Journal of Amasya University the Institute of Sciences and Technology 4(2) (2023) 100-107

JOURNAL OF AMASYA UNIVERSITY THE INSTITUTE OF SCIENCES AND TECHNOLOGY (JAUIST)

ISSN: 2717-8900 https://doi.org/10.54559/jauist.1400386

Statistical Riesz and Nörlund convergence for sequences of fuzzy numbers

Burak Arslan[1,*], **Samira Jalali**[2], **Serdar Enginoğlu**[3]

[1,2,3]Department of Mathematics, Faculty of Science, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye

Abstract — Nuray and Savaş proposed statistical convergence of fuzzy number sequences. Afterward, Tripathy and Baruah presented Riesz and Nörlund convergence for sequences of fuzzy numbers. This paper defines statistical Riesz and Nörlund convergence of fuzzy number sequences. It then shows that if a sequence of fuzzy numbers is convergent, then it is statistical Riesz/Nörlund convergent, but the converse is not always true. Finally, this paper discusses the need for further research.

Keywords: *Statistical convergence, statistical Riesz convergence, statistical Nörlund convergence, sequences of fuzzy numbers*

Subject Classification (2020): 40A35, 46S40

1. Introduction

Thus far, many studies, such as de la Vallée-Poussin, Cesàro, Riesz, and Nörlund convergence [1-3], have been conducted on sequences of fuzzy numbers [4,5]. New convergence types for nonconvergent sequences of fuzzy numbers via Cesàro, Riesz, and Nörlund means have been proposed in these studies. Another useful type of convergence introduced for fuzzy number sequences is statistical convergence [6]. Afterward, statistical convergence and statistical Cesàro and p-Cesàro convergence [7-9] have been investigated. This study defines statistical Riesz and Nörlund convergence for fuzzy number sequences.

Section 2 of the present study provides some basic definitions to be required in the next section. Section 3 defines statistical Riesz and Nörlund convergence of fuzzy number sequences. Moreover, it shows that convergent sequences are statistical Riesz/Nörlund convergent, but the converse is not always correct. Finally, we discuss the need for further research.

2. Preliminaries

This section presents some basic notions to be needed for the following section.

Definition 2.1. A fuzzy set μ over $\mathbb{R}$ is called a fuzzy number if

i. there exists an $x \in \mathbb{R}$ such that $\mu(x) = 1$

ii. $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$

iii. for all $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|x - a| < \delta \Rightarrow \mu(x) - \mu(a) < \varepsilon$

iv. the closure of $\{x \in \mathbb{R}: \mu(x) > 0\}$, denoted by $\overline{\operatorname{supp}(\mu)}$, in the usual topology of $\mathbb{R}$ is compact

*Corresponding Author

[1]tburakarslan@gmail.com; [2]samira.jalali83@gmail.com; [3]serdarenginoglu@gmail.com
Article History: Received: 5 Dec 20223 — Accepted: 25 Dec 2023 — Published: 31 Dec 2023

Throughout this paper, the set of all the fuzzy numbers over $\mathbb{R}$ is denoted by $FN(\mathbb{R})$.

Definition 2.2. The α-level set $[\mu]_\alpha$ of $\mu \in FN(\mathbb{R})$ is defined by

$$[\mu]_\alpha := \begin{cases} \{x \in \mathbb{R}: \mu(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ \overline{\text{supp}(\mu)}, & \alpha = 0 \end{cases}$$

Proposition 2.3. Let $\mu \in FN(\mathbb{R})$. Then, the set $[\mu]_\alpha$, denoted by $[\mu^-(\alpha), \mu^+(\alpha)]$, is a closed, bounded, and non-empty interval for all $\alpha \in [0,1]$.

Proposition 2.4. The function D defined by, for all $\mu, \nu \in FN(\mathbb{R})$,

$$D(\mu,\nu) \coloneqq \sup_{\alpha \in [0,1]} \max\{|\mu^-(\alpha) - \nu^-(\alpha)|, |\mu^+(\alpha) - \nu^+(\alpha)|\}$$

is a metric on $FN(\mathbb{R})$, and $(FN(\mathbb{R}), D)$ is a complete metric space.

Proposition 2.5. Let $\mu, \nu, \eta, \omega \in FN(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,

i. $D(\lambda\mu, \lambda\nu) = |\lambda| D(\mu,\nu)$

ii. $D(\mu + \nu, \eta + \nu) = D(\mu, \eta)$

iii. $D(\mu + \nu, \eta + \omega) \leq D(\mu, \eta) + D(\nu, \omega)$

Definition 2.6. A sequence (u_k) of fuzzy numbers is a function u from $\mathbb{N}$ to $FN(\mathbb{R})$. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called the k^{th} term of the sequence.

