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Effect of fractional time derivatives to pressure-driven flow through the horizontal microchannel

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Abstract — This research applies fractional time derivatives to fluid flow through a horizontal microchannel. It uses fractional time derivatives with the Laplace transform technique and method of undetermined coefficient to analyze and obtain solutions of the governing equations in the Laplace domain. To this end, the solutions are reversed in the time domain using Riemann-sum approximation methods. In order to obtain the solutions for the pressure-driven flow, the time factional derivative in the Caputo sense is employed. Here, the influence of each governing parameter is explained with a line graph. Results show that with the decreases in fractional order (α), the velocity decreases within the interval $0 < \alpha < 1$. The fluid velocity increases and decreases as the Knudsen number (kn) changes. Besides, transient wall-skin frictions for different times (t) and Knudsen number (kn) with a fixed value of fractional order (α) are observed.

Keywords: *Fractional time derivative, pressure-driven flow, Knudsen number, Laplace transform, Couette flow*

Subject Classification (2020): 76D55, 34A08

1. Introduction

Fractional calculus is a generalization of ordinary differential and integral of non-integer order α. It was first introduced by L'Hospital and Leibnitz in 1695 after they proposed what would happen if the ordinary derivative of integer order was changed to fractional order by L'Hospital. Then, Leibnitz first used the notation $d^{(1/2)}y$ in 1697. Many mathematicians have suggested their interest in its application, as Lacroix 1819 mentioned fractional derivatives in his text on differential and integral calculus. Euler and Fourier mentioned fractional derivatives but did not give any application or example. The first and popular definition, the Riemann-Liouville definition, was proposed by Riemann and Liouville, after which Caputo proposed a second popular definition called the Caputo Fractional Derivative. There are many definitions of Fractional calculus, such as Jumarie, Wely, Eudelyikober, Hadamard, and the Riesz fractional derivative (see, for example, Srivastava and Saxena [1] and Kaur [2]).

In recent decades, many researchers have been devoted to its applications in science and engineering. Fractional calculus has been recognized as a practical modeling approach in fields such as electrical networks, electrochemistry, and viscoelastic deformation, solving linear, nonlinear partial fractional differential equations (see Farid et al. [3]). According to Ali et al. [4], Newtonian and non-Newtonian fluids depend on their deformation. Newtonian fluids are fluids that obey Newton's law of viscosities. In contrast,

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non-Newtonian fluids do not follow Newton's law of viscosities, having many applications in fields such as industries, medical treatment, and engineering worldwide.

A micro-channel is defined in Djordjevic [5] as a flow channel with a hydraulic diameter of less than 1 mm and characterized by the rarefaction effect, which includes the Knudsen number $kn = \lambda/l$, where λ is the mean free path of the molecules and l is the characteristic length scale (height h of the channel or the radius an of the tube), then the Knudsen number is the quantity that helps to know which fluid dynamic formula to use to model a situation in both statistical mechanics or continuum mechanics for the continuum hypothesis to be valid, the Knudsen number must be less than 0.1 for pressure driven. Saqib et al. [6] explain that the temperature and velocity fields can be reduced for any value of α between the interval $0 < \alpha < 1$ with memory and heredity profile, which are more generally flexible and reliable, in the presence of variation in the thermal boundary layer when increasing volume fraction of carbon nanotubes (CNTs) the temperature profile increases and decreases with increases of fractional order α in both cases.

Ellahia et al. [7] change more than one parameter in the channel and find that the fluid velocity decreases as β increases the fixed value of the channel length L. In the same paper, they also found that for a fixed value of β and growing L, the velocity also increases. Farooq et al. [8] studied the generalized Couette flow by varying the various parameters of temperature distribution and velocity, where it was observed that as the fluid moved from a fixed plate to a movable plate, the velocity and temperature increased. Arif et al. [9] open channel for Couple Stress Fluid (CSF) using integral transform (Laplace and Fourier) for the comparative analysis of Caputo (C), Caputo-Fabrizio (CF), Atangana Baleanu (AB), and classical CSF where it is observed the velocity of the CSF of C, CF has less influence of fluid dynamics than the velocity of the CSF concerning AB, which clearly shows that AB fractional derivatives has a better memory effect than C, CF and increasing the constant pressure gradient of CSF improves the velocity. Maitia et al. [10] discovered the impact of the Caputo-Fabrizio derivative of the fractional order model on blood flow, where they found that increasing the value of the fractional parameter value decreased velocity and temperature when memory counters average speed of blood flow medium effect to drive faster. The velocity profile improves as the stress jump coefficient increases. When the stress jump coefficient β is close to zero, non-Newtonian characteristics become more effective, and β approaches infinity, the model becomes a Newtonian when viscosity increases. Blood flow decreases with a decrease in β. In the same paper, they also find that the shear stress increases in the wall due to chemical reactions.

