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Research Article

On a new approach in the space of measurable functions

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ABSTRACT. In this paper, we present a new modulus of continuity for locally integrable function spaces which is effected by the natural structure of the L_p space. After basic properties of it are expressed, we provide a quantitative type theorem for the rate of convergence of convolution type integral operators and iterates of them. Moreover, we state their global smoothness preservation property including the new modulus of continuity. Finally, the obtained results are performed to the Gauss-Weierstrass operators.

Keywords: Convolution type integral operators, measurable functions, weighted modulus of continuity.

2020 Mathematics Subject Classification: 41A36, 41A25.

1. INTRODUCTION

The study of quantitative type theorems for an approximation process is one of the research topics in Approximation Theory. Quantitative type theorems are significant tools to identify not only the convergence of a sequence of operators into an identity operator but also the rate of this convergence in a unique theorem. On the other hand, the modulus of continuity represents considerable tools for obtaining quantitative estimates of the error of approximation for positive processes. They can be defined in more special functions related to a wide class of function space. Gadjiev et al. in [27], motivated us to write this paper, was presented a new approximation process. The authors took into account weighted local integrable function space which contains classical $L_p(\mathbb{R})$ space and obtained the Korovkin type theorem on this space. Thus, some results regarding the Korovkin type approximation theorem in the space $L_p[a,b]$ of the Lebesgue integrable functions on a compact interval are generalized the results on unbounded intervals. Also, in [25], rates of A- statistical convergence of operators in the space of locally integrable functions are handled. The main advantage in considering weighted local integrable functions space, any function that is bounded with respect to the corresponding norm of the space, can be unbounded for the usual L_p norm. This allows us to widen the class of functions for which we consider the above approximation problems. In fact, in the literature, approximation results have been primarily considered either $L_p[a,b]$ or $L_p(\mathbb{R})$ space (see [17], [24], [28] and [26]), for a more general space of functions, for instance Orlicz spaces, see [23, 4].

On the other hand, Mellin transformations play major roles not only in mathematics but also in engineering, computer science, physics, etc. Their significance arises from their applications to real-life problems. For example, they are concerned with signal processing problems as in the classical Shannon Sampling Theorem, but exponentially spaced (see e.g., [18], [19], [21],

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[15]). Comprehensive approach to Mellin transforms and connections with the Mellin convolution operators were improved in [20]. The singular integrals of Mellin convolution type were first introduced by W. Kolbe and R. J. Nessel in [29]. Butzer and Jansche [20] comprehensively analyzed Mellin transformation. They defined the Mellin convolution and gained some significant results. It plays a prominent role in the Mellin analysis, like the conventional convolution operators in the Fourier analysis. These convolution integrals are used to characterize the behavior of solutions of certain boundary value problems in wedge-shaped regions. In [31], quantitative theorems on linear approximation processes of convolution operators in Banach spaces are given. Butzer and Jansche [20] extensively studied them, in connection with the L_p convergence. Later, Bardaro and Mantellini [11] concerned Mellin convolution operators of type

$$(T_w f)(s) = \int_0^\infty K_w(t) f(ts) \frac{dt}{t}, \quad s \in \mathbb{R}^+,$$

where f belongs to domain of the operator T_w and $K_w : (0, \infty) \to \mathbb{R}$ is a set of the kernels. Compared with the usual classical convolution, the translation operator is replaced by a dilation operator, and Lebesgue measure by the Haar measure $\mu = dt/t$ of the multiplicative group $\mathbb{R}^+$. This makes fully independent the operator from the classical convolution operators over the line group. We will denote by $L_p(\mu, \mathbb{R}^+) = L_p(\mu)$, $1 \le p < +\infty$, the Lebesgue spaces with respect to the measure μ and by $L_\infty(\mu)$ the space of all the essentially bounded functions. We will denote by $\|f\|_p$ and $\|f\|_\infty$ the corresponding norms.