Across this study, the set of all the sequences of fuzzy numbers is denoted by $w(F)$.

Definition 2.7. A sequence $(u_k) \in w(F)$ is called convergent to $u \in FN(\mathbb{R})$ if, for all $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $D(u_k, u) < \varepsilon$, for all $k \geq n_0$.

Hereinafter, the set of all the convergent sequences of fuzzy numbers is denoted by $c(F)$.

Definition 2.8. [3] Let $(u_k) \in w(F)$, (q_k) be a sequence of non-negative real numbers, not all zero and $q_1 > 0$, and $Q_n \coloneqq q_1 + q_2 + \cdots + q_n$, for all $n \in \mathbb{N}$. If $\lim_n R_n(q,u) = u_0 \in FN(\mathbb{R})$, then (u_k) is called Riesz-convergent to fuzzy number u_0 and denoted by R-$\lim_k u_k = u_0$ or $u_k \overset{R}{\to} u_0$ where

$$R_n(q,u) = \frac{1}{Q_n} \sum_{k=1}^{n} q_k u_k, \quad n \in \mathbb{N}$$

Definition 2.9. [3] Let $(u_k) \in w(F)$, (q_k) be a sequence of non-negative real numbers, not all zero and $q_1 > 0$, and $Q_n \coloneqq q_1 + q_2 + \cdots + q_n$, for all $n \in \mathbb{N}$. If $\lim_n N_n(q,u) = u_0 \in FN(\mathbb{R})$, then (u_k) is called Nörlund-convergent to fuzzy number u_0 and denoted by N-$\lim_k u_k = u_0$ or $u_k \overset{N}{\to} u_0$ where

$$N_n(q,u) = \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k+1} u_k, \quad n \in \mathbb{N}$$

From now on, the set of all the Riesz and Nörlund convergent sequences of fuzzy numbers are denoted by $Rc(F)$ and $Nc(F)$, respectively.

Definition 2.10. The natural density of a set $K \subseteq \mathbb{N}$ is defined by $\delta(K) \coloneqq \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$ where $|\,.\,|$ denotes the cardinality of a set.

Definition 2.11. [6] A sequence $(u_k) \in w(F)$ is called statistical convergent (or briefly *st*-convergent) to $u_0 \in FN(\mathbb{R})$ and denoted by *st*-$\lim_k u_k = u_0$ or $u_k \overset{st}{\to} u_0$ if

$$\text{for all } \varepsilon > 0 \text{ and for all } k \text{ except for a set of natural density zero, } D(u_k, u_0) < \varepsilon$$

or

$$\text{for all } \varepsilon > 0,\ \delta(\{k \leq n : D(u_k, u_0) \geq \varepsilon\}) = 0$$

or

$$\text{there exists a subsequence } (u_{n_k}) \text{ such that } \lim_k \frac{k}{n_k} = 1 \text{ and } u_{n_k} \to u_0$$

Proposition 2.12. Let $(u_k) \in c(F)$. Then, $(u_k) \in sc(F)$, $(u_k) \in Rc(F)$, and $(u_k) \in Nc(F)$.

3. Statistical Riesz and Nörlund Convergence for Fuzzy Number Sequences

This section proposes statistical Riesz and Nörlund convergence of sequences of fuzzy numbers and investigates their properties.

Definition 3.1. Let $(u_k) \in w(F)$ and (u_{n_k}) be a Riesz convergent subsequence of (u_k) to $u_0 \in FN(\mathbb{R})$ such that $\lim_k \frac{k}{n_k} = 1$. Then, (u_k) is called statistical Riesz convergent (or briefly *stR*-convergent) to u_0 and denoted by *stR*-$\lim_k u_k = u_0$ or $u_k \overset{stR}{\longrightarrow} u_0$. In other words,

$$stR\text{-}\lim_k u_k = u_0 \Leftrightarrow \exists (u_{n_k}) \ni R\text{-}\lim_{n_k} u_{n_k} = u_0 \wedge \lim_k \frac{k}{n_k} = 1$$

Throughout this study, the set of all the *stR*-convergent sequences of fuzzy numbers is denoted by $stRc(F)$.

