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A New Type of Generalized Ernst Numbers

Nazmiye Gönül Bilgin[1*], Ece Gülşah Çolak[1] and Yüksel Soykan[1]

[1] *Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak, Turkey*

[*] *Corresponding author*

Abstract

This study presents the Gaussian generalized Ernst numbers as a new complex recursive number sequence. We also give Binet's formula, Simson's formulas and generating function for this sequence and we touch on Gaussian Ernst and Gaussian Ernst-Lucas numbers. Besides, we establish some identities and matrices associated with these sequence. This study's contribution to the literature is the constructed of an important generalization of generalized Ernst numbers that can be applied to different fields and the establishment of important equations regarding these numbers.

Keywords: *Ernst numbers; Gaussian Ernst numbers; Gaussian Ernst-Lucas numbers; Gaussian generalized Ernst numbers.*

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1. Introduction and Preliminaries

In the literature, there are lots of recent studies on the sequences of Gaussian numbers. Gaussian numbers were defined by Gauss in 1832. A Gaussian integer is showed with $z = a+ib$, $a,b \in \mathbb{Z}$ and the set of them is shown with $\mathbb{Z}[i]$. The norm of this integer is its product with its conjugate, therefore $N(z) = a^2+b^2$, see [10] for details. On the other hand, the concept of Fibonacci numbers have been carried out to the complex sense by Horadam [16] in 1963 and the complex Fibonacci numbers called with Gaussian Fibonacci numbers have been defined by him. Then Jordan [17], in 1965, considered the complex Fibonacci numbers $\{GF_n\}$ and the complex Lucas numbers $\{GL_n\}$ called as Gaussian Lucas numbers by writing $GF_n = F_n + iF_{n-1}$ where $\{F_n\}$ is the Fibonacci sequence and $GL_n = L_n + iL_{n-1}$ where $\{L_n\}$ is the Lucas sequence, respectively. Here, $\{F_n\}$ and $\{L_n\}$ are given with $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ with $F_1 = 1$, $F_0 = 0$ and $L_1 = 1$, $L_0 = 2$ initial values, respectively. Later on, Harman [15] in 1981 and Pethe and Horadam [23] in 1986 presented several properties of Gaussian Fibonacci numbers. Gaussian versions of the other sequences of numbers except Fibonacci were studied later. For example, Aşçı and Gürel [1], in 2013, worked on Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers. Next example is exhibited by Halıcı and Öz [13] in 2016, investigating Gaussian Pell and Gaussian Pell-Lucas numbers. For the complex sequences given with the third order recurrence relation, we can see the work about Gaussian generalized Tribonacci and Tribonacci-Lucas numbers written by Soykan et. al. [28] in 2018. They defined the Gaussian generalized Tribonacci numbers$\{GV_n\}$ by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} \tag{1.1}$$

with the initial conditions

$$GV_0 = V_0 + i(V_2 - V_1 - V_0),\ GV_1 = V_1 + iV_0,\ GV_2 = V_2 + iV_1$$

not all being zero where $\{V_n\}$ is a generalized Tribonacci sequence given with the relation from [24]

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} \tag{1.2}$$

with initial conditions V_0, V_1 and V_2 arbitrary real numbers. (1.1) and (1.2) are third order linear recurrence relations. Equivalently, this Gaussian sequence can be also defined by

$$GV_n = V_n + iV_{n-1}. \tag{1.3}$$

Generalized Tribonacci numbers or (r,s,t)-numbers have been worked by many authors, see [3, 4, 7, 9, 20, 24, 26, 27, 31, 36, 38]. Apart from these, there have been several studies about Gaussian sequences which are defined recursively, see [2, 5, 6, 8, 11, 12, 14, 18, 19, 21, 22, 25, 32, 33, 34, 35, 37], however, considering the Ernst numbers, we see that Gaussian version of this series has not yet been defined. This

Email addresses: **nazmiyegonul@beun.edu.tr (Nazmiye Gönül Bilgin), egsavkli@beun.edu.tr (Ece Gülşah Çolak), yuksel_soykan@hotmail.com (Yüksel Soykan)**

study has been prepared to fill this gap. From this point of view, we establish Gaussian generalized Ernst numbers and construct the Gaussian Ernst and Gaussian Ernst-Lucas numbers in its special situations. Then we present their properties such as Binet's formula, summation formulas, generating function, matrix formulation and identities. First of all, let us give the basic information necessary for the numbers we will define.
A generalized Ernst sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined with the third-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3}. \tag{1.4}$$

The initial values W_0,W_1,W_2 are not all zero. You can refer to [29] for more detailed information on generalized Ernst numbers.

