Gültekin Çitil, H. (2024). Second-Order Fuzzy Differential Equation with Variable Coefficients. *The Black Sea Journal of Sciences*, 14(1), 272-280.

The Black Sea Journal of Sciences, 14(1), 272-280, 2024. DOI: 10.31466/kfbd.1401325

KFBD

Karadeniz Fen Bilimleri Dergisi
The Black Sea Journal of Sciences
ISSN (Online): 2564-7377 https://dergipark.org.tr/tr/pub/kfbd

Araştırma Makalesi / Research Article

Second-Order Fuzzy Differential Equation with Variable Coefficients

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Abstract

This paper is on a fuzzy initial value problem for second-order fuzzy differential equation with variable coefficients. The solution of the problem is solved via fuzzy Laplace transform method. Example is given to illustrate the problem. To interpret the problem and see the results, the graphics of the problem are drawn for each alpha level set.

Keywords: Fuzzy differential equation, Fuzzy function, Fuzzy problem.

Değişken Katsayılı İkinci-Mertebe Fuzzy Diferansiyel Denklem

Öz

Bu çalışma, değişken katsayılı ikinci-mertebe fuzzy diferansiyel denklem için bir fuzzy başlangıç değer problemi üzerinedir. Problemin çözümü fuzzy Laplace dönüşüm yöntemi ile çözülmüştür. Problemi açıklamak için örnek verilmiştir. Problemi yorumlamak ve sonuçları görmek için, her bir alfa seviye seti için problemin grafikleri çizilmiştir.

Anahtar Kelimeler: Fuzzy diferansiyel denklem, Fuzzy fonksiyon, Fuzzy problem.

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Geliş/Received: 06.12.2023 **Kabul/Accepted:** 22.02.2024 **Yayın/Published:** 15.03.2024

1. Introduction

Many studies have been done by several authors in fuzzy differential equations (Khastan and Nieto, 2010; Akın at al., 2016; Gültekin Çitil, 2020; Eljaoui and Melliani, 2023). There are different approaches to interpret the concept of a solution to fuzzy differential equations. The most popular approach is using the Hukuhara differentiability (Buckley and Feuring, 2000; Nieto at. al., 2006; Gültekin Çitil, 2020). The other approach is the strongly generalized differentiability. This