Definition 3.2. Let $(u_k) \in w(F)$ and (u_{n_k}) be a Nörlund convergent subsequence of (u_k) to $u_0 \in FN(\mathbb{R})$ such that $\lim_k \frac{k}{n_k} = 1$. Then, (u_k) is called statistical Nörlund convergent (or briefly *stN*-convergent) to u_0 and denoted by *stN*-$\lim_k u_k = u_0$ or $u_k \overset{stN}{\longrightarrow} u_0$. In other words,

$$stN\text{-}\lim_k u_k = u_0 \Leftrightarrow \exists (u_{n_k}) \ni N\text{-}\lim_{n_k} u_{n_k} = u_0 \wedge \lim_k \frac{k}{n_k} = 1$$

Across this study, the set of all the *stN*-convergent sequences of fuzzy numbers is denoted by $stNc(F)$.

Theorem 3.3. Let $(u_k) \in stc(F)$. Then, $(u_k) \in stRc(F)$.

Proof.

Let $u_k \overset{st}{\to} u_0$. Then, there exists a (u_{n_k}) such that $\lim_k \frac{k}{n_k} = 1$ and $u_{n_k} \to u_0$. From Proposition 2.12, $u_{n_k} \overset{R}{\to} u_0$.

□

The converse of Theorem 3.3 is not always correct.

Example 3.4. Let $(w_k) \in w(\mathbb{R})$ defined by

$$w_k(x) = \begin{cases} v_k, & \exists n \in \mathbb{N} \ni k = n^2 \\ u_k, & \forall n \in \mathbb{N}, k \neq n^2 \end{cases}$$

such that

$$u_k(x) = \begin{cases} \frac{k-2+x}{k}, & x \in [2-k, 2] \\ \frac{k+2-x}{k}, & x \in (2, 2+k] \\ 0, & \text{otherwise} \end{cases}$$

and

$$v_k(x)=\begin{cases} x-k, & x\in[k,k+1] \\ k+2-x, & x\in(k+1,k+2] \\ 0, & \text{otherwise}\end{cases}$$

Then, the α-level sets of u_k and v_k, for all $k\in\mathbb{N}$ and for all $\alpha\in[0,1]$, are as follows:

$$[u_k]_\alpha=[k\alpha-(k-2),(k+2)-k\alpha]$$

and

$$[v_k]_\alpha=[\alpha+k,k+2-\alpha]$$

Therefore, (w_k) is not convergent and also not statistical convergent because

$$\lim_k[u_k]_\alpha=\lim_k[2-k(1-\alpha),2+k(1-\alpha)]=\infty$$

Consider a sequence of real numbers $(q_k)=\left(\frac{k}{2^k}\right)$. Thus,

$$stR\text{-}\lim_k w_k=R\text{-}\lim_k u_k=\lim_k R_k(q,u)$$

where

$$R_k(q,u)=\begin{cases}\dfrac{2\left(2^k-1\right)(x+1)-k(k+x+2)}{6(2^k-1)-k(k+4)}, & x\in\left[\dfrac{k(k+1)}{2^{k+1}-(k+2)}-1,2\right] \\ \dfrac{k(-k+x-6)-2\left(2^k-1\right)(x-5)}{6(2^k-1)-k(k+4)}, & x\in\left(2,\dfrac{k(k+1)}{(k+2)-2^{k+1}}+5\right] \\ 0 & \text{otherwise}\end{cases}$$

and its α-level sets, for all $\alpha\in[0,1]$,

$$[R_k(q,u)]_\alpha=\left[3\alpha-1-\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)},5-3\alpha+\frac{(\alpha-1)k(k+1)}{2^{k+1}-(k+2)}\right]$$

because

$$[u_1]_\alpha=[\alpha+1,3-\alpha]$$

$$[u_2]_\alpha=[2\alpha,4-2\alpha]$$

$$[u_3]_\alpha=[3\alpha-1,5-3\alpha]$$

$$[u_4]_\alpha=[4\alpha-2,6-4\alpha]$$

$$[u_5]_\alpha=[5\alpha-3,7-5\alpha]$$

$$[u_6]_\alpha=[6\alpha-4,8-6\alpha]$$

$$\vdots$$

$$[u_k]_\alpha=[k\alpha-(k-2),(k+2)-k\alpha]$$

and

$$[R_1(q,u)]_\alpha=\frac{q_1u_1}{q_1}=u_1=[\alpha+1,3-\alpha]$$

$$[R_2(q,u)]_\alpha=\frac{q_1u_1+q_2u_2}{q_1+q_2}=\left[\frac{6\alpha+2}{4},\frac{14-6\alpha}{4}\right]$$

$$[R_3(q,u)]_\alpha=\frac{q_1u_1+q_2u_2+q_3u_3}{q_1+q_2+q_3}=\left[\frac{21\alpha+1}{11},\frac{43-21\alpha}{11}\right]$$