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + W_{-(n-2)} - \frac{1}{2}W_{-(n-3)}$$

is the negative index expansion of $\{W_n\}_{n\geq 0}$ for $n = 1,2,3,....$ So, for all integers n, recurrence (1.4) is valid .
The characteristic equation of $\{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by

$$x^3 - 2x^2 - x + 2 = (x-2)(x+1)(x-1) = 0. \tag{1.5}$$

Here, α,β,γ are the roots of Equation (1.5) and $\alpha = 2$, $\beta = -1, \gamma = 1$. Also, for every integer n, the generalized Ernst numbers $W_n(W_0,W_1,W_2)$ is written in Binet's form with

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{p_1\alpha^n}{3} + \frac{p_2\beta^n}{6} - \frac{p_3\gamma^n}{2} \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} p_1 &= W_2 - W_1(\beta+\gamma) + \beta\gamma W_0 = W_2 - W_0, \\ p_2 &= W_2 - W_1(\alpha+\gamma) + \alpha\gamma W_0 = W_2 - 3W_1 + 2W_0, \\ p_3 &= W_2 - W_1(\alpha+\beta) + \alpha\beta W_0 = W_2 - W_1 - 2W_0. \end{aligned} \tag{1.7}$$

Then,

$$W_n = \frac{-W_0+W_2}{3}\alpha^n + \frac{2W_0-3W_1+W_2}{6}\beta^n - \frac{-2W_0-W_1+W_2}{2}\gamma^n$$

and

$$\begin{aligned} \alpha+\beta+\gamma &= 2, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\ \alpha\beta\gamma &= -2, \\ \alpha-\beta &= 3. \end{aligned}$$

We give two special cases of $\{W_n\}$: $W_n(0,1,2) = E_n$ is the sequence of Ernst numbers and $W_n(3,2,6) = H_n$ is the sequence of Ernst-Lucas numbers are given with the next relations

$$\begin{aligned} E_n &= 2E_{n-1} + E_{n-2} - 2E_{n-3}, \text{where } E_2 = 2, E_1 = 1, E_0 = 0, \\ H_n &= 2H_{n-1} + H_{n-2} - 2H_{n-3}, \text{ where } H_2 = 6,\ H_1 = 2,\ H_0 = 3, \end{aligned}$$

and the expansion to negative subscripts is done by the next description:

$$\begin{aligned} E_{-n} &= \frac{1}{2}E_{-(n-1)} + E_{-(n-2)} - \frac{1}{2}E_{-(n-3)}, \\ H_{-n} &= \frac{1}{2}H_{-(n-1)} + H_{-(n-2)} - \frac{1}{2}H_{-(n-3)}. \end{aligned}$$

On the other hand, since we know the p_1, p_2 and p_3 terms in the Binet's formula of the generalized Ernst numbers from (1.7), then we get the Binet's formula of the sequences of Ernst and Ernst-Lucas numbers as follows:

$$\begin{aligned} E_n &= \frac{2}{3}\alpha^n - \frac{1}{6}\beta^n - \frac{1}{2}, \\ H_n &= \alpha^n + \beta^n + 1. \end{aligned}$$

After giving the necessary basic definitions, we are now ready for the main part.

2. Gaussian Generalized Ernst Numbers

Gaussian generalized Ernst numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 2GW_{n-1} + GW_{n-2} - 2GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$GW_0 = W_0 + i\frac{1}{2}(W_0 + 2W_1 - W_2),\ GW_1 = W_1 + iW_0,\ GW_2 = W_2 + iW_1,$$

not all being zero. The sequence $\{GW_n\}_{n\geq 0}$ is extended to negative subscripts with defining

$$GW_{-n} = \frac{1}{2}GW_{-(n-1)} + GW_{-(n-2)} - \frac{1}{2}GW_{-(n-3)}$$

for $n = 1, 2, 3, ...$ So, for all integers n, (2.1) holds. Here, for $n \geq 0$

$$GW_n = W_n + iW_{n-1} \tag{2.2}$$

and for $n \geq 1$

$$GW_{-n} = W_{-n} + iW_{-n-1}.$$

The first few generalized Gaussian Ernst numbers with negative and positive subscript are given in the next table:

n	GW_n	GW_{-n}
0	$W_0 + i\frac{1}{2}(W_0 + 2W_1 - W_2)$	$W_0 + i\frac{1}{2}(W_0 + 2W_1 - W_2)$
1	$W_1 + iW_0$	$\frac{1}{2}(W_0 + 2W_1 - W_2) + i\frac{1}{4}(5W_0 - W_2)$
2	$W_2 + iW_1$	$\frac{1}{4}(5W_0 - W_2) + i\frac{1}{8}(5W_0 + 8W_1 - 5W_2)$
3	$-2W_0 + W_1 + 2W_2 + iW_2$	$\frac{1}{8}(5W_0 + 8W_1 - 5W_2) + i\frac{1}{16}(21W_0 - 5W_2)$
4	$iW_1 - (4+2i)W_0 + (5+2i)W_2$	$\frac{1}{16}(21W_0 - 5W_2) + i\frac{1}{32}(21W_0 + 32W_1 - 21W_2)$
5	$W_1 - (10+4i)W_0 + (10+5i)W_2$	$\frac{1}{32}(21W_0 + 32W_1 - 21W_2) + i\frac{1}{64}(85W_0 - 21W_2)$
6	$iW_1 - (20+10i)W_0 + (21+10i)W_2$	$\frac{1}{64}(85W_0 - 21W_2) + i\frac{1}{128}(85W_0 + 128W_1 - 85W_2)$
7	$W_1 - (42+20i)W_0 + (42+21i)W_2$	$\frac{1}{128}(85W_0 + 128W_1 - 85W_2) + i\frac{1}{256}(341W_0 - 85W_2)$
8	$iW_1 - (84+42i)W_0 + (85+42i)W_2$	$\frac{1}{256}(341W_0 - 85W_2) + i\frac{1}{512}(341W_0 + 512W_1 - 341W_2)$
9	$W_1 - (170+84i)W_0 + (170+85i)W_2$	$\frac{1}{512}(341W_0 + 512W_1 - 341W_2) + i\frac{1}{1024}(1365W_0 - 341W_2)$
10	$iW_1 - (340+170i)W_0 + (341+170i)W_2$	$\frac{1}{1024}(1365W_0 - 341W_2) + i\frac{1}{2048}(1365W_0 + 2048W_1 - 1365W_2)$
11	$W_1 - (682+340i)W_0 + (682+341i)W_2$	$\frac{1}{2048}(1365W_0 + 2048W_1 - 1365W_2) + i\frac{1}{4096}(5461W_0 - 1365W_2)$
12	$iW_1 - (1364+682i)W_0 + (1365+682i)W_2$	$\frac{1}{4096}(5461W_0 - 1365W_2) + i\frac{1}{8192}(5461W_0 + 8192W_1 - 5461W_2)$
13	$W_1 - (2730+1364i)W_0 + (2730+1365i)W_2$	$\frac{1}{8192}(5461W_0 + 8192W_1 - 5461W_2) + i\frac{1}{16384}(21\,845W_0 - 5461W_2)$

Table 1: Some values of the generalized Gaussian Ernst numbers with negative and positive subscripts

We give two special cases of GW_n : $GW_n(0, 1, 2+i) = GE_n$ and $GW_n(3 + \frac{1}{2}i, 2+3i, 6+2i) = GH_n$ are the sequences of Gaussian Ernst and Gaussian Ernst-Lucas numbers. Gaussian Ernst numbers are defined with

$$GE_n = 2GE_{n-1} + GE_{n-2} - 2GE_{n-3} \tag{2.3}$$

with the initial conditions

$$GE_0 = 0, GE_1 = 1, GE_2 = 2 + i$$

and Gaussian Ernst-Lucas numbers are defined with

$$GH_n = 2GH_{n-1} + GH_{n-2} - 2GH_{n-3} \tag{2.4}$$

with the initial conditions

$$GH_0 = 3 + \frac{1}{2}i, GH_1 = 2 + 3i, GH_2 = 6 + 2i.$$

Note that for $n \geq 1$

$$GE_{-n} = E_{-n} + iE_{-n-1}$$

and

$$GH_{-n} = H_{-n} + iH_{-n-1}.$$

In Table 2, a few values of Gaussian Ernst and Gaussian Ernst-Lucas sequence are listed.
Next, for Gaussian generalized Ernst numbers, we give Binet's formula which is a way of writing the terms of the sequence in terms of the roots of the characteristic equation (1.5).