Mamedov [30] developed the approximation theory by Mellin convolution operators T_w by considering the logarithmic Taylor formula, Mellin derivatives, logarithmic uniform continuity and logarithmic moment of kernel function K_w, which makes probable us to have better order of approximation. In [8] and [9], the authors introduced a suitable linear combination of Mellin type operators to accelerate convergence. A crucial contribution for the Voronovskaja type results for singular integral operators of Mellin convolution given by C. Bardaro and I. Mantellini in [14], [11] and [7]. Another approach to gain better approximation order, Bardaro and Mantellini [12] considered linear combinations of Mellin type operators using the iterated kernels. Also, in [10], [16] and [5], some contributions on sampling operators in Mellin-Lebesgue spaces were obtained recently. Angeloni and Vinti, both in [2] and [3], studied Mellin integral operators in the space of functions of bounded variation in the multidimensional setting via the notion of variation for multivariate functions. In the recent past, in [32], Ozsarac et al. defined a new generalization of Mellin convolution operators that preserve logarithmic functions, and investigated the weighted approximation properties of the operators.

Pointwise convergence for type linear singular integrals in periodic case or in the line group was thoroughly worked in the classical book by P.L. Butzer and R.J. Nessel [22], where in particular an almost everywhere convergence is acquired using the notion of the Lebesgue point of a function $f \in L_p$, $1 \le p \le +\infty$. Also, in [13], pointwise convergence theorems for nonlinear Mellin convolution operators are verified.

In [27], even if, for what concerns Korovkin type results on this concept have been proved, quantitative type results of approximation have not been yet studied. In this paper, we tackle the above problems for convolution type operators. Such operators are studied by many mathematicians due to their various application in different domains of mathematics and physics (see [22]).

For this intention, in this paper, with the motivation of [6], we present a new modulus of continuity whose structure is compatible with the nature of the locally integrable function space to measure the rate of convergence. Also, the global smoothness preservation property of the convolution-type operators is proved. This property is also used to obtain a quantitative type theorem for the convolution type operator with an iterated kernel instead of a basic kernel.

Now, we express the notion of locally integrable function in Mellin setting. In the course of this paper, we will use the weight function ω defined by $\omega(x) = 1 + \log^2 x$, $x \in \mathbb{R}^+$. Then, we will denote by $X_{p,\omega}(loc)$ the space of all locally integrable functions, that is the space of all measurable functions f satisfying the inequality

$$\left(\frac{1}{2\log h} \int\limits_{x/h}^{xh} |f(s)|^p \frac{ds}{s} \right)^{1/p} \leq M_f \omega(x), x \in \mathbb{R}^+,$$

where M_f is a positive constant which depends on the function f, $p > 1$ and $h > 1$ is any positive constant.

To simplify statement, we need the followings. For any real numbers a and b $(a < b)$, we write

$$\|f; X_p(a,b)\| = \left(\frac{1}{\log \frac{b}{a}} \int\limits_a^b |f(s)|^p \frac{ds}{s} \right)^{1/p}.$$

$X_{p,\omega}(loc)$ is a linear normed space with the norm

$$\begin{aligned} \|f\|_{X_{p,\omega}} &= \sup_{x \in \mathbb{R}^+} \frac{\left(\frac{1}{2\log h} \int\limits_{x/h}^{xh} |f(s)|^p \frac{ds}{s} \right)^{1/p}}{\omega(x)} \\ &= \sup_{x \in \mathbb{R}^+} \frac{\|f; X_p(xh, x/h)\|}{\omega(x)}. \end{aligned} \tag{1.1}$$

It is clear that

$$L_p(\mathbb{R}^+) \subset X_{p,\omega}(loc),$$

where $L_p(\mathbb{R}^+)$ is the Lebesgue space with respect to the measure ds/s. Let $X_{p,\omega}^{k_f}(loc)$ be the subspace of all functions $f \in X_{p,\omega}(loc)$ for which there exists a constant k_f such that

$$\lim_{x \to \infty} \frac{\|f - k_f \omega; X_{p,\omega}(xh, x/h)\|}{\omega(x)} = 0.$$

In the case of $k_f = 0$, we will write $X_{p,\omega}^0(loc)$.