The governing equation for this present study is in fractional order, focusing on pressure-driven flow through a horizontal channel with many applications in applied science and engineering, such as biological research, geophysical engineering, DNA sequencing for drug delivery, and micro-electro-mechanical systems (MECSs). In the presence of pressure gradient and upper plate motion of the boundary layer (generalized Couette flow), the steady and unsteady flow has been studied related to skin friction on the pressure gradient and dependence on the interface velocity (see Kaurangini and Jha [11]).

2. Analytical Solution

The governing equations [12] are derived from the classical equations and modified by replacing the ordinary time derivatives with the fractional calculus operators. This generalization allows us to define non-integer order integrals or derivatives precisely. Figure 1 manifests the fluid flow in the microchannel is induced by pressure-driven flow.

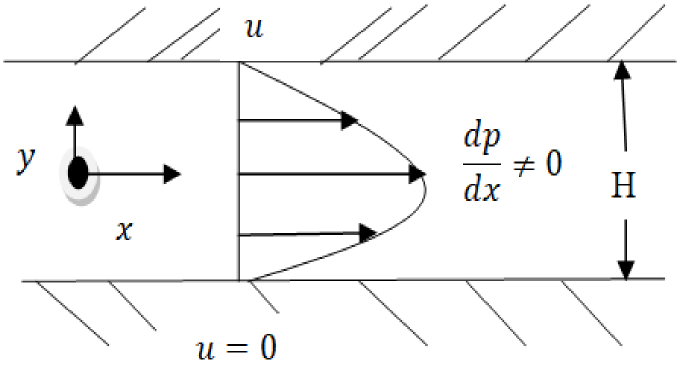

Figure 1. Schematic diagram of pressure-driven flow

The governing equation [12] for the flow is as follows:

$$D_t^{\alpha} u(y,t) = \gamma \frac{\partial^2 u(y,t)}{\partial y^2} + p \tag{2.1}$$

with initial and boundary conditions and $D_t^{\alpha} u(y,t)$ is the Caputo fractional derivative,

$$t \le 0: u = 0, \text{for all } y \tag{2.2}$$

and

$$t > 0: u(y,t) = \begin{cases} +\beta_v k_n \dfrac{du}{dy}, & y = 0 \\ -\beta_v k_n \dfrac{du}{dy}, & y = H \end{cases} \tag{2.3}$$

with the analysis technique mentioned above, we have the following solution approach:

Taking the Laplace transform of both sides of (2.1) together with (2.2), we have

$$\frac{d^2 \overline{U}(y,s)}{dy^2} - \frac{s^{\alpha} \overline{U}(y,s)}{\gamma} = -\frac{p}{\gamma s} \tag{2.4}$$

Solving (2.4) by the method of undetermined coefficient to obtain the general solution,

$$\overline{U}(y,s) = c_1 \cosh k_1 y + c_2 \sinh k_1 y + k_2 \tag{2.5}$$

Similarly, we apply the Laplace transform to (2.3), the boundary conditions become

$$\frac{1}{s^2} > 0: \overline{U}(y,s) = \begin{cases} +\beta_v k_n \dfrac{d}{dy} \overline{U}(y,s), & y = 0 \\ -\beta_v k_n \dfrac{d}{dy} \overline{U}(y,s), & y = H \end{cases} \tag{2.6}$$