n	GE_n	GE_{-n}	GH_n	GH_{-n}
0	0		$3+\frac{1}{2}i$	
1	1	$-\frac{1}{2}i$	$2+3i$	$\frac{1}{2}+\frac{9}{4}i$
2	$2+i$	$-\frac{1}{2}-\frac{1}{4}i$	$6+2i$	$\frac{9}{4}+\frac{1}{8}i$
3	$5+2i$	$-\frac{1}{4}-\frac{5}{8}i$	$8+6i$	$\frac{1}{8}+\frac{33}{16}i$
4	$10+5i$	$-\frac{5}{8}-\frac{5}{16}i$	$18+8i$	$\frac{33}{16}+\frac{1}{32}i$
5	$21+10i$	$-\frac{5}{16}-\frac{21}{32}i$	$32+18i$	$\frac{1}{32}+\frac{129}{64}i$
6	$42+21i$	$-\frac{21}{32}-\frac{21}{64}i$	$66+32i$	$\frac{129}{64}+\frac{1}{128}i$
7	$85+42i$	$-\frac{21}{64}-\frac{85}{128}i$	$128+66i$	$\frac{1}{128}+\frac{513}{256}i$
8	$170+85i$	$-\frac{85}{128}-\frac{85}{256}i$	$258+128i$	$\frac{513}{256}+\frac{1}{512}i$
9	$341+170i$	$-\frac{85}{256}-\frac{341}{512}i$	$512+258i$	$\frac{1}{512}+\frac{2049}{1024}i$
10	$682+341i$	$-\frac{341}{512}-\frac{341}{1024}i$	$1026+512i$	$\frac{2049}{1024}+\frac{1}{2048}i$
11	$1365+682i$	$-\frac{341}{1024}-\frac{1365}{2048}i$	$2048+1026i$	$\frac{1}{2048}+\frac{8193}{4096}i$
12	$2730+1365i$	$-\frac{1365}{2048}-\frac{341}{1024}i$	$4098+2048i$	$\frac{8193}{4096}+\frac{1}{8192}i$
13	$5461+2730i$	$-\frac{1365}{4096}-\frac{5461}{8192}i$	$8192+4098i$	$\frac{1}{8192}+\frac{32769}{16384}i$
14	$10922+5461i$	$-\frac{5461}{8192}-\frac{5461}{16384}i$	$16386+8192i$	$\frac{32769}{16384}+\frac{1}{32768}i$
15	$21845+10922i$	$-\frac{5461}{16384}-\frac{21845}{32768}i$	$32768+16386i$	$\frac{1}{32768}+\frac{131073}{65536}i$

Table 2: Some positive and negative subscript values of Gaussian Ernst and Gaussian Ernst-Lucas numbers

Theorem 2.1. *The Binet's formula with regards to Gaussian generalized Ernst numbers is*

$$GW_n = \left(\frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\right)+i\left(\frac{p_1\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{p_2\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{p_3\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}\right)$$

where p_1, p_2 and p_3 are as in (1.7).

Proof. The proof is obtained from (1.6) and (1.7). □

From Theorem 2.1, the next results are obtained as specific situations:

Corollary 2.2. *The Binet's formulas of the Gaussian Ernst and Gaussian Ernst-Lucas numbers are given by*

$$GE_n = \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right)+i\left(\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}\right)$$

and

$$GH_n = (\beta^n+\alpha^n+\gamma^n)+i\left(\beta^{n-1}+\alpha^{n-1}+\gamma^{n-1}\right)$$

respectively.

Now, we will give a theorem about the generating function of Gaussian generalized Ernst numbers.

Theorem 2.3. *The generating function of Gaussian generalized Ernst numbers is defined with*

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1-GW_0)x^2}{1-2x-x^2+2x^3}.$$

Proof. Let $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$
be the generating function of Gaussian generalized Ernst numbers. So, we use the definition (2.1) and substract $2xf_{GW_n}(x)$, $x^2 f_{GW_n}(x)$ and $-2x^3 f_{GW_n}(x)$ from $f_{GW_n}(x)$ we have

$$\begin{aligned}
(1-2x-x^2+2x^3)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 2x\sum_{n=0}^{\infty} GW_n x^n - x^2\sum_{n=0}^{\infty} GW_n x^n + 2x^3\sum_{n=0}^{\infty} GW_n x^n \\
&= \sum_{n=0}^{\infty} GW_n x^n - 2\sum_{n=0}^{\infty} GW_n x^{n+1} - \sum_{n=0}^{\infty} GW_n x^{n+2} + 2\sum_{n=0}^{\infty} GW_n x^{n+3} \\
&= \sum_{n=0}^{\infty} GW_n x^n - 2\sum_{n=1}^{\infty} GW_{n-1} x^n - \sum_{n=2}^{\infty} GW_{n-2} x^n + 2\sum_{n=3}^{\infty} GW_{n-3} x^n \\
&= (GW_0+GW_1x+GW_2x^2) - 2(GW_0x+GW_1x^2) - GW_0x^2 + \sum_{n=3}^{\infty}(GW_n-2GW_{n-1}-GW_{n-2}+2GW_{n-3})x^n \\
&= GW_0+GW_1x+GW_2x^2-2GW_0x-2GW_1x^2-GW_0x^2 \\
&= GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1-GW_0)x^2.
\end{aligned}$$

If we rearrange the above equation, we have

$$f_{GW_n}(x) = \frac{GW_0+(GW_1-2GW_0)x+(GW_2-2GW_1-GW_0)x^2}{1-2x-x^2+2x^3}.$$

□

From Theorem 2.3, the following result can be obtained.

Corollary 2.4. *The generating functions of Gaussian Ernst numbers and Gaussian Ernst-Lucas numbers are defined with*

$$f_{GE_n}(x)=\frac{x+ix^2}{1-2x-x^2+2x^3}$$

and

$$f_{GH_n}(x)=\frac{-\left(1+\frac{9}{2}i\right)x^2-(4-2i)x+3+\frac{1}{2}i}{1-2x-x^2+2x^3}$$

respectively.