2. Definition of New Weighted Modulus of Continuity

In this part, to obtain the rate of convergence of approximation, we introduce a new type weighted modulus of continuity for function $f \in X_{p,\omega}(loc)$. Firstly, the new weighted modulus of continuity has some properties that are similar to the properties of the classical modulus of continuity. Using the weighted modulus of continuity, we obtain estimates of approximation of function $f \in X_{p,\omega}(loc)$ with respect to weighted norm. For each $f \in X_{p,\omega}(loc)$, we set

$$\Omega_{X,\omega}(f;\delta) = \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts)-f(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)}$$

$$= \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \frac{\|f(ts)-f(t);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)}, \tag{2.2}$$

where $\delta > 0$. It is clear that $\Omega_{X,\omega}(f;\delta)$ is a non-negative and non-decreasing function. First, we show that $\Omega_{X,\omega}$ is bounded.

Lemma 2.1. *For any $f \in X_{p,\omega}(loc)$ and $\delta > 0$, we have*

$$\Omega_{X,\omega}(f;\delta) \leq 3\|f\|_{X_{p,\omega}}.$$

Proof. Using the inequality $\omega(xs) \leq 2\omega(x)\,\omega(s)$, we obtain by (1.1)

$$\begin{aligned}\Omega_{X,\omega}(f;\delta) &\leq \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \frac{\|f(\cdot s);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)} \\ &+ \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \frac{\|f;X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)} \\ &\leq 3\|f\|_{X_{p,\omega}}.\end{aligned}$$

□

Lemma 2.2. *For any non-negative real numbers λ and δ, the following relation*

$$\Omega_{X,\omega}(f;\lambda\delta) \leq (1+\lambda)\,\Omega_{X,\omega}(f;\delta) \tag{2.3}$$

holds.

Proof. We take into account $\delta > 0$. For any positive integer n, we may write

$$\begin{aligned}\Omega_{X,\omega}(f;n\delta) &= \sup_{|\log s|\leq n\delta} \sup_{x\in\mathbb{R}^+} \frac{\|f(ts)-f(t);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s)} \\ &= \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \frac{\|f(ts^n)-f(t);X_{p,\omega}(xh,x/h)\|}{\omega(x)\,\omega(s^n)} \\ &\leq \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \sum_{k=1}^{n} \frac{\left\|f\left(ts^k\right)-f\left(ts^{k-1}\right);X_{p,\omega}(xh,x/h)\right\|}{\omega(x)\,\omega(s^n)} \\ &= \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \sum_{k=1}^{n} \frac{\left\|f\left(ts^k\right)-f\left(ts^{k-1}\right);X_{p,\omega}(xh,x/h)\right\|}{\omega(x)\,\omega(s)} \\ &\times \frac{\omega(s)}{\omega(s^n)} \\ &\leq n\Omega_{X,\omega}(f;\delta).\end{aligned}$$

Since $\Omega_{X,\omega}(f;\delta)$ is non-decreasing function of δ, the inequality

$$\Omega_{X,\omega}(f;\lambda\delta) \leq \Omega_{X,\omega}(f;([\lambda]+1)\,\delta) \leq (\lambda+1)\,\Omega_{X,\omega}(f;\delta)$$

holds for $\lambda > 0$, where $[\cdot]$ means the integer part. □

Theorem 2.1. *If $f \in X_{p,\omega}^{k_f}(loc)$, then $\lim_{\delta\to 0}\Omega_{X,\omega}(f;\delta) = 0$.*