Using (2.5) on (2.6), we obtained the following solutions:

$$\overline{U}(y,s) = k_7 \cosh k_1 y + k_6 \sinh k_1 y + k_2 \tag{2.7}$$

where

$$\begin{cases} k_1 = \left(\frac{s^\alpha}{\gamma}\right)^{\frac{1}{2}} \\ k_2 = \frac{P}{s^{\alpha+1}} \\ k_3 = \cosh k_1 H + \beta_v k_n k_1 \sinh k_1 H \\ k_4 = \sinh k_1 H + \beta_v k_n k_1 \cosh k_1 H \\ k_5 = \beta_v k_n k_1 k_3 + k_4 \\ k_6 = \frac{k_2(k_3 - 1)}{k_5} \\ k_7 = \beta_v k_n k_1 k_6 - k_2 \end{cases} \tag{2.8}$$

3. Skin Friction for Pressure Driven Flow

From (2.7), the skin frictions at the wall of the channel are obtained

$$\hat{\tau}_0 = \left.\frac{d\bar{U}(y,s)}{dy}\right|_{y=0} = (k_1 k_7 \sinh k_1 y + k_1 k_6 \cosh k_1 y)|_{y=0} = K_1 K_6 \tag{3.1}$$

$$\hat{\tau}_1 = \left.\frac{d\bar{U}(y,s)}{dy}\right|_{y=1} = (k_1 k_7 \sinh k_1 y + k_1 k_6 \cosh k_1 y)|_{y=1} = k_1 k_7 \sinh k_1 + k_1 k_6 \cosh k_1 \tag{3.2}$$

4. Results and Discussions

4.1. Numerical Results

This was done to simulate numerical solutions for transient skin friction at different walls using the computational software MATLAB R2014a. We obtained the following results.

Table 1. Transient skin frictions at the walls for different time t and Knudsen number kn of Pressure driven flow

	$\beta v = 0.5, a = 0.5$ $p = 2, \gamma = 1, kn = 0.0$		$\beta v = 0.5, a = 0.5$ $Y = 1, kn = 0.04$		$\beta v = 0.5, a = 0.5, p = 2$ $Y = 1, kn = 0.08$	
t	τ_0	τ_1	τ_0	τ_1	τ_0	τ_1
0.1	0.0140	−0.0140	0.0134	−0.0134	0.0129	−0.0129
0.2	0.0310	−0.0310	0.0299	−0.0299	0.0289	−0.0289
0.3	0.0487	−0.0487	0.0473	−0.0473	0.0460	−0.0460
0.4	0.0670	−0.0670	0.0652	−0.0652	0.0636	−0.0636
0.5	0.0856	−0.0856	0.0835	−0.0835	0.0815	−0.0815
0.6	0.1044	−0.1044	0.1020	−0.1020	0.0998	−0.0998
0.7	0.1234	−0.1234	0.1208	−0.1208	0.1183	−0.1183
0.8	0.1425	−0.1425	0.1397	−0.1397	0.1370	−0.1370
0.9	0.1618	−0.1618	0.1587	−0.1587	0.1558	−0.1558
1.0	0.1812	−0.1812	0.1779	−0.1779	0.1747	−0.1747

4.2. 2D-Plots Presentation

Figures 2-5 describe the velocity profiles for various parameters, such as fractional α, pressure p, Knudsen number kn, and time t, caused by the pressure gradient. Figure 2 shows that the reduction of fractional order reduces the velocity of the fluid, which implies that the velocity can be slowed down by decreasing the fractional order. In addition, it shows the advantages of fractional derivatives over integer derivatives. Figure 3 shows that the transient velocity increases with time, which implies that with a constant driving force, the velocity can increase as time goes on. Figure 4 shows that as the Knudsen number increased, the fluid velocity increased, indicating that enlarging the length scale allowed the mean free path to enlarge and the velocity to increase. Similarly, Figure 5 shows the velocity profiles for different Knudsen numbers kn. Figure 6 depicts the variation of fractional order α with velocity U, which that the velocity of the fluid decreases when the fractional order decreases with the interval of 0.3, and between 0.6 and 0.3, the velocity reduces unlike from 0.9 to 0.6, which clearly shows the effect of the fractional order. Figure 7 shows the variation of velocity U with time t, which shows that increasing time makes velocity also increase and converge at both walls, which leads the velocity to decrease between 0.5 to 0.8, and as time goes on, the velocity will be steady. It should be noted that the fluid velocity slows down as the Knudsen number decreases. Table 1 shows that skin friction increases evenly on both walls but in opposite directions with increasing time and Knudsen number.

