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SOME INTEGRAL INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE φ−CONVEX BY USING FRACTIONAL INTEGRALS

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Abstract. In this paper, we obtain new estimates on generalization of Hermite-Hadamard type inequalities for functions whose second derivatives is φ−convex via fractional integrals.

1. Introduction

The following inequality is called the Hermite-Hadamard inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$. If f is concave, then both inequalities hold in the reversed direction .

The inequality (1.1) was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality is known as the Hermite-Hadamard inequality, because this inequality was found by Mitrinovic Hermite and Hadamard' note in Mathesis in 1974.

The inequality (1.1) is studied by many authors, see ([1]-[7], [9]-[11], [12], [15]-[21]) where further references are listed.

Firstly, we need to recall some concepts of convexity concerning our work.

Definition 1.1. [6] A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I if inequality

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{1.2}$$

holds for all $a, b \in I$ and $t \in [0, 1]$.

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Definition 1.2. [8] Let $s \in (0,1]$. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be $s-$convex in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b), \tag{1.3}$$

holds for all $a, b \in I$ and $t \in [0,1]$.

Tunç and Yildirim in [21] introduced the following definition as follows:

Definition 1.3. *A function f:$I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $MT(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the inequality*;

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y).$$

Dragomir in [3] introduced the following definition as follows:

Definition 1.4. [3] Let $\varphi : (0,1) \to (0,\infty)$ be a measurable function. We say that the function $f : I \to [0,\infty)$ *is a* $\varphi-$convex function on the interval I if for $x, y \in I$, we have

$$f(tx + (1-t)y) \leq t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y).$$

Remark 1.1. According to definition 4, the followings hold for the special choose of φ (t):

For $\varphi(t) \equiv 1$, we obtain the definition of convexness in the classical sense,
for $\varphi(t) = t^{s-1}$, we obtain the definition of $s-$ convexness,
for $\varphi(t) = \dfrac{1}{2\sqrt{t(1-t)}}$, we obtain the definition of $MT-$convexness.

Now, we give some definitions and notations of fractional calculus theory which are used later in this paper. Samko et al. in [14] used the following definitions as follows:

Definition 1.5. [14] The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a \tag{1.4}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b \tag{1.5}$$

where $f \in L_1[a,b]$, respectively. Note that, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{\alpha} f(x) = f(x)$.

Definition 1.6. [14] The Euler Beta function is defined as follows:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \ x, y > 0.$$

The incomplete beta function is defined as follows:

$$\beta(a,x,y) = \int_0^a t^{x-1}(1-t)^{y-1} dt, \ x, y > 0, \ 0 < \alpha < 1.$$

In [13], Jaekeun Park established the following lemma which is necessary to prove our main results:

Lemma 1.1. *Let $f\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I such that $f''\in L_1[a,b]$, where $a,b\in I$ with $a<b$. Then, for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$, we have*

$$\begin{aligned} S_f(x,\lambda,\alpha;a,b) &= \tfrac{(x-a)^{\alpha+2}}{b-a}\int_0^1 t\left(\lambda-t^{\alpha}\right)f''\left(tx+(1-t)a\right)dt \\ &+\tfrac{(b-x)^{\alpha+2}}{b-a}\int_0^1 t\left(\lambda-t^{\alpha}\right)f''\left(tx+(1-t)b\right)dt. \end{aligned}$$

2. Main results

Throughout this paper, we use S_f as follows;

$$\begin{aligned} S_f(x,\lambda,\alpha;a,b) &\equiv (1-\lambda)\left\{\tfrac{(b-x)^{\alpha+1}-(x-a)^{\alpha+1}}{b-a}\right\}f'(x) \\ &+(1+\alpha-\lambda)\left\{\tfrac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}\right\}f(x) \\ &+\lambda\left\{\tfrac{(x-a)^{\alpha}(f(a)+(b-x)^{\alpha}f(b)}{b-a}\right\} \\ &-\tfrac{\Gamma(\alpha+2)}{b-a}\left\{J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right\}, \end{aligned}$$

for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$.