Proof. Because of $f \in X_{p,\omega}^{k_f}(loc)$, $\lim\limits_{x\to\infty} \frac{\|f-k_f\omega; X_{p,\omega}(xh,x/h)\|}{\omega(x)} = 0$, for all $\varepsilon > 0$, there exists a positive real number x_0 such that for all $x > x_0$

$$\|f - k_f\omega; X_{p,\omega}(xh, x/h)\| < \varepsilon\omega(x).$$

Let $x_1 > x_0 + \delta$. Let us divide the norm into two parts. Then

$$\begin{aligned}
\Omega_{X,\omega}(f;\delta) &= \sup_{|\log s|\leq\delta} \sup_{x\in\mathbb{R}^+} \frac{\|f(ts) - f(t); X_{p,\omega}(xh, x/h)\|}{\omega(x)\,\omega(s)} \\
&\leq \sup_{|\log s|\leq\delta} \sup_{0<x\leq x_1} \frac{\|f(ts) - f(t); X_{p,\omega}(xh, x/h)\|}{\omega(x)\,\omega(s)} \\
&+ \sup_{|\log s|\leq\delta} \sup_{x>x_1} \frac{\|f(ts) - f(t); X_{p,\omega}(xh, x/h)\|}{\omega(x)\,\omega(s)} \\
&\leq \omega_X(f;\delta) + \sup_{|\log s|\leq\delta} \sup_{x>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)} \\
&+ \sup_{|\log s|\leq\delta} \sup_{x>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(t) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)},
\end{aligned}$$

where

$$\omega_X(f;\delta) = \sup_{|\log s|\leq\delta} \sup_{|x|\leq x_1} \|f(ts) - f(t); X_{p,\omega}(xh, x/h)\|.$$

It is shown that in [20, page:340], for each $\varepsilon > 0$, there exists $h > 1$ such that for all $0 < s < x_1$. Then for $x > x_1$ and $|\log s| \leq \delta$, with the elementary calculations, we get

$$\begin{aligned}
\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p} &\leq \left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(ts) - k_f\omega(ts)|^p \frac{dt}{t}\right)^{1/p} \\
&+ k_f\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |\omega(ts) - \omega(t)|^p \frac{dt}{t}\right)^{1/p} \\
&\leq \left(\frac{1}{2\log h}\int\limits_{xs/h}^{xsh} |f(t) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p} \\
&+ 4k_f|\log s|\left(|\log x| + \log h + |\log s|\right).
\end{aligned}$$

For $x > x_1$ and $|\log s| \leq \delta$, we obtain

$$\Omega_{X,\omega}(f;\delta) \leq \omega_X(f;\delta) + \sup_{|\log s|\leq\delta} \sup_{|x|>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{xs/h}^{xsh} |f(t) - k_f\omega(t)|^p \frac{dt}{t}\right)^{1/p}}{\omega(x)\,\omega(s)}$$
$$+ \sup_{|\log s|\leq\delta} \sup_{|x|>x_1} \frac{4k_f |\log s| (|\log x| + \log h + |\log s|)}{\omega(x)\,\omega(s)}$$
$$+ \sup_{|\log s|\leq\delta} \sup_{|x|>x_1} \frac{\left(\frac{1}{2\log h}\int\limits_{x/h}^{xh} |f(t) - k_f\omega(t)| \frac{dt}{t}\right)^{1/p}}{\omega(x)} \frac{1}{\omega(s)}$$

and

$$\Omega_{X,\omega}(f;\delta) \leq \omega_X(f;\delta) + \varepsilon + 4k_f\delta(1+\delta+\log h) + \varepsilon.$$

As $[x_1/h, x_1h]$ is compact interval, we get $\lim\limits_{\delta\to 0} \omega_X(f;\delta) = 0$. Therefore, we have $\lim\limits_{\delta\to 0} \Omega_{X,\omega}(f;\delta) < 2\varepsilon$. Since the inequality is true for each $\varepsilon > 0$, desired result is attained. □

3. Approximation Properties

Let $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a kernel function homogenous degree 0, i.e.

$$K(\lambda s, \lambda t) = K(s,t)$$

for every $\lambda, s, t > 0$. We will assume that K is globally measurable $K(s,.) \in L_1(\mathbb{R}^+)$ with

$$\int\limits_{\mathbb{R}^+} K(s,t)\frac{dt}{t} = 1, \quad s \in \mathbb{R}^+.$$

For a given $j \in \mathbb{N}$, we define logarithmic and absolute logarithmic moment of order j of the function K, respectively by