3. Sum Formulas

In the next theorem, some sum formulas for the Gaussian generalized Ernst numbers are given.

Theorem 3.1. *We have the following sum formulas for Gaussian generalized Ernst numbers:*

(a) $\sum_{k=0}^{n} GW_k = -(\frac{1}{2}n+1)GW_{n+2}+\frac{1}{2}(n+3)GW_{n+1}+(n+3)GW_n+GW_2-\frac{3}{2}GW_1-2GW_0.$

(b) $\sum_{k=0}^{n} GW_{2k} = -\frac{1}{3}(n+1)GW_{2n+2}+\frac{4}{3}(n+2)GW_{2n}+\frac{1}{3}GW_2-\frac{5}{3}GW_0.$

(c) $\sum_{k=0}^{n} GW_{2k+1} = -\frac{2}{3}(n+1)GW_{2n+2}+\frac{1}{3}(3n+7)GW_{2n+1}+\frac{2}{3}(n+1)GW_{2n}+\frac{2}{3}GW_2-\frac{4}{3}GW_1-\frac{2}{3}GW_0.$

(d) $\sum_{k=0}^{n} GW_{-k} = \frac{1}{2}(-(n+1)GW_{-n+2}+nGW_{-n+1}+2(n+1)GW_{-n}+GW_2).$

(e) $\sum_{k=0}^{n} GW_{-2k} = \frac{1}{6}(8(n+1)GW_{-2n-2}-2(n+2)GW_{-2n}+2GW_2).$

(f) $\sum_{k=0}^{n} GW_{-2k+1} = \frac{1}{6}(4(n+2)GW_{-2n-2}+2(3n+2)GW_{-2n-1}-4(n+2)GW_{-2n}+4GW_2+2GW_1-4GW_0).$

Proof. Once we take $r=2, s=1, t=-2$ in [30] we obtain the sum formulas of generalized Ernst numbers. Then, if we modify the sum formulas to the Gaussian version, we get the sum formulas above of Gaussian generalized Ernst numbers. □

As special cases of Theorem 3.1, we have the following corollary.

Corollary 3.2. *For $n\geq 1$, we have the following sum formulas for Gaussian Ernst and Gaussian Ernst-Lucas numbers:*

(a) $\sum_{k=1}^{n} GE_k = \frac{-1}{2}\left((n+3)GE_{n+3}-(n+4)GE_{n+2}-2(n+3)GE_{n+1}-1-2i\right).$

(b) $\sum_{k=1}^{n} GE_{2k+1} = \frac{1}{6}(6n+20)GE_{2n+1}+\frac{2}{3}(n+2)GE_{2n}-\frac{2}{3}(n+2)GE_{2n+2}+\frac{1}{3}+\frac{4}{3}i.$

(c) $\sum_{k=1}^{n} GE_{2k} = \frac{4}{3}GE_{2n}(n+3)-\frac{1}{3}(n+2)GE_{2n+2}+\frac{2}{3}(2+i).$

(d) $\sum_{k=1}^{n} GH_k = GH_{n+1}(n+3)+\frac{1}{2}(n+4)GH_{n+2}-\frac{1}{2}(n+3)GH_{n+3}-3-\frac{7}{2}i.$

(e) $\sum_{k=1}^{n} GH_{2k+1} = \frac{1}{6}(6n+20)GH_{2n+1}+\frac{2}{3}(n+2)GH_{2n}-\frac{2}{3}(n+2)GH_{2n+2}-\frac{2}{3}-5i.$

(f) $\sum_{k=1}^{n} GH_{2k} = \frac{4}{3}GH_{2n}(n+3)-\frac{1}{3}GH_{2n+2}(n+2)-5-\frac{1}{6}i.$

4. Some Identities Between Gaussian Ernst and Gaussian Ernst-Lucas Numbers

In this section, we give some identities with regards to Gaussian Ernst and Gaussian Ernst-Lucas numbers.