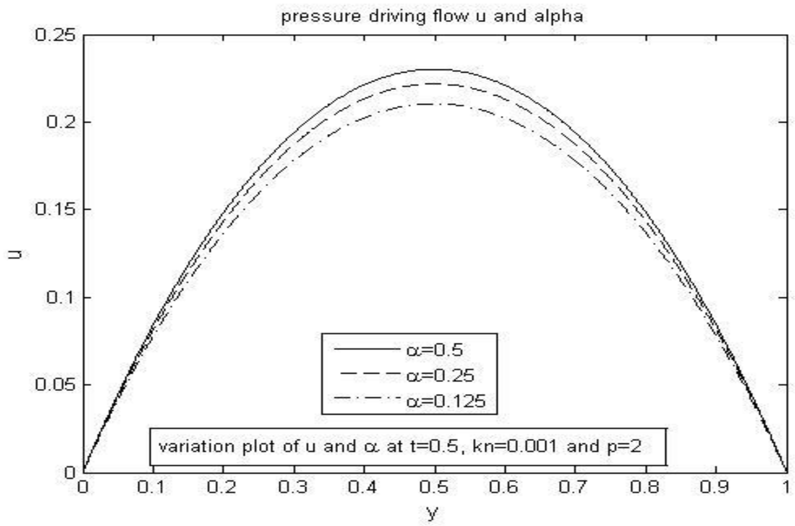

Figure 2. Variation of fractional order α against velocity $U$when the fluid flow as a result of pressure gradient

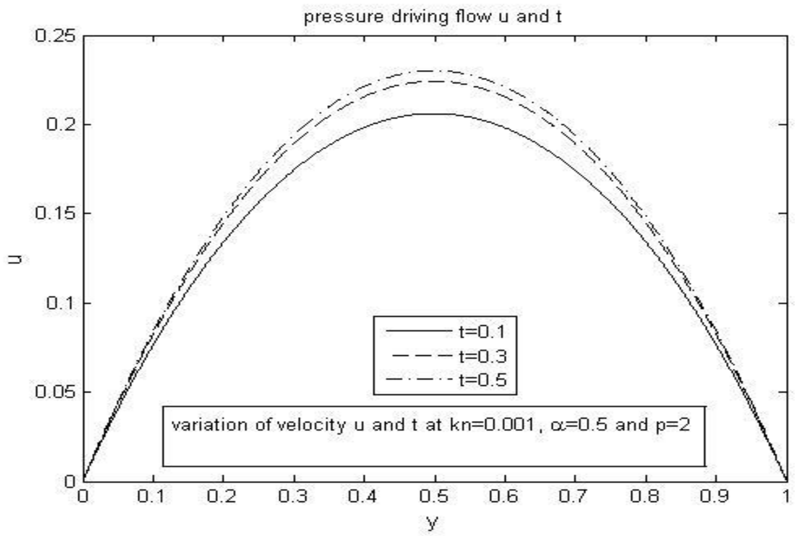

Figure 3. Variation of time t against velocity U when the fluid flows as a result of pressure gradient

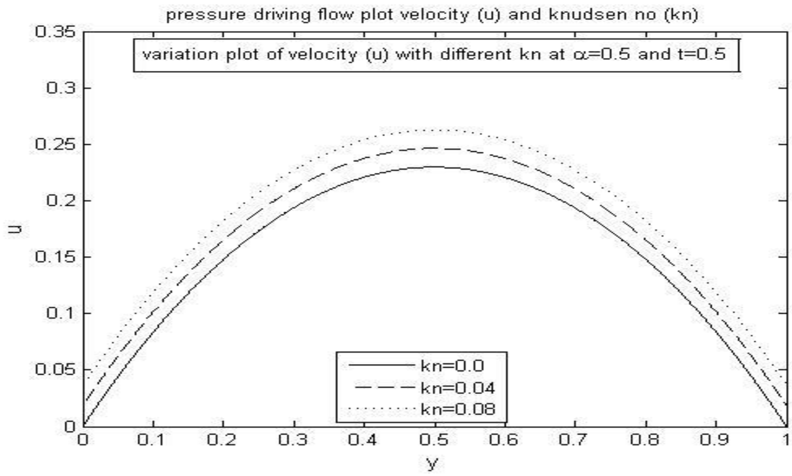

Figure 4. Variation of Knudsen number kn against velocity U as a result of pressure gradient