(3.4) $$m_j(K) := \int\limits_{\mathbb{R}^+} K(s,t)\log^j\left(\frac{t}{s}\right)\frac{dt}{t}$$

and

(3.5) $$M_j(K) := \int\limits_{\mathbb{R}^+} |K(s,t)| \left|\log^j\left(\frac{t}{s}\right)\right| \frac{dt}{t}.$$

Also, we define Mellin Fejer kernel (K_w) for all $w > 0$ generated by K putting

$$K_w(s,t) = wK(s^w, t^w), \ \ s,t \in \mathbb{R}.$$

It is easy to see that

(3.6) $$\int\limits_{\mathbb{R}^+} K_w(s,t)\frac{dt}{t} = 1.$$

Let us regard the convolution type singular integral operator

(3.7) $$(T_wf)(s) = \int\limits_{\mathbb{R}^+} K_w(s,t) f(t)\frac{dt}{t}$$

for every $f : \mathbb{R}^+ \to \mathbb{R}$ in the domain of the operators T_w.

Lemma 3.3. *T_w be defined by (3.7). If $f \in X_{p,\omega}\left(loc\right)$, then we have*

$$\left\| T_w f \right\|_{X_{p,\omega}} \leq 2\left(M_0\left(K_w\right) + M_2\left(K_w\right)\right) \left\| f \right\|_{X_{p,\omega}}.$$

Proof. Taking into account the operator defined by (3.7), we can write

$$\begin{aligned}
\left\| T_w f \right\|_{X_{p,\omega}} &= \sup_{x\in\mathbb{R}^+} \frac{\left\| T_w f; X_{p,\omega}\left(xh, x/h\right) \right\|}{\omega\left(x\right)} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \left(\frac{1}{2\log h} \int\limits_{x/h}^{xh} \left| \int\limits_{\mathbb{R}^+} K_w\left(s,t\right) f\left(t\right) \frac{dt}{t} \right|^p \frac{ds}{s} \right)^{1/p} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \left(\frac{1}{2\log h} \int\limits_{x/h}^{xh} \left| \int\limits_{\mathbb{R}^+} K_w\left(1,t\right) f\left(ts\right) \frac{dt}{t} \right|^p \frac{ds}{s} \right)^{1/p}.
\end{aligned}$$

From Minkowski inequality, we obtain

$$\begin{aligned}
\left\| T_w f \right\|_{X_{p,\omega}} &\leq \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \int\limits_{x\in\mathbb{R}^+} \left(\frac{1}{2\log h} \int\limits_{x/h}^{xh} \left| f\left(ts\right) \right|^p \frac{ds}{s} \right)^{1/p} \left| K_w\left(1,t\right) \right| \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \int\limits_{x\in\mathbb{R}^+} \left(\frac{1}{2\log h} \int\limits_{xt/h}^{xht} \left| f\left(s\right) \right|^p \frac{ds}{s} \right)^{1/p} \left| K_w\left(1,t\right) \right| \frac{dt}{t} \\
&\leq \left\| f \right\|_{X_{p,\omega}} \sup_{x\in\mathbb{R}^+} \frac{1}{\omega\left(x\right)} \int\limits_{\mathbb{R}^+} \omega\left(tx\right) \left| K_w\left(1,t\right) \right| \frac{dt}{t} \\
&\leq 2 \left\| f \right\|_{X_{p,\omega}} \int\limits_{\mathbb{R}^+} \omega\left(t\right) \left| K_w\left(1,t\right) \right| \frac{dt}{t} \\
&= 2 \left\| f \right\|_{X_{p,\omega}} \left(1 + \int\limits_{\mathbb{R}^+} \log^2 t \left| K_w\left(1,t\right) \right| \frac{dt}{t} \right).
\end{aligned}$$

From (3.5) for $j = 2$, we get desired result. □

Our main results are following:

Theorem 3.2. *Let T_w be defined by (3.7) and $\Omega_{X,\omega}\left(f;\delta\right)$ be defined (2.2). If $f \in X_{p,\omega}\left(loc\right)$, then we have*

$$\left\| T_w f - f \right\|_{X_{p,\omega}} \leq P_w \Omega_{X,\omega}\left(f; \left(M_2\left(K_w\right)\right)^{1/2}\right),$$

where $P_w := 1 + M_2\left(K_w\right) + \sqrt{2}\sqrt{1 + M_4\left(K_w\right)}$.

