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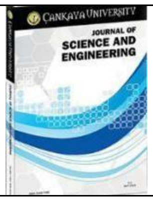

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Extended Hypergeometric Function as a Solution to Unsteady Fluid Flow through Porous Horizontal Channel using SUM Transform

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Keywords

Laplace Transform, SUM transform, Extended hypergeometric function, Fluid flow Injection velocity.

Abstract

The objective of this paper is to obtain solution to unsteady fluid flow through porous horizontal channel with injection and suction velocities using SUM integral transform. The solution is represented in terms of extended special function that contained two Fox-Wright functions in its kernel.

1. Introduction

Special functions are obtained as solution to differential equations, for example, the confluent hypergeometric equation is an important differential equation that arises in optics, electrodynamics, waves, diffusion, fluid flow, string theory, heat transfer, general relativity, graphic design, quantum mechanics and quantum physics, is given in [1] by

$$z\frac{d^2u}{dz^2} + (b - z)\frac{du}{dz} - au = 0, \tag{1}$$

with a and b as constants. The first and second standard solutions of equation (1) using Frobenius methods are given as, respectively:

$$M(a;b;z) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r}\frac{z^r}{r!},$$

and

$$U(a;b;z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a;b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(1+a-b;2-b;z), b \notin \mathbb{Z}. \tag{2}$$

The solution in equation (2) has the following important relation [2, 3]:

$$U(a;b;z) = z^{1-b} U(1+a-b;2-b;z), \tag{3}$$

and

$$U(a;b;z) = z^{-a}\ {}_2F_0(a, 1+a-b; -; -z^{-1}), \tag{4}$$

where ${}_2F_0(a,b;-;z)$ is a generalized hypergeometric function.

Recent in 2023, the following generalized Gauss hypergeometric function defined using two Fox-Wright is presented in [3,4] as

$$\begin{aligned} {}^{\Psi}F_{p,q}^{\omega,\varpi}(a,b;c;z) &= {}^{\Psi}F_{p,q}^{\omega,\varpi}\left[\begin{matrix}(B_i,b_i)_{1,\mu} \\ (D_j,d_j)_{1,\xi}\end{matrix}\middle|\begin{matrix}(E_m,e_m)_{1,\rho} \\ (G_n,g_n)_{1,\varrho}\end{matrix}\middle| a,b;c;z\right] \\ &= \sum_{r=0}^{\infty} (a)_r \frac{{}^{\Psi}B_{p,q}^{\omega,\varpi}(c+r,c-b)}{B(c,c-b)}\frac{z^r}{r!}, \end{aligned} \tag{5}$$

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Where $Re(c) > Re(b), |z| < 1$ and

$$ {}^{\Psi}B_{p,q}^{\omega,\varpi}(x,y) = {}^{\Psi}B_{p,q}^{\omega,\varpi}\left[\begin{matrix}(B_i,b_i)_{1,\mu}\\(D_j,d_j)_{1,\xi}\end{matrix}\middle|\begin{matrix}(E_m,e_m)_{1,\rho}\\(G_n,g_n)_{1,\varrho}\end{matrix}\middle| x,y\right] $$

$$ = \int_0^1 t^{x-1}(1-t)^{y-1} \ {}_{\mu}\Psi_{\xi}\left(-\frac{p}{t^{\omega}}\right) \ {}_{\rho}\Psi_{\varrho}\left(-\frac{q}{(1-t)^{\varpi}}\right)dt, \quad (6) $$

and ${}_{\mu}\Psi_{\xi}(z)$ is the Fox-Wright function [5].
The following integral transform was studied in [6, 7]

$$ S_a\{f(t)\}_{(s)} = \frac{1}{s^r}\int_0^{\infty} f(t)a^{-st}dt = F_a(s), \quad (7) $$

where $t \geq 0, r \in \mathbb{Z}, a > 0, n_1 \leq s \leq n_2, n_1, n_2 > 0$ and $f(t)$ is sectionally continuous and exponential order.