Theorem 2.1. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function. Assume also that $f:I\subset[0,\infty)\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I such that $f''\in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''|^q$ is $\varphi-$convex on $[a,b]$ for some fixed $q\geq 1$, then for any $x\in[a,b]$, $t,\lambda\in[0,1]$ and $\alpha>0$,*

$$\begin{aligned} |S_f(x,\lambda,\alpha,t,\varphi;a,b)| &\leq A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{A_2(\alpha,\lambda,t,\varphi)\left|f''(x)\right|^q\right.\right. \\ &\left.+A_3(\alpha,\lambda,t,\varphi)\left|f''(a)\right|^q\right\}^{\frac{1}{q}} \\ &\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{A_2(\alpha,\lambda,t,\varphi)\left|f''(x)\right|^q+A_3(\alpha,\lambda,t,\varphi)\left|f''(b)\right|^q\right\}^{\frac{1}{q}}\right]. \end{aligned} \tag{2.1}$$

The above inequality for fractional integrals holds, where

$$\begin{aligned} A_1(\alpha,\lambda) &= \tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}, \\ A_2(\alpha,\lambda,t,\varphi) &= \int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|t\varphi(t)\,dt, \\ A_3(\alpha,\lambda,t,\varphi) &= \int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|(1-t)\varphi(1-t)\,dt. \end{aligned}$$

Proof. By using Lemma 1.1, the power mean inequality, we get

(2.2)
$$\begin{aligned}
&|S_f(x,\lambda,\alpha,t,\varphi;a,b)| \\
&\leq \tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\
&+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)b)|\,dt\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\lambda)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$A_1(\alpha,\lambda)=\int_0^1 |t(\lambda-t^\alpha)|\,dt=\left(\frac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\frac{\lambda}{2}\right).$$

Since $|f''|^q$ is $\varphi-$convex on $[a,b]$, we have

$$\begin{aligned}
I_1 &= \int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)a)|^q\,dt \\
&\leq \int_0^1 |t(\lambda-t^\alpha)|\left\{t\varphi(t)\,|f''(x)|^q+(1-t)\,\varphi(1-t)\,|f''(a)|^q\right\}dt \qquad (2.3)\\
&= A_2(\alpha,\lambda,t,\varphi)\,|f''(x)|^q+A_3(\alpha,\lambda,t,\varphi)\,|f''(a)|^q,
\end{aligned}$$

and similarly, we can obtain

$$\begin{aligned}
I_2 &= \int_0^1 |t(\lambda-t^\alpha)|\,|f''(tx+(1-t)b)|^q\,dt \\
&\leq \int_0^1 |t(\lambda-t^\alpha)|\left\{t\varphi(t)\,|f''(x)|^q+(1-t)\,\varphi(1-t)\,|f''(b)|^q\right\}dt \qquad (2.4)\\
&= A_2(\alpha,\lambda,t,\varphi)\,|f''(x)|^q+A_3(\alpha,\lambda,t,\varphi)\,|f''(b)|^q,
\end{aligned}$$

where

$$\begin{aligned}
A_2(\alpha,\lambda,t,\varphi) &= \int_0^1 |t(\lambda-t^\alpha)|\,t\varphi(t)\,dt, \\
A_3(\alpha,\lambda,t,\varphi) &= \int_0^1 |t(\lambda-t^\alpha)|\,(1-t)\,\varphi(1-t)\,dt.
\end{aligned}$$

By substituting (2.3) and (2.4) in (2.2), we get

$$\begin{aligned}
&\left|S_f\left(x,\lambda,\alpha,t,\varphi;a,b\right)\right| \\
&\leq \left(\tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}\right)^{1-\frac{1}{q}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\left|f''\left(x\right)\right|^q\int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|t\varphi\left(t\right)dt\right.\right. \\
&\left.+\left|f''\left(a\right)\right|^q\int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|\left(1-t\right)\varphi\left(1-t\right)dt\right\}^{\frac{1}{q}} \\
&+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\left|f''\left(x\right)\right|^q\int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|t\varphi\left(t\right)dt\right. \\
&\left.\left.+\left|f''\left(b\right)\right|^q\int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|\left(1-t\right)\varphi\left(1-t\right)dt\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Thus the proof is completed. □