Proof. We attain

$$\left(T_w f\right)\left(s\right) - f\left(s\right) = \int\limits_{\mathbb{R}^+} K_w\left(s,t\right)\left(f\left(t\right) - f\left(s\right)\right) \frac{dt}{t}.$$

We conclude

$$\begin{aligned}
\|T_w f - f\|_{X_{p,\omega}} &= \sup_{x\in\mathbb{R}^+} \frac{\|T_w f - f; X_{p,\omega}(xh, x/h)\|}{\omega(x)} \\
&= \sup_{x\in\mathbb{R}^+} \frac{1}{\omega(x)} \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \int_{\mathbb{R}^+} (f(t) - f(s)) K_w(s,t) \frac{dt}{t} \right|^p \frac{ds}{s} \right)^{1/p} \\
&\leq \int_{\mathbb{R}^+} \sup_{x\in\mathbb{R}^+} \frac{1}{\omega(x)} \left(\frac{1}{2\log h} \int_{x/h}^{xh} |(f(ts) - f(s))|^p \frac{ds}{s} \right)^{1/p} |K_w(1,t)| \frac{dt}{t} \\
&= \int_{\mathbb{R}^+} \Omega_{X,\omega}(f; |\log t|)\, \omega(t)\, |K_w(1,t)| \frac{dt}{t}.
\end{aligned}$$

From (2.3), for any $\delta > 0$, we can write

$$\|T_w f - f\|_{X_{p,\omega}} \leq \Omega_{X,\omega}(f;\delta) \int_{\mathbb{R}^+} \left(1 + \frac{|\log t|}{\delta}\right) \omega(t)\, |K_w(1,t)| \frac{dt}{t}.$$

Using Cauchy-Schwarz inequality and (3.5), we obtain

$$\begin{aligned}
&\|T_w f - f\|_{X_{p,\omega}} \\
\leq& \Omega_{X,\omega}(f;\delta) \left(1 + M_2(K_w) + \frac{1}{\delta} \left(\int_{\mathbb{R}} \log^2 t\, |K_w(1,t)| \frac{dt}{t} \right)^{1/2} \left(\int_{\mathbb{R}} \omega^2(t)\, |K_w(1,t)| \frac{dt}{t} \right)^{1/2} \right) \\
=& \Omega_{X,\omega}(f;\delta) \left(1 + M_2(K_w) + \frac{\sqrt{2}}{\delta} (M_2(K_w))^{1/2} \sqrt{1 + M_4(K_w)} \right).
\end{aligned}$$

If we choose $\delta = (M_2(K_w))^{1/2}$, then we have desired result. □

The global smoothness preservation property of the operator $T_w f$ is following:

Theorem 3.3. *Let T_w be defined by (3.7) and let $\Omega_{X,\omega}(f;\delta)$ be defined (2.2). If $f \in X_{p,\omega}(loc)$ and $\delta > 0$, then we get*

$$\Omega_{X,\omega}(T_w f;\delta) \leq 2(M_0(K_w) + M_2(K_w))\, \Omega_{X,\omega}(f;\delta).$$

Proof. We have

$$\begin{aligned}
J_t(x) :&= \frac{\|T_w f(\cdot z) - T_w f(\cdot); X_{p,\omega}(xh, x/h)\|}{\omega(x)\,\omega(z)} \\
&= \left(\frac{1}{2\log h} \int_{x/h}^{xh} \frac{|T_w f(uz) - T_w f(u)|^p}{\omega(x)\,\omega(z)} \frac{du}{u} \right)^{1/p} \\
&= \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \int_{\mathbb{R}^+} \frac{[f(uzt) - f(tu)]}{\omega(x)\,\omega(z)} K_w(1,t) \frac{dt}{t} \right|^p \frac{du}{u} \right)^{1/p}
\end{aligned}$$

$$\leq \int_{\mathbb{R}^+} \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \frac{[f(uzt) - f(ut)]}{\omega(x)\omega(z)} \right|^p \frac{du}{u} \right)^{1/p} |K_w(1,t)| \frac{dt}{t}.$$