Definition 1: [7] If a function $f(t)$ is m-times continuously differentiable on $[0,\infty)$ and of exponential order $\partial\ (>0)$, then $S_a\{f'(t)\}_{(s)}, S_a\{f''(t)\}_{(s)}, \cdots, S_a\{f^{(m)}(t)\}_{(s)}$ exist for $Re(s) > \frac{\partial}{log(a)}$ and

$$ S_a\{f^{(m)}(t)\}_{(s)} = [slog(a)]^m S_a\{f(t)\}_{(s)} - \frac{1}{s^r}\sum_{w=0}^{(m-1)}[slog(a)]^{m-w-1}f^{(m-1)-w}(0). \quad (8) $$

Definition 2: [6] Suppose a function $u(x,t)$ is defined for $x \in [a,b], t > 0$ and $S_a\{u(x,t)\}_{(s)} = U_a(x,s)$, then

$$ S_a\left\{\frac{\partial u(y,t)}{\partial t}\right\}_{(s)} = -\frac{U(y,0)}{s^r} + [slog(a)]U_a(y,s), \quad (9) $$

$$ S_a\left\{\frac{\partial^2 u(y,t)}{\partial t^2}\right\}_{(s)} = -\frac{1}{s^r}\frac{\partial U(y,0)}{\partial t} - [slog(a)]\frac{U(y,0)}{s^r} + [slog(a)]^2 U_a(y,s), \quad (10) $$

$$ S_a\left\{\frac{\partial u(y,t)}{\partial y}\right\}_{(s)} = \frac{dU_a(y,s)}{dy}, \quad (11) $$

$$ S_a\left\{\frac{\partial^2 u(y,t)}{\partial y^2}\right\}_{(s)} = \frac{d^2U_a(y,s)}{d^2y}. \quad (12) $$

Many of the known results in the literature utilized Laplace integral transform to solve problems related to fluid flows, see for example [8-13]. In this article the SUM integral transform will be utilized to obtain solution of unsteady fluid flow through a horizontal channel with injection and suction velocities.

2. Main Results

Theorem 3: The following result holds:

$$ S_a\left\{t^{-\frac{3}{2}}\exp\left(-\frac{\varphi^2}{4t}\right)\right\}_{(s)} = \frac{2\sqrt{\pi}}{\varphi s^n}\exp\left(-\varphi\sqrt{slog(a)}\right), \quad (13) $$

where φ is a constant denoting the diffusion coefficient, this function occurs frequently in diffusion problems.

Proof: Using the SUM transform in equation (7), we get

$$ S_a\left\{t^{-\frac{3}{2}}\exp\left(-\frac{\varphi^2}{4t}\right)\right\}_{(s)} = \frac{1}{s^n}\int_{t=0}^{\infty} t^{-\frac{3}{2}}\exp\left[-\left(stlog(a) + \frac{\varphi^2}{4t}\right)\right]dt. \quad (14) $$

Letting $stlog(a) = \phi^2$ in equation (14), we have

$$ S_a\left\{t^{-\frac{3}{2}}\exp\left(-\frac{\varphi^2}{4t}\right)\right\}_{(s)} = \frac{2\sqrt{slog(a)}}{s^n}\int_0^{\infty}\exp\left[-\left(\phi^2 + \frac{\varphi^2[slog(a)]}{4\phi^2}\right)\right]\frac{d\phi}{\phi^2}. \quad (15) $$

Substituting $\frac{\varphi^2[slog(a)]}{4} = \eta^2$ in (15), we have

$$ S_a\left\{t^{-\frac{3}{2}}\exp\left(-\frac{\varphi^2}{4t}\right)\right\}_{(s)} = \frac{2\sqrt{slog(a)}}{s^n}\int_0^{\infty}\exp\left[-\left(\phi^2 + \frac{\eta^2}{\phi^2}\right)\right]\frac{d\phi}{\phi^2}. \quad (16) $$

Putting $\frac{\eta}{\phi} = \omega$ in (16), we obtain

$$ S_a\left\{t^{-\frac{3}{2}}\exp\left(-\frac{\varphi^2}{4t}\right)\right\}_{(s)} = \frac{2\sqrt{slog(a)}}{\eta s^n}\Psi, \quad (17) $$

where

$$ \Psi = \int_0^{\infty}\exp\left[-\left(\frac{\phi^2}{\omega^2} + \omega^2\right)\right]d\omega. \quad (18) $$

Also

$$ \frac{d\Psi}{d\phi} = -2\phi\int_0^{\infty}\exp\left[-\left(\frac{\phi^2}{\omega^2} + \omega^2\right)\right]\frac{d\omega}{\omega^2}. \quad (19) $$

Similarly, putting $\frac{\phi}{\omega}=\eta$ in equation (19) leads us to

$$\frac{d\Psi}{d\phi}=-2\int_0^{\infty}\exp\left[-\left(\frac{\phi^2}{\eta^2}+\eta^2\right)\right]d\eta=-2\Psi. \quad (20)$$

Solving equation (19) using elementary method, we have

$$\Psi(\eta)=\mathrm{A}\exp(-2\eta). \quad (21)$$

And since $\Psi(0)=\frac{\sqrt{\pi}}{2}$, we have

$$\Psi(\eta)=\frac{\sqrt{\pi}}{2}\exp(-2\eta). \quad (22)$$

Putting equation (22) into (17), we obtained the required result in (13).