Corollary 2.1. *Let $\varphi\left(t\right)=1$ in Theorem 2.1, then we get the following inequality:*

$$\begin{aligned}
&\left|S_f\left(x,\lambda,\alpha;a,b\right)\right| \\
&\leq \left(\tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}\right)^{1-\frac{1}{q}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{A_2\left(\alpha,\lambda\right)\left|f''\left(x\right)\right|^q+A_3\left(\alpha,\lambda\right)\left|f''\left(a\right)\right|^q\right\}\right. \\
&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{A_2\left(\alpha,\lambda\right)\left|f''\left(x\right)\right|^q+A_3\left(\alpha,\lambda\right)\left|f''\left(b\right)\right|^q\right\}\right].
\end{aligned}$$

Where

$$A_2\left(\alpha,\lambda\right) = \int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|tdt=\frac{3-\left(\alpha+3\right)\lambda+2\alpha\lambda^{1+\frac{3}{\alpha}}}{3\left(\alpha+3\right)}$$

and

$$\begin{aligned}
A_3\left(\alpha,\lambda\right) &= \int_0^1\left|t\left(\lambda-t^{\alpha}\right)\right|\left(1-t\right)dt \\
&= \frac{\alpha\lambda^{1+\frac{2}{\alpha}}}{\alpha+2}-\frac{2\lambda^{1+\frac{3}{\alpha}}}{3\left(\alpha+3\right)}+\frac{\alpha\lambda}{6}-\frac{\alpha}{\left(\alpha+2\right)\left(\alpha+3\right)}.
\end{aligned}$$

Corollary 2.2. *If we choose $\varphi\left(t\right)=1$ and $x=\frac{a+b}{2}$ in Theorem 2.1, we can obtain the corollary 2.2, 2.3, 2.4 in* [13]*, respectively for $\lambda=\frac{1}{3}$, $\lambda=0$, $\lambda=1$.*

Corollary 2.3. *Let $\varphi\left(t\right)=t^{s-1}$ in Theorem 2.1, then we have*

$$\begin{aligned}
&\left|S_f\left(x,\lambda,\alpha,t,\varphi;a,b\right)\right| \\
&\leq \left(\tfrac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\tfrac{\lambda}{2}\right)^{1-\frac{1}{q}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\left|f''\left(x\right)\right|^qA_4\left(\alpha,\lambda,s\right)+\left|f''\left(a\right)\right|^qA_5\left(\alpha,\lambda,t,\varphi\right)\right\}^{\frac{1}{q}}\right. \\
&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\left|f''\left(x\right)\right|^qA_4\left(\alpha,\lambda,s\right)+\left|f''\left(b\right)\right|^qA_5\left(\alpha,\lambda,t,\varphi\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Where

$$A_4(\alpha,\lambda,s) = 2\frac{\lambda^{\frac{s+2}{\alpha}+1}}{s+2} - 2\frac{\lambda^{\frac{s+2}{\alpha}+1}}{\alpha+s+2} + \frac{1}{\alpha+s+2}$$

$$A_5(\alpha,\lambda,t,\varphi) = \lambda\beta\left(\lambda^{\frac{1}{\alpha}},2,s+1\right) - \beta\left(\lambda^{\frac{1}{\alpha}},\alpha+2,s+1\right)$$
$$+\beta\left(1-\lambda^{\frac{1}{\alpha}},\alpha+2,s+1\right) - \lambda\beta\left(1-\lambda^{\frac{1}{\alpha}},2,s+1\right).$$

Theorem 2.2. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function. For $f: I \subset [0,\infty)\to\mathbb{R}$ be a twice differentiable function on the interior I^0 assume also that $f'' \in L_1[a,b]$, where $a,b\in I^0$ with $a<b$. If $|f''|^q$ is $\varphi-$convex on $[a,b]$ for some fixed $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$ the following inequality holds*