Lemma 4.1. *For all integers n, a few basic relationships between $\{GE_n\}$ and $\{GH_n\}$ can be defined as follows:*

(a) $GE_n=\frac{4}{9}GH_{n+2}-\frac{1}{6}GH_{n+1}-\frac{7}{9}GH_n.$

(b) $GE_n=\frac{13}{18}GH_{n+1}-\frac{1}{3}GH_n-\frac{8}{9}GH_{n-1}.$

(c) $GE_n=\frac{10}{9}GH_n-\frac{1}{6}GH_{n-1}-\frac{13}{9}GH_{n-2}.$

(d) $GH_n=\frac{1}{2}GE_{n+2}+2GE_{n+1}-\frac{9}{2}GE_n.$

(e) $GH_n=3GE_{n+1}-4GE_n-GE_{n-1}.$

(f) $GH_n=2GE_n+2GE_{n-1}-6GE_{n-2}.$

Proof. Now, we can show (a) by writing

$$GE_n = aGH_{n+2} + bGH_{n+1} + cGH_n$$

and solving the system of equations

$$\begin{aligned} GE_0 &= aGH_2 + bGH_1 + cGH_0 \\ GE_1 &= aGH_3 + bGH_2 + cGH_1 \\ GE_2 &= aGH_4 + bGH_3 + cGH_2. \end{aligned}$$

So, we determine $a = \frac{4}{9}, b = -\frac{1}{6}, c = -\frac{7}{9}$. Others can be obtained in a similar way. □

5. Special Identities

In this section, we will exhibit some identities for Gaussian generalized Ernst numbers, Gaussian Ernst numbers and Gaussian Ernst-Lucas numbers such as Simson's identity, Catalan's identity, d'Ocagne identity, Gelin-Cesàro identity and Melham identity. Firstly, we give the Simson's formula as follows.

Theorem 5.1. *(Simson's identity)For all integers n, we have*

$$\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} = (-2)^{n-2}(2+i)(W_0 - W_2)(-2W_0 - W_1 + W_2)(2W_0 - 3W_1 + W_2).$$

Proof. We use the strong induction over n. For $n = 1$ it is true. Now, let the identity be true for $n = 1, 2, ..., k$. Then we have the next identity:

$$\begin{vmatrix} GW_{k+2} & GW_{k+1} & GW_k \\ GW_{k+1} & GW_k & GW_{k-1} \\ GW_k & GW_{k-1} & GW_{k-2} \end{vmatrix} = (-2)^{k-2}(2+i)(W_0 - W_2)(-2W_0 - W_1 + W_2)(2W_0 - 3W_1 + W_2).$$

Now, we must show that the identity is also correct for $n = k+1$.We get the required result as follows using the recurrence of the sequence, induction hypothesis and the determinant.

$$\begin{aligned} \begin{vmatrix} GW_{k+3} & GW_{k+2} & GW_{k+1} \\ GW_{k+2} & GW_{k+1} & GW_k \\ GW_{k+1} & GW_k & GW_{k-1} \end{vmatrix} &= \begin{vmatrix} 2GW_{k+2} + GW_{k+1} - 2GW_k & GW_{k+2} & GW_{k+1} \\ 2GW_{k+1} + GW_k - 2GW_{k-1} & GW_{k+1} & GW_k \\ 2GW_k + GW_{k-1} - 2GW_{k-2} & GW_k & GW_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} 2GW_{k+2} & GW_{k+2} & GW_{k+1} \\ 2GW_{k+1} & GW_{k+1} & GW_k \\ 2GW_k & GW_k & GW_{k-1} \end{vmatrix} + \begin{vmatrix} GW_{k+1} & GW_{k+2} & GW_{k+1} \\ GW_k & GW_{k+1} & GW_k \\ GW_{k-1} & GW_k & GW_{k-1} \end{vmatrix} + \begin{vmatrix} -2GW_k & GW_{k+2} & GW_{k+1} \\ -2GW_{k-1} & GW_{k+1} & GW_k \\ -2GW_{k-2} & GW_k & GW_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} -2GW_k & GW_{k+2} & GW_{k+1} \\ -2GW_{k-1} & GW_{k+1} & GW_k \\ -2GW_{k-2} & GW_k & GW_{k-1} \end{vmatrix} = -2\begin{vmatrix} GW_k & GW_{k+2} & GW_{k+1} \\ GW_{k-1} & GW_{k+1} & GW_k \\ GW_{k-2} & GW_k & GW_{k-1} \end{vmatrix} \\ &= -2\begin{vmatrix} GW_{k+2} & GW_{k+1} & GW_k \\ GW_{k+1} & GW_k & GW_{k-1} \\ GW_k & GW_{k-1} & GW_{k-2} \end{vmatrix} \\ &= (-2)^{k-1}(2+i)(W_0 - W_2)(-2W_0 - W_1 + W_2)(2W_0 - 3W_1 + W_2). \end{aligned}$$

So, the identity is also correct for $n = k+1$. □

We can immediately reach the next corollary using Theorem 5.1.

Corollary 5.2. *For all integers n, the Simson's identity of Gaussian Ernst and Gaussian Ernst-Lucas numbers are given as*

$$\begin{vmatrix} GE_{n+2} & GE_{n+1} & GE_n \\ GE_{n+1} & GE_n & GE_{n-1} \\ GE_n & GE_{n-1} & GE_{n-2} \end{vmatrix} = (-2)^{n-1}(2+i)$$

and

$$\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{vmatrix} = 9(-2)^n(2+i)$$

respectively.