$$[R_4(q,u)]_\alpha = \frac{q_1u_1 + q_2u_2 + q_3u_3 + q_4u_4}{q_1 + q_2 + q_3 + q_4} = \left[\frac{58\alpha - 6}{26}, \frac{110 - 58\alpha}{26}\right]$$

$$[R_5(q,u)]_\alpha = \frac{q_1u_1 + q_2u_2 + q_3u_3 + q_4u_4 + q_5u_5}{q_1 + q_2 + q_3 + q_4 + q_5} = \left[\frac{141\alpha - 27}{57}, \frac{255 - 141\alpha}{57}\right]$$

$$[R_6(q,u)]_\alpha = \frac{q_1u_1 + q_2u_2 + q_3u_3 + q_4u_4 + q_5u_5 + q_6u_6}{q_1 + q_2 + q_3 + q_4 + q_5 + q_6} = \left[\frac{318\alpha - 78}{120}, \frac{558 - 318\alpha}{120}\right]$$

$$\vdots$$

$$[R_k(q,u)]_\alpha = \left[\frac{(3\cdot 2^{k+1} - k^2 - 4k - 6)\alpha - (2^{k+1} - k^2 - 2k - 2)}{2^{k+1} - (k+2)}, \frac{5\cdot 2^{k+1} - k^2 - 6k - 10 - (3\cdot 2^{k+1} - k^2 - 4k - 6)\alpha}{2^{k+1} - (k+2)}\right]$$

$$= \left[3\alpha - 1 - \frac{(\alpha-1)k(k+1)}{2^{k+1} - (k+2)}, 5 - 3\alpha + \frac{(\alpha-1)k(k+1)}{2^{k+1} - (k+2)}\right]$$

Thus,

$$\lim_k [R_k(q,u)]_\alpha = \lim_k \left[3\alpha - 1 - \frac{(\alpha-1)k(k+1)}{2^{k+1} - (k+2)}, 5 - 3\alpha + \frac{(\alpha-1)k(k+1)}{2^{k+1} - (k+2)}\right]$$

$$= [3\alpha - 1, 5 - 3\alpha]$$

Hence,

$$\lim_k R_k(q,u) = \begin{cases} \frac{x+1}{3}, & x \in [-1,2] \\ \frac{5-x}{3}, & x \in (2,5] \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$Rst\text{-}\lim_k w_k = \begin{cases} \frac{x+1}{3}, & x \in [-1,2] \\ \frac{5-x}{3}, & x \in (2,5] \\ 0, & \text{otherwise} \end{cases}$$

Consequently, although (w_k) is not convergent and not statistical convergent, (w_k) is statistical Riesz convergent.

Corollary 3.5. Let $(u_k) \in c(F)$. Then, $(u_k) \in stRc(F)$.

Theorem 3.6. Let $(u_k) \in sc(F)$. Then, $(u_k) \in stNc(F)$.

The proof is similar to the proof of Theorem 3.3. The converse of Theorem 3.6 is not always correct.

Example 3.7. Consider (w_k) provided in Example 3.4 and $(q_k) = (2^k)$. Thus,

$$stN\text{-}\lim_k w_k = N\text{-}\lim_k u_k = \lim_k N_k(q,u)$$

where

$$N_k(q,u)=\begin{cases}\dfrac{(2^k-1)x-k}{2^{k+1}-(k+2)}, & x\in\left[\dfrac{k}{2^k-1},2\right]\\ \dfrac{-(2^k-1)x+k}{2^{k+1}-2-k}+2, & x\in\left(2,4-\dfrac{k}{2^k-1}\right]\\ 0 & \text{otherwise}\end{cases}$$

and its α-level sets, for all $\alpha\in[0,1]$,

$$[N_k(q,u)]_\alpha=\left[2\alpha+\frac{k(1-\alpha)}{2^k-1},4-2\alpha+\frac{k(\alpha-1)}{2^k-1}\right]$$

because

$$[u_1]_\alpha=[\alpha+1,3-\alpha]$$

$$[u_2]_\alpha=[2\alpha,4-2\alpha]$$

$$[u_3]_\alpha=[3\alpha-1,5-3\alpha]$$

$$[u_4]_\alpha=[4\alpha-2,6-4\alpha]$$

$$[u_5]_\alpha=[5\alpha-3,7-5\alpha]$$

$$[u_6]_\alpha=[6\alpha-4,8-6\alpha]$$

$$\vdots$$

$$[u_k]_\alpha=[k\alpha-(k-2),(k+2)-k\alpha]$$

and

$$[N_1(q,u)]_\alpha=\frac{q_1u_1}{q_1}=u_1=[\alpha+1,3-\alpha]$$

$$[N_2(q,u)]_\alpha=\frac{q_2u_1+q_1u_2}{q_1+q_2}=\left[\frac{4\alpha+2}{3},\frac{10-4\alpha}{3}\right]$$