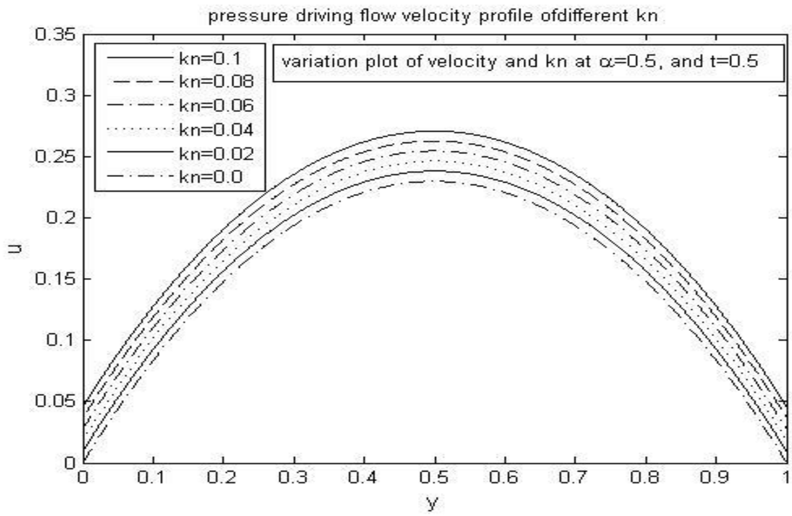

Figure 5. Variation of Knudsen number kn against velocity U as a result of pressure gradient

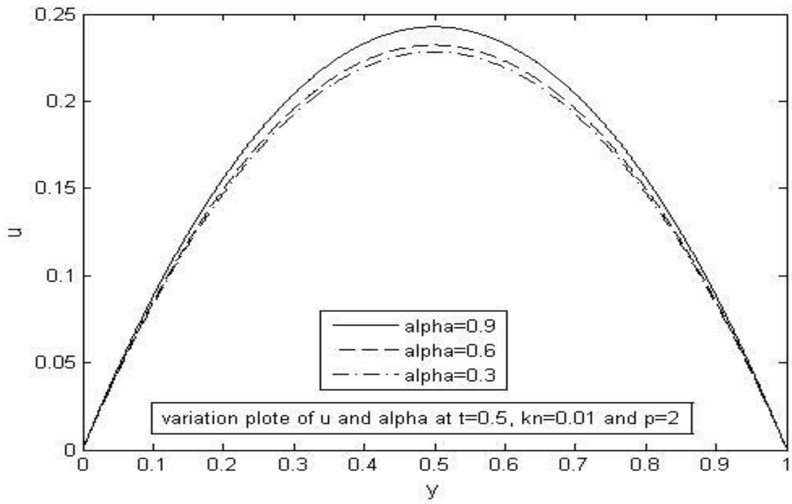

Figure 6. Variation of fractional order α against velocity U when the fluid flows as a result of pressure gradient

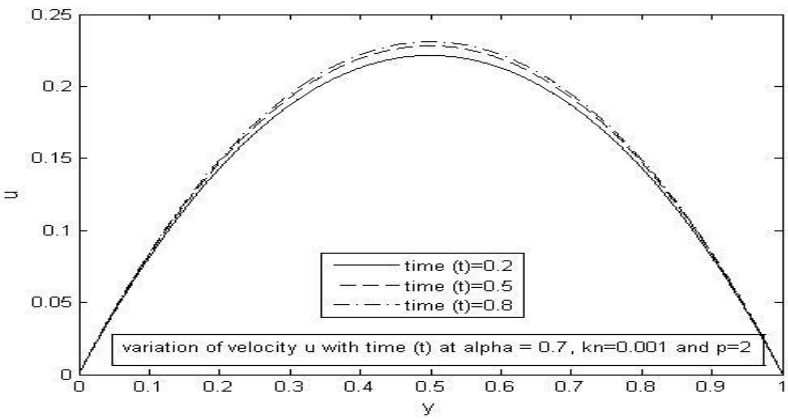

Figure 7. Variation of time t against velocity U when the fluid flows as a result of pressure gradient