Corollary 4: If $a=e$ and $r=0$ it reduces in equation (13) it reduces to the following well-known result [14]:

$$L\left\{t^{-\frac{3}{2}}\exp\left(-\frac{\varphi^2}{4t}\right)\right\}_{(s)}=\frac{2\sqrt{\pi}}{\varphi}\exp\left(-\varphi\sqrt{s}\right). \quad (23)$$

2.1. Fluid Flow through Porous Horizontal Channel with Injection Viscosity

The If x is the distance along a two-dimensional porous plate with velocity component u and y is the distance normal to the plate with velocity component v. Assuming that accelerating fluid particles are added from the boundary layer through the porous plate by injection. Also, assuming that the velocity component u will be equal to zero at the surface of the plate for all time, approach a function $f(t)$ as $y\to\infty$, and equal $f(0)$ at time $t=0$, then the following system described the model:

$$\frac{\partial u}{\partial t}+v_I\frac{\partial u}{\partial y}=\frac{\partial f}{\partial t}+v\frac{\partial^2 u}{\partial y^2} \quad 0<y<\infty,\quad t>0 \quad (24)$$
$$u(y,0)=f(0),\ 0<y<\infty$$
$$u(0,t)=0, u(y,t)=f(t) \text{ as } y\to\infty,\ t>0$$

where v_I is the constant injection velocity and v is the kinematic viscous of the fluid.

Applying the SUM transform to (24), utilizing (25)-(28), we have

$$S_a\left\{\frac{\partial u(y,t)}{\partial t}\right\}_{(s)}=\frac{1}{s^r}\int_0^{\infty}\frac{\partial u(y,t)}{\partial t}a^{-st}dt$$
$$=\frac{1}{s^r}\left\{\lim_{k\to\infty}\left(\left[u(y,t)a^{-st}\right]_{t=0}^{k}+[slog(a)]\int_0^k u(y,t)a^{-st}dt\right)\right\}$$
$$=[slog(a)]U_a(y,s)-\frac{U(x,0)}{s^r}=[slog(a)]U_a(x,s)-\frac{f(0)}{s^r}, \quad (25)$$

$$S_a\left\{\frac{\partial u(x,ts)}{\partial y}\right\}_{(s)}=\frac{dU_a(y,s)}{dy}, \quad (26)$$

$$S_a\left\{\frac{\partial^2 u(y,t)}{\partial^2 y}\right\}_{(s)}=\frac{d^2U_a(y,s)}{d^2 y} \quad (27)$$

$$S_a\left\{\frac{\partial f(t)}{\partial t}\right\}_{(s)}=[slog(a)]F_a(s)-\frac{f(0)}{s^r} \quad (28)$$

We have

$$v\frac{d^2U_a(y,s)}{dy^2}-v_I\frac{dU_a(y,s)}{dy}-[slog(a)]U_a(y,s)=-[slog(a)]F_a(s), \quad (29)$$

with the condition: $U_a(0,s)=0,\ U_a(y,s)\to F_a(s)$ as $y\to\infty$.

The solution of $U_a(y,s)$ with the boundary condition is

$$U_a(y,s)=F_a(s)\left\{1-\exp\left(-\left[\frac{1}{2}\sqrt{\beta+\delta[slog(a)]}-\alpha\right]y\right)\right\}. \quad (30)$$

where $\alpha=\frac{v_I}{2v}$, $\beta=\left(\frac{v_s}{v}\right)^2$ and $\gamma=\frac{4}{v}$.