Using the inequality $\omega(xt) \leq 2\omega(x)\omega(t)$, Minkowski's integral inequality for two dimensional spaces and identity (3.5) for $j = 2$, we gain

$$J_t(x) \leq \int_{\mathbb{R}^+} \left(\frac{1}{2\log h} \int_{x/h}^{xh} \left| \frac{[f(uzt) - f(ut)]}{\omega(x)\omega(z)} \right|^p \frac{du}{u} \right)^{1/p} |K_w(1,t)| \frac{dt}{t}$$

$$= \int_{\mathbb{R}^+} \left(\frac{1}{2\log h} \int_{tx/h}^{txh} \left| \frac{\omega(xt)}{\omega(x)\omega(xt)\omega(z)} [f(vz) - f(v)] \right|^p \frac{dv}{v} \right)^{1/p} |K_w(1,t)| \frac{dt}{t}.$$

Then, we have

$$\Omega_{X,\omega}(T_w f;\delta) \leq 2\Omega_{X,\omega}(f;\delta) \int_{\mathbb{R}^+} \omega(t) |K_w(1,t)| \frac{dt}{t}$$

$$= 2(M_0(K_w) + M_2(K_w))\, \Omega_{X,\omega}(f;\delta).$$

Hence, the proof is fulfilled. □

4. Iterations of T_w

Given the function K we define for every $n \in \mathbb{N}$ the iterated kernel of order n of K as in [12], in the following way: for $n = 2$,

$$K^2(s,t) := \int_{\mathbb{R}^+} K(s,z) K(z,t) \frac{dz}{z}$$

and for $n > 2$

$$K^n(s,t) := \int_{\mathbb{R}^+} K(s,z) K^{n-1}(z,t) \frac{dz}{z}.$$

Similarly, endowed the function K_w, we define for every $n \in \mathbb{N}$, the iterated kernel of order n of K_w in the following way: for $n = 2$,

$$K_w^2(s,t) := \int_{\mathbb{R}^+} K_w(s,z) K_w(z,t) \frac{dz}{z}$$

and for $n > 2$

$$K_w^n(s,t) := \int_{\mathbb{R}^+} K_w(s,z) K_w^{n-1}(z,t) \frac{dz}{z}.$$

We gain for every $n \in \mathbb{N}$

$$\int_{\mathbb{R}^+} K_w^n(s,t) \frac{dt}{t} = 1.$$

Also, we have

$$m_j(K_w^n) = \frac{1}{w^j} m_j(K^n)$$

and

$$M_j\left(K_w^n\right)=\frac{1}{w^j}M_j\left(K^n\right),\ \ j\in\mathbb{N}.$$

In the same method, let us consider $n-$iterations of T_w defined by

$$\left(T_w^n f\right)(s)=\int\limits_{\mathbb{R}^+} K_w^n\left(s,t\right)f\left(t\right)\frac{dt}{t}. \tag{4.8}$$

We start with the following

Theorem 4.4. *T_w^n be defined by (4.8). If $f\in X_{p,\omega}\left(loc\right)$, then we have, for every $n\in\mathbb{N}$*

$$\left\|T_w^n f-f\right\|_{X_{p,\omega}}\leq P_w\left[\sum_{k=0}^{n-1}\left[2\left(1+M_2\left(K_w\right)\right)\right]^k\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right),$$

where P_w is as in Theorem 3.2.