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$
$$\leq B^{\frac{1}{p}}(\alpha,\lambda,p)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\left(|f''(x)|^q+|f''(a)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right. \tag{2.5}$$
$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\left(|f''(x)|^q+|f''(b)|^q\right)\int_0^1 t\varphi(t)\,dt\right\}^{\frac{1}{q}}\right],$$

where

$$B(\alpha,\lambda,p) = \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\left\{\Gamma(1+p)\,\Gamma\left(\tfrac{1+p+\alpha}{\alpha}\right)\ \left({}_2F_1\left(1,1+p,2+p+\tfrac{1+p}{\alpha},1\right)\right)\right.$$
$$\left.+\beta\left(1+p,-\tfrac{1+p+\alpha p}{\alpha}\right) - \beta\left(\lambda,1+p,-\tfrac{1+p+\alpha p}{\alpha}\right)\right\},$$

also, for $0<b<c$ and $|z|<1$, ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b,c,z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt.$$

Proof. By using Lemma 1.1 and the Hölder inequality, we have the below inequality

$$|S_f(x,\lambda,\alpha,t,\varphi;a,b)|$$
$$\leq \tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}$$
$$+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1 |t(\lambda-t^\alpha)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}} \tag{2.6}$$
$$=\left(\int_0^1 |t(\lambda-t^\alpha)|^p\right)^{\frac{1}{p}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left(\int_0^1 |f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left(\int_0^1 |f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right].$$

Since $|f''|$ is $\varphi-$convex on $[a,b]$, we have

$$\begin{aligned} \int_0^1 \left|f''\left(tx+(1-t)\,a\right)\right|^q dt &\leq \int_0^1 t\varphi\left(t\right)\left|f''\left(x\right)\right|^q dt \\ &+\int_0^1 \left(1-t\right)\varphi\left(1-t\right)\left|f''\left(a\right)\right|^q dt \\ &= \left(\left|f''\left(x\right)\right|^q+\left|f''\left(a\right)\right|^q\right)\int_0^1 t\varphi\left(t\right)dt, \end{aligned} \tag{2.7}$$

and using same technique, we get

$$\begin{aligned} \int_0^1 \left|f''\left(tx+(1-t)\,b\right)\right|^q dt &\leq \int_0^1 t\varphi\left(t\right)\left|f''\left(x\right)\right|^q dt \\ &+\int_0^1 \left(1-t\right)\varphi\left(1-t\right)\left|f''\left(b\right)\right|^q dt \\ &= \left(\left|f''\left(x\right)\right|^q+\left|f''\left(b\right)\right|^q\right)\int_0^1 t\varphi\left(t\right)dt. \end{aligned} \tag{2.8}$$

On the other hand, we can obtain the following equality;

$$\begin{aligned} B\left(\alpha,\lambda,p\right) &= \int_0^1 \left|t\left(\lambda-t^\alpha\right)\right|^p dt \\ &= \int_0^{\lambda^{\frac{1}{\alpha}}} \left\{t(\lambda-t^\alpha)\right\}^p dt+\int_{\lambda^{\frac{1}{\alpha}}}^1 \left\{t\left(t^\alpha-\lambda\right)\right\}^p dt \\ &= C_1\left(\alpha,\lambda,p\right)+C_2\left(\alpha,\lambda,p\right). \end{aligned} \tag{2.9}$$