The next identity is Catalan's identity which will be demonstrated for only Gaussian Ernst and Gaussin Ernst-Lucas numbers.

Theorem 5.3. *(Catalan's identity) Let n and m be any integer. In this case, the following identities for the Gaussian Ernst numbers and Gaussian Ernst-Lucas numbers, respectively, hold:*

$$\begin{aligned} GE_{n+m}GE_{n-m}-GE_n^2 &= \frac{1}{3}(-1)^n\left(2^n\left(\left(1-\frac{1}{3}i\right)\right)+\left(\frac{1}{2}(-1)^{-m}-1\right)\right)+\left(\frac{1}{3}+i\right)2^n\left(1-\frac{1}{2}\left(2^m+2^{-m}\right)\right) \\ &+\frac{1}{6}(-1)^{m+n}\left(1-2^n\left(1-\frac{1}{3}i\right)\left(2^{-m}+2^m\right)\right) \end{aligned}$$

and

$$\begin{aligned} GH_{n+m}GH_{n-m}-GH_n^2 &= \left(\frac{3}{2}-\frac{1}{2}i\right)(-1)^n2^n\left((-1)^m2^{-m}+(-1)^{-m}2^m\right)+2(-1)^n\left((-1)^m+(-1)^{-m}-2\right) \\ &+2^n\left(\frac{1}{2}(1+3i)\left(2^{-m}+2^m\right)-\left((3-i)(-1)^n+(1+3i)\right)\right). \end{aligned}$$

Cassini's identity is a special case of Catalan's identity. Therefore, the following corollary can be given as a result of Theorem 5.3.

Corollary 5.4. *(Cassini's identity)Let n be any integer. Then, for the Gaussian Ernst and Gaussian Ernst-Lucas numbers the following identities are true;*

$$GE_{n+1}GE_{n-1}-GE_n^2=2^{n-2}\left((3-i)(-1)^n-\left(\frac{1}{3}+i\right)\right)-\frac{2}{3}(-1)^n$$

and

$$GH_{n+1}GH_{n-1}-GH_n^2=2^{n-2}\left((1+3i)-9(3-i)(-1)^n\right)-8(-1)^n.$$

The last identities which we will present are the identities for Gaussian Ernst and Gaussian Ernst-Lucas numbers as follows.

Theorem 5.5. *Let n and m be any integers. Then for the Gaussian Ernst and Gaussian Ernst-Lucas numbers the following identities are true:*

(a) *(d'Ocagne identity)*

$$GE_{m+1}GE_n-GE_mGE_{n+1} = \left(\frac{1}{2}-\frac{1}{6}i\right)\left((-1)^m2^n-(-1)^n2^m\right)+\frac{1}{3}\left((-1)^n-(-1)^m\right)+\left(\frac{1}{6}+\frac{1}{2}i\right)\left(2^n-2^m\right)$$

and

$$GH_{m+1}GH_n-GH_mGH_{n+1} = \left(\frac{9}{2}-\frac{3}{2}i\right)\left((-1)^n2^m-(-1)^m2^n\right)+4\left((-1)^n-(-1)^m\right)+\left(\frac{1}{2}+\frac{3}{2}i\right)\left(2^m-2^n\right).$$

(b) *(Gelin-Cesàro identity)*

$$\begin{aligned} GE_{n+2}GE_{n+1}GE_{n-1}GE_{n-2}-GE_n^4 &= 2^n\left(\frac{29}{24}\left(1-\frac{1}{3}i\right)-2^n\left(\frac{11}{6}-\frac{11}{8}i\right)+\frac{11}{24}2^{2n}\left(1-\frac{13}{9}i\right)\right) \\ &+2^n(-1)^n\left(\frac{251}{216}\left(\frac{1}{3}+i\right)-\frac{2}{3}\left(1+\frac{4}{3}i\right)2^n+\frac{1}{8}\left(\frac{13}{9}+i\right)2^{2n}\right) \\ &-(-1)^n\left(\frac{1}{3}i-\frac{1}{27}\left(i-\left(\frac{1}{3}+i\right)2^n\right)\right)-\frac{2}{9} \end{aligned}$$

and

$$\begin{aligned} GH_{n+2}GH_{n+1}GH_{n-1}GH_{n-2}-GH_n^4 &= -2^{3n}\left(\left(\frac{117}{32}+\frac{81}{32}i\right)(-1)^n+\left(\frac{99}{32}-\frac{143}{32}i\right)\right) \\ &-2^n\left(\left(\frac{129}{2}-\frac{43}{2}i\right)+(-1)^n\left((18+24i)2^n+\left(\frac{23}{2}+\frac{69}{2}i\right)\right)\right) \\ &-2^{2n}\left(\frac{85}{2}-\frac{255}{8}i\right)-32. \end{aligned}$$