Taking inverse SUM transform of equation (30) using (13) and the convolution theory, we have

$$u(y,t)=f(t)-\frac{y}{4}\sqrt{\frac{\gamma}{\pi}}\exp(\alpha y)\left\{\int_0^{\mathrm{t}}\mathrm{f(t-u)u}^{-\frac{3}{2}}\exp\left(-\left[\frac{\gamma y^2}{16u}-\frac{\beta}{\gamma}u\right]\right)du\right\}. \quad (31)$$

By the Maclaurin series expansion of $\exp\left(\frac{\beta}{\gamma}u\right)$ and putting $u=\frac{t}{p+1}$ *and* $f(t)=u_0$, then equation (31) can be expressed in term of extended hypergeometric function (5) as

$$u(y,t) = u_0\left\{1 - \frac{y}{4}\sqrt{\frac{t}{\gamma\pi}}\exp\left(\alpha y - \frac{\gamma y^2}{16t}\right)\sum_{r=0}^{\infty}\frac{\left(\frac{\beta}{\gamma}t\right)^r}{r!}\ {}^{\Psi}F_{0,0}^{1,1}\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle|\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| r+\frac{1}{2},1;-;-\frac{16t}{\gamma y^2}\right]\right\}.$$

2.2. Fluid Flow through Porous Horizontal Channel with Suction Viscosity

If x is the distance along a two-dimensional porous plate with velocity component u and y is the distance normal to the plate with velocity component v. Assuming that decelerating fluid particles are removed from the boundary layer through the porous plate by suction. Also, assuming that the velocity component u will be equal zero at the surface of the plate for all time, approach a function $f(t)$ as $y \to \infty$, and equal $f(0)$ at time $t = 0$, then the following system described the model:

$$\frac{\partial u}{\partial t} - v_s\frac{\partial u}{\partial y} = \frac{\partial f}{\partial t} + v\frac{\partial^2 u}{\partial y^2} \quad 0 < y < \infty, \quad t > 0 \tag{32}$$

$$u(y,0) = f(0),\ \ 0 < y < \infty$$

$$u(0,t) = 0, u(y,t) = f(t) \text{ as } y \to \infty,\ \ t > 0$$

where v_s is the constant suction velocity and v is the kinematic viscous of the fluid.

Applying the SUM transform to equation (32), utilizing (25)-(28) and after some simplifications, we have

$$v\frac{d^2u_a(y,s)}{dy^2} + v_s\frac{du_a(y,s)}{dy} - [slog(a)]u_a(y,s) = -[slog(a)]F_a(s). \tag{33}$$

Solving equation (33), using undetermined coefficient method and applying the boundary condition, we have

$$u_a(y,s) = F_a(s)\left\{1 - \exp\left(-\left[\alpha + \frac{1}{2}\sqrt{\beta + \delta[slog(a)]}\right]y\right)\right\}, \tag{34}$$

Where $\alpha = \frac{v_s}{2v}, \beta = \left(\frac{v_s}{v}\right)^2$ and $\gamma = \frac{4}{v}$.

Then, taking inverse SUM transform on equation (34) using (13) and the convolution theory, we have

$$u(y,t) = f(t) - \frac{y}{4}\sqrt{\frac{\gamma}{\pi}}\exp(-\alpha y)\left\{\int_0^t \mathrm{f(t-u)u}^{-\frac{3}{2}}\exp\left(-\left[\frac{\beta}{\gamma}u + \frac{\gamma y^2}{16u}\right]\right)du\right\}. \tag{35}$$

Applying the Maclaurin series expansion of $\exp\left(-\frac{\beta}{\gamma}u\right)$ and putting $u = \frac{t}{p+1}$, then if $f(t) = u_0$, then equation (35) can be expressed in term of the extended hypergeometric function (5) by

$$u(y,t) = u_0\left\{1 - \frac{y}{4}\sqrt{\frac{t}{\gamma\pi}}\exp\left[-\left(\alpha y + \frac{\gamma y^2}{16t}\right)\right]\sum_{r=0}^{\infty}\frac{\left(-\frac{\beta}{\gamma}t\right)^r}{r!}\ {}^{\Psi}F_{0,0}^{1,1}\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle|\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| r+\frac{1}{2},1;-;-\frac{16t}{\gamma y^2}\right]\right\}. \tag{36}$$

3. Conclusions

In this paper, the solutions to unsteady fluid flow problems in a porous horizontal channel with injection and suction velocities are considered. These solutions were obtained utilizing the SUM integral transform method, which is consistent to the existing solutions obtained in the literature using other methods like the Laplace integral transform (refer to [15] for more information). The models in this study can be extended to include the impact of time fractional derivatives using approaches such as Caputo, Caputo-Fabrizio, Atangana-Baleanu and Phrabakar fractional derivatives. This research has the potential to greatly benefit the study of soil water flow in soil science and improve productivity in horizontal well drilling operations in the oil and gas industry.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Muhammad Lawan Kaurangini: Conceptualization, Supervision, Reviewing, and Corrections
Umar Muhammad Abubakar: Methodology, Writing, and Editing

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