$$[N_3(q,u)]_\alpha=\frac{q_3u_1+q_2u_2+q_1u_3}{q_1+q_2+q_3}=\left[\frac{11\alpha+3}{7},\frac{25-11\alpha}{7}\right]$$

$$[N_4(q,u)]_\alpha=\frac{q_4u_1+q_3u_2+q_2u_3+q_1u_4}{q_1+q_2+q_3+q_4}=\left[\frac{26\alpha+4}{15},\frac{56-26\alpha}{15}\right]$$

$$[N_5(q,u)]_\alpha=\frac{q_5u_1+q_4u_2+q_3u_3+q_2u_4+q_1u_5}{q_1+q_2+q_3+q_4+q_5}=\left[\frac{57\alpha+5}{31},\frac{119-57\alpha}{31}\right]$$

$$[N_6(q,u)]_\alpha=\frac{q_6u_1+q_5u_2+q_4u_3+q_3u_4+q_2u_5+q_1u_6}{q_1+q_2+q_3+q_4+q_5+q_6}=\left[\frac{120\alpha+6}{63},\frac{246-120\alpha}{63}\right]$$

$$\vdots$$

$$[N_k(q,u)]_\alpha=\left[\frac{(2^{k+1}-2-k)\alpha+k}{2^k-1},\frac{2^{k+2}-4-k-(2^{k+1}-2-k)\alpha}{2^k-1}\right]$$

$$=\left[2\alpha+\frac{k(1-\alpha)}{2^k-1},4-2\alpha+\frac{k(\alpha-1)}{2^k-1}\right]$$

Thus,

$$\lim_k[N_k(q,u)]_\alpha=[2\alpha,4-2\alpha]$$

Hence,

$$\lim_k N_k(q,u) = \begin{cases} \frac{x}{2}, & x \in [0,2] \\ \frac{4-x}{2}, & x \in (2,4] \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$stN\text{-}\lim_k w_k = \begin{cases} \frac{x}{2}, & x \in [0,2] \\ \frac{4-x}{2}, & x \in (2,4] \\ 0, & \text{otherwise} \end{cases}$$

Consequently, although (w_k) is not convergent and not statistical convergent, (w_k) is statistical Nörlund convergent.

Corollary 3.8. Let $(u_k) \in c(F)$. Then, $(u_k) \in stNc(F)$.

4. Conclusion

This paper proposed statistical Riesz and Nörlund convergence of sequences of fuzzy numbers. It then showed that if a sequence of fuzzy numbers is convergent, then it is statistical Riesz/Nörlund convergent, and the converse is not always correct by two examples. In the future, the Tauberian conditions for a statistical Riesz/Nörlund convergent sequence to be convergent/statistical convergent and Korovkin-type theorems can be studied.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] V. Loku, N. L. Braha, M. Et, A. Tato, *Tauberian theorems for the generalized de la Vallée-Poussin mean-convergent sequences of fuzzy numbers*, Bulletin of Mathematical Analysis and Applications 9 (2) (2017) 45–56.

[2] S. Enginoğlu, S. Demiriz, *A comparison with the convergent, Cesàro convergent and Riesz convergent sequences of fuzzy numbers*, in: İ. Yüksek, F. Yiğit, H. Baraçlı (Eds.), FUZZYSS'15 The 4th International Fuzzy Systems Symposium, İstanbul, 2015, pp. 416–419.

[3] B. C. Tripathy, A. Baruah, *Nörlund and Riesz mean of sequences of fuzzy real numbers*, Applied Mathematics Letters 23 (5) (2010) 651–655.

[4] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL 28 (1) (1986) 28–37.

[5] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets and Systems 33 (1) (1989) 123–126.

[6] F. Nuray, E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Mathematica Slovaca 45 (3) (1995) 269–273.

[7] J-S. Kwon, *On statistical and p-Cesàro convergence of fuzzy numbers*, Korean Journal of Computational and Applied Mathematics 7 (1) (2000) 195–203.

[8] Y. Altın, M. Mursaleen, H. Altınok, *Statistical summability (C,1) for sequences of fuzzy real numbers and a Tauberian theorem*, Journal of Intelligent & Fuzzy Systems 21 (6) (2010) 379–858.

[9] S. Aytar, S. Pehlivan, *Statistical convergence of sequences of fuzzy numbers and sequences of α-cuts*, International Journal of General Systems 37 (2) (2008) 231–237.