Proof. For $n=2$, we obtain

$$T_w^2 f\left(u\right)-f\left(u\right)=T_w\left(T_w f\right)\left(u\right)-\left(T_w f\right)\left(u\right)+\left(T_w f\right)\left(u\right)-f\left(u\right).$$

Using Theorem 3.2, we have

$$\left\|T_w^2 f-f\right\|_{X_{p,\omega}}\leq P_w\Omega_{X,\omega}\left(T_w f;\left(M_2\left(K_w\right)\right)^{1/2}\right)+P_w\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right).$$

Using Theorem 3.3, we achieve

$$\left\|T_w^2 f-f\right\|_{X_{p,\omega}}\leq P_w\left[2\left(1+M_2\left(K_w\right)\right)+1\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right).$$

By induction, we gain

$$\left\|T_w^n f-f\right\|_{X_{p,\omega}}\leq P_w\left[\sum_{k=0}^{n-1}\left[2\left(1+M_2\left(K_w\right)\right)\right]^k\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right).$$

□

Now, we can denote following result which expresses the difference of $n-$iterations and itself of T_w.

Corollary 4.1. *T_w^n be defined by (4.8) and T_w be defined by (3.7). If $f\in X_{p,\omega}\left(loc\right)$, then we get, for every $n\in\mathbb{N}$*

$$\left\|T_w^n f-T_w f\right\|_{X_{p,\omega}}\leq P_w\left[\sum_{k=1}^{n-1}\left[2\left(1+M_2\left(K_w\right)\right)\right]^k\right]\Omega_{X,\omega}\left(f;\left(M_2\left(K_w\right)\right)^{1/2}\right),$$

where P_w is as in Theorem 3.2.

5. Application

This section is allocated to some example. The results obtained in the previous sections can be applied to the Gauss-Weierstrass operators. The recent results related to the Gauss-Weierstrass operators also can be found in [1]. Let $K:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ be a function defined by

$$K\left(s,t\right)=\frac{1}{2\sqrt{\pi}}\exp\left(-\left(\frac{1}{2}\log\frac{t}{s}\right)^2\right)$$

(See [22]). It is easy to check that

$$\int_{\mathbb{R}} K(s,t)\,\frac{dt}{t}=1.$$

The Mellin-Fejer kernel generated by K is given by

$$K_w(s,t)=\frac{w}{2\sqrt{\pi}}\exp\left(-\left(\frac{w}{2}\log\frac{t}{s}\right)^2\right).$$

The corresponding Mellin-Gauss-Weierstrass operator is given by

$$(G_w f)(s)=\frac{w}{2\sqrt{\pi}}\int_{\mathbb{R}^+}\exp\left(-\left(\frac{w}{2}\log\frac{t}{s}\right)^2\right)f(t)\,\frac{dt}{t}.$$

If j is even, we get the moment of order $2j$ of the function G_w

(5.9) $$m_j(K)=M_j(K)=2^{j/2}(j-1)!!,$$

where in the case $n!!=3.5...n$ with n is odd. For the n-iterated kernels, we have the formula

$$G_w^n(s,t)=\frac{w}{2\sqrt{n}\sqrt{\pi}}\exp\left(-\left(\frac{w}{2\sqrt{n}}\log\frac{s}{t}\right)^2\right)$$

(see [12]). We have by Theorem 3.2 and (5.9), the following:

Corollary 5.2. *Let $\Omega_{X,\omega}(f;\delta)$ be defined (2.2). If $f\in X_{p,\omega}(loc)$, then we get*

$$\|G_w f-f\|_{X_{p,\omega}}\le P_w\Omega_{X,\omega}\left(f;\frac{\sqrt{2}}{w}\right),$$

where $P_w:=1+\frac{2}{w^2}+\sqrt{2}\sqrt{1+\frac{12}{w^4}}$.

We have by Theorem 4.4 and (5.9), the following:

Corollary 5.3. *If $f\in X_{p,\omega}(loc)$, then we get*

$$\|G_w^n f-f\|_{X_{p,\omega}}\le\left(1+\frac{2}{w^2}+\sqrt{2}\sqrt{1+\frac{12}{w^4}}\right)\left[\sum_{k=0}^{n-1}\left[2\left(1+\frac{2}{w^2}\right)\right]^k\right]\Omega_{p,\omega}\left(f;\frac{\sqrt{2}}{w}\right).$$

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