By letting $\lambda-t^\alpha=u$ and $t^\alpha=u$, respectively, we have

$$\begin{aligned} C_1\left(\alpha,\lambda,p\right) &= \int_0^{\lambda^{\frac{1}{\alpha}}} \left\{t\left(\lambda-t^\alpha\right)\right\}^p dt \\ &= \frac{1}{\alpha}\int_0^\lambda u^p\left(\lambda-u\right)^{\frac{1+p-\alpha}{\alpha}} du \\ &= \frac{1}{\alpha}\int_0^1 \lambda^p y^p \lambda^{\frac{1+p-\alpha}{\alpha}}\left(1-y\right)^{\frac{1-\alpha+p}{\alpha}}\lambda dy \\ &= \frac{\lambda^{\frac{p\alpha+1+p}{\alpha}}}{\alpha}\int_0^1 y^p\left(1-y\right)^{\frac{1+p}{\alpha}}\left(1-y\right)^{-1}dy \\ &= \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\Gamma\left(1+p\right)\Gamma\left(\tfrac{1+p+\alpha}{\alpha}\right){}_2F_1\left(1,1+p,2+p+\tfrac{1+p}{\alpha},1\right), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} C_2\left(\alpha,\lambda,p\right) &= \int_{\lambda^{\frac{1}{\alpha}}}^1 \left\{t\left(t^\alpha-\lambda\right)\right\}^p dt \\ &= \frac{1}{\alpha}\int_{\lambda^u}^1 \frac{1+p-\alpha}{\alpha}\left(u-\lambda\right)^p du \\ &= \frac{\lambda^{\frac{1+p+\alpha p}{\alpha}}}{\alpha}\left\{\beta\left(1+p,-\tfrac{1+p+\alpha p}{\alpha}\right)-\beta\left(\lambda,1+p,-\tfrac{1+p+\alpha p}{\alpha}\right)\right\}. \end{aligned} \tag{2.11}$$

By substituting (2.7), (2.8), (2.9), (2.10) and (2.11)in (2.6), we get

$$\begin{aligned}&\left|S_f\left(x,\lambda,\alpha,t,\varphi;a,b\right)\right|\\&\leq B^{\frac{1}{p}}\left(\alpha,\lambda,p\right)\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\left(\left|f''\left(x\right)\right|^q+\left|f''\left(a\right)\right|^q\right)\int_0^1 t\varphi\left(t\right)dt\right\}^{\frac{1}{q}}\right.\\&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\left(\left|f''\left(x\right)\right|^q+\left|f''\left(b\right)\right|^q\right)\int_0^1 t\varphi\left(t\right)dt\right\}^{\frac{1}{q}}\right],\end{aligned}$$

thus, the proof is completed. □

Corollary 2.4. *Let $\varphi(t)=1$ in Theorem 2.2, then we get the following inequality for any $x\in[a,b]$, $\lambda\in[0,1]$ and $\alpha>0$;*

$$\begin{aligned}&\left|S_f\left(x,\lambda,\alpha,t,\varphi;a,b\right)\right|\\&\leq\left(\int_0^1\left|t\left(\lambda-t^\alpha\right)\right|^p dt\right)^{\frac{1}{p}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\tfrac{\left(\left|f''(x)\right|^q+\left|f''(a)\right|^q\right)}{2}\right\}^{\frac{1}{q}}\right.\\&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\tfrac{\left(\left|f''(x)\right|^q+\left|f''(b)\right|^q\right)}{2}\right\}^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 2.5. *If we choose $\varphi(t)=1$ and $x=\frac{a+b}{2}$ in Theorem 2.2, we can obtain the corollary 2.6, 2.7, 2.8 in* [13]*, respectively for $\lambda=\frac{1}{3}$, $\lambda=0$, $\lambda=1$.*

Corollary 2.6. *Let $\varphi(t)=t^{s-1}$ in Theorem 2.2, then we obtain*

$$\begin{aligned}&\left|S_f\left(x,\lambda,\alpha,t,\varphi;a,b\right)\right|\\&\leq\left(\int_0^1\left|t\left(\lambda-t^\alpha\right)\right|^p dt\right)^{\frac{1}{p}}\left[\tfrac{(x-a)^{\alpha+2}}{b-a}\left\{\tfrac{\left(\left|f''(x)\right|^q+\left|f''(a)\right|^q\right)}{s+1}\right\}^{\frac{1}{q}}\right.\\&\left.+\tfrac{(b-x)^{\alpha+2}}{b-a}\left\{\tfrac{\left(\left|f''(x)\right|^q+\left|f''(b)\right|^q\right)}{s+1}\right\}^{\frac{1}{q}}\right].\end{aligned}$$

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