(c) *(Melham identity)*

$$\begin{aligned} GE_{n+1}GE_{n+2}GE_{n+6}-GE_{n+3}^3 &= \frac{1}{9}(1-i)-2^n\frac{28}{3}(1-2i)+2^{2n}\frac{100}{9}(1-7i)-\frac{1}{3}(-1)^n(1+i) \\ &+2^n\frac{26}{9}(-1)^n(2+i)-2^{2n}\frac{4}{3}(-1)^n(7+i) \end{aligned}$$

and

$$\begin{aligned} GH_{n+1}GH_{n+2}GH_{n+6}-GH_{n+3}^3 &= 8\left((1+i)(-1)^n-1+i\right)+2^{n+2}\left(13(2+i)(-1)^n-34(1-2i)\right) \\ &+2^{2n+1}\left(9(7+i)(-1)^n-25(1-7i)\right). \end{aligned}$$

6. Matrix Formulation of GW_n

We get some important identities for the special sequences with the matrix method. The third-order square matrix C is defined as:

$$C = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det C = -2$. Note that

$$M_n = \begin{pmatrix} E_{n+1} & E_n - 2E_{n-1} & -2E_n \\ E_n & E_{n-1} - 2E_{n-2} & -2E_{n-1} \\ E_{n-1} & E_{n-2} - 2E_{n-3} & -2E_{n-2} \end{pmatrix} \tag{6.1}$$

and

$$M_n = C^n,$$

i.e.,

$$C^n = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} E_{n+1} & E_n - 2E_{n-1} & -2E_n \\ E_n & E_{n-1} - 2E_{n-2} & -2E_{n-1} \\ E_{n-1} & E_{n-2} - 2E_{n-3} & -2E_{n-2} \end{pmatrix}.$$

Also we have

$$C^n = \frac{1}{18}\begin{pmatrix} 8H_{n+3} - 3H_{n+2} - 14H_{n+1} & -6H_{n+2} + 9H_{n+1} + 6H_n & -16H_{n+2} + 6H_{n+1} + 28H_n \\ 8H_{n+2} - 3H_{n+1} - 14H_n & -6H_{n+1} + 9H_n + 6H_{n-1} & -16H_{n+1} + 6H_n + 28H_{n-1} \\ 8H_{n+1} - 3H_n - 14H_{n-1} & -6H_n + 9H_{n-1} + 6H_{n-2} & -16H_n + 6H_{n-1} + 28H_{n-2} \end{pmatrix}.$$

Consider the matrices Y_E, N_E defined by as follows:

$$\begin{aligned} Y_E &= \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ -\frac{1}{2}i & i & 1+\frac{1}{2}i \end{pmatrix}, \\ N_E &= \begin{pmatrix} GE_{n+1} & GE_n - 2GE_{n-1} & -2GE_n \\ GE_n & GE_{n-1} - 2GE_{n-2} & -2GE_{n-1} \\ GE_{n-1} & GE_{n-2} - 2GE_{n-3} & -2GE_{n-2} \end{pmatrix}. \end{aligned}$$

The following theorem shows the relationships between C^n, Y_E, and N_E.

Theorem 6.1. *For every integers n, we get*

$$C^n Y_E = N_E.$$

Proof. It is clear from the matrix multiplication. □

We define matrices Y_H, N_H by as follows:

$$\begin{aligned} Y_H &= \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & i \\ -\frac{1}{2}i & i & \frac{1}{2}i+1 \end{pmatrix} \\ N_H &= \frac{1}{18}\begin{pmatrix} 8GH_{n+3} - 3GH_{n+2} - 14GH_{n+1} & -6GH_{n+2} + 9GH_{n+1} + 6GH_n & -16GH_{n+2} + 6GH_{n+1} + 28GH_n \\ 8GH_{n+2} - 3GH_{n+1} - 14GH_n & -6GH_{n+1} + 9GH_n + 6GH_{n-1} & -16GH_{n+1} + 6GH_n + 28GH_{n-1} \\ 8GH_{n+1} - 3GH_n - 14GH_{n-1} & -6GH_n + 9GH_{n-1} + 6GH_{n-2} & -16GH_n + 6GH_{n-1} + 28GH_{n-2} \end{pmatrix} \end{aligned}$$

The following theorem shows the relationships between C^n, Y_H and N_H.

Theorem 6.2. *For every integers n, we have*

$$C^n Y_H = N_H.$$

Proof. It is clear from the matrix multiplication. □

7. Conclusion

The important properties of the Gaussian version of the generalized Ernst numbers having been newly introduced to the literature are investigated. In this context, it can be evaluated as a considerable source for those who will work with special sequences. In this work, matrices, important equations and formulas have been acquired and they include important results that can be applied to daily life problems.

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