5. Conclusion

In this paper, the effect of varying the governing parameters was considered to study the velocity profile induced by pressure-driven flow in the microchannel. The transient skin friction uniformly increases at both walls but in opposite directions. It has been discovered that the velocity of a fluid flow can be controlled by adjusting the fractional order (α). The advantages of fractional derivatives over integer derivatives have been studied. It has been observed that the velocity can increase with a constant driven force as time goes on. Enlarging the length scale enlarged the mean free path and increased velocity. It can be seen that with a continuous decrease of the Knudsen number kn, at $\beta v = -0.5$, the velocity increases but remains constant at $\beta v = 0.0$ while at $\beta v = 0.5$, the velocity decreases. The results obtained from this study are significant for industries, as they contribute to a better comprehension of various applications such as oil reservoirs, nuclear reactors, groundwater flow, filtration, and geothermal systems. Furthermore, these findings present opportunities for further investigation through the inclusion of porous mediums, suction, and injection velocities.

Abbreviations

D	fractional derivative
α	fractional order
t	time
u	velocity of fluid flow
y	dimensionless y coordinate
p	dimensionless pressure gradient
kn	Knudsen number
β	stress jump coefficient
ρ	fluid density
λ	molecular mean free path
v_f	kinematics viscosity of the fluid
y'	dimensional y-coordinate
u_f	dimensionless velocity in the clear fluid region
u'_f	dimensional velocity in the clear fluid region
u_t	transient velocity
$\frac{\delta p'}{\delta z'}$	dimensional pressure gradient
β_v	dimensionless variable

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] H. M. Srivastava, R. K. Saxena, *Operations of fractional integration and their applications*, Applied Mathematics and Computation 118 (1) (2001) 1–52.

[2] D. Kaur, *A detail study on fractional calculus*, in: S. Negi, A. Kumar, T. Singh (Eds.), International Multidisciplinary Conference, Chandigarh, 2022, pp. 54–59.

[3] G. Farid, N. Latif, M. Anwar, A. Imran, M. Ozair, M. Nawaz, *On applications of Caputo k-fractional derivatives*, Advances in Difference Equations 2019 (2019) 439 16 pages.

[4] F. Ali, Z. Ahmad, M. Arif, I. Khan, K. S. Nisar, *A time fractional model of generalized Couette flow of couple stress nanofluid with heat and mass transfer: Applications in engine oil*, IEEE Access 8 (2020) 146944–146966.

[5] V. Djordjevic, *Modeling of the slip boundary condition in micro-channel/pipe flow via fractional derivative*, Monograph of Academy of Nonlinear Sciences. Advances in Nonlinear Sciences II-Sciences and Applications, Belgrade 2 (2008) 136–158.

[6] M. Saqib, A. R. M. Kasim, N. F. Mohammad, D. L. C. Ching, S. Shafie, *Application of fractional derivative without singular and local kernel to enhanced heat transfer in CNTs nanofluid over an inclined plate*, Symmetry 12 (5) (2020) 768 22 pages.

[7] R. Ellahi, T. Hayat, F. M. Mahomed, *Generalized Couette flow of a third-grade fluid with slip: the exact solutions*, Zeitschrift für Naturforschung A 65 (12) (2010) 1071–1076.

[8] M. Farooq, A. Khan, R. Nawaz, S. Islam, M. Ayaz, Y.-M. Chu, *Comparative study of generalized couette flow of couple stress fluid using optimal homotopy asymptotic method and new iterative method*, Scientific Reports 11 (2021) 3478 20 pages.

[9] M. Arif, P. Kumam, W. Kumam, M. B. Riaz, D. Khan, *A comparative analysis of multiple fractional solutions of generalized Couette flow of couple stress fluid in a channel*, Heat Transfer 51 (8) (2022) 7348–7368.

[10] S. Maiti, S. Shaw, G. C. Shit, *Fractional order model for thermochemical flow of blood with Dufour and Soret effects under magnetic and vibration environment*, Colloids and Surfaces B: Biointerfaces, 197 (2021) 1–18.

[11] M. L. Kaurangini, B. K. Jha, *Unsteady generalized Couette flow in composite microchannel*, Applied Mathematics and Mechanics 32 (2011) 23–32.

[12] S. Chen, L. Zheng, C. Li, J. Sui, *Time-space dependent fractional viscoelastic MHD fluid flow and heat transfer over accelerating plate with slip boundary*, Thermal Science 21 (6 Part A) (2017) 2337–2345.