)) \quad (7.1)$$

for all $X,Y,Z,W \in \Gamma(TM)$

We choose a local field of orthonormal frames $\{E_1,\dots,E_m,E_{m+1},\dots,E_{2m},\xi_1,\dots,\xi_s\}$ of $TM$, where $D = sp\{E_1,\dots,E_m\}$ and $D^\perp = sp\{E_{m+1},\dots,E_{2m}\}$.

Now, let begin with the following theorem:

**Theorem 7.1** *Let M be CR-submanifolds of an S-space form* $\widetilde{M}(c)$ *with a semi symmetric metric connection. Then*
$$\bar{R}(X,Y,Z,W) = \frac{c-s}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W)) \quad (7.2)$$

for all $X,Y,Z,W \in \Gamma(D^\perp)$.

*Proof.* For all $X,Y,Z,W \in \Gamma(D^\perp)$, by making use of (7.1), we obtain
$$\bar{R}(X,Y,Z,W) = \frac{c+3s}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} + s\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W))$$
$$= \frac{c-s}{4}g(X,W)g(Y,Z) + \frac{s-c}{4}g(X,Z)g(Y,W) + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W))$$
which give us (7.2).

As a consequence of Theorem 7.1, we can give the following corollary,

**Corollary 7.1** *Let M be CR-submanifolds of an S-space form* $\widetilde{M}(c)$ *with a semi symmetric metric connection. and for all* $X,Y,Z,W \in \Gamma(D^\perp)$. *Let* $D^\perp$ *be a totally geodesic. Then M is flat if and only if c=s.*

**Theorem 7.2** *Let M be CR-submanifolds of an S-space form* $\widetilde{M}(c)$ *with a semi symmetric metric connection. and for all* $X,Y \in \Gamma(D^\perp)$. *If* $D^\perp$ *is totally geodesic, Then the scalar curvature of* $D^\perp$ *is given by*
$$\bar{\tau}_{D^\perp} = \frac{c-s}{4}m(m-1),$$

where $\bar{\tau}$ is the scalar curvature.

*Proof.* For all $X,Y \in \Gamma(D^\perp)$ using (7.2), we get
$$\bar{S}(X,Y) = \sum_{\alpha=1}^{s} R(E_i,X,Y,E_i) = \frac{c-s}{4}(m-1)g(X,Y),$$

where $\bar{S}$ is Ricci tensor.

**Theorem 7.3** Let *M be CR-submanifolds of an S-space form* $\widetilde{M}(c)$ *with a semi symmetric metric connection. . Then the scalar curvature determined by D is given*

$$\bar{\tau}_{\mathrm{D}} = \frac{c-s}{4} m(m+2).$$

*Proof.* For all $X, Y \in \Gamma(\mathrm{D})$ from (7.2), we have
$\bar{R}(X,Y,Z,W) = \frac{c+3s}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}$
$+\frac{c-s}{4}\{g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W)\}$
$+s\{g(\varphi Z,X)g(Y,W) - g(X,W)g(\varphi Z,Y) + g(Y,\varphi Z)g(\varphi X,W) + g(X,Z)g(Y,W) - g(Y,Z)g(X,W) - g(X,Z)g(\varphi Y,W)\} + g\big(h(X,Z),h(Y,W)\big) - g\big(h(Y,Z),h(X,W)\big)$

Then, if $S$ is Ricci tensor field of $M$ then we have
$\bar{S}(X,Y) = \frac{c-s}{4}(m+2)g(X,Y) + s(2-m)g(X,\varphi Y).$

**Theorem 7.4** Let *M be CR-submanifolds of an S-space form* $\widetilde{M}(c)$ *with a semi symmetric metric connection. Then, φ-sectional curvature of D is 2s-c if and only if D is totally geodesic.*

*Proof.* By the use of (7.1), we have
$\bar{R}(X,\varphi X,X,\varphi X) = \frac{c+3s}{4}\{g(X,\varphi X)g(\varphi X,X) - g(X,X)g(\varphi X,\varphi X)$
$+\frac{c-s}{4}\{g(X,\varphi^2 X)g(\varphi X,\varphi X) - g(X,\varphi X)g(\varphi X,\varphi^2 X) - 2g(X,\varphi^2 X)g(X,\varphi^2 X)\}$
$+s\{g(\varphi X,X)g(\varphi X,\varphi X) - g(X,\varphi X)g(\varphi X,\varphi X) + g(\varphi X,\varphi X)g(\varphi X,\varphi X) + g(X,X)g(\varphi X,\varphi X) - g(\varphi X,X)g(X,\varphi X) - g(X,X)g(\varphi^2 X,\varphi X)\} + g\big(h(X,X),h(\varphi X,\varphi X)\big) - g\big(h(\varphi X,X),h(X,\varphi X)\big)$

for all $X \in \Gamma(\mathrm{D})$. Then, we obtain
$$\bar{R}(X,\varphi X,X,\varphi X) = -c + 2s - 2g\big(h(X,X),h(X,X)\big).$$

**Proposition 7.1** *Let M be CR-submanifolds of an S-manifold with a semi symmetric metric connection. Then,*
$$\bar{R}(X,Y,Z,W) = 0$$

for all $X, Y \in \Gamma(\mathrm{D} \oplus sp\{\xi_1,\dots,\xi_s\})$ and $Z, W \in \Gamma(\mathrm{D}^\perp)$.

*Proof.* Let *M* be *CR*-submanifolds of an *S*-manifold with a semi symmetric metric connection $\widetilde{M}$. Then for all $Z, W \in \Gamma(\mathrm{D}^\perp)$,
$$\varphi Z, \varphi W \in \varphi\mathrm{D}^\perp \subset \mathrm{TM}^\perp.$$

Using (7.1), we finish the proof of the proposition.

**Proposition 7.2** *Let M be CR-submanifolds of an S-manifold with a semi symmetric metric connection. Then,*
$$\bar{R}(X,Y,Z,W) = 0$$
for all $X, Y \in \Gamma(\mathrm{D})$ and $Z, W \in \Gamma(\mathrm{D}^\perp \oplus sp\{\xi_1,\dots,\xi_s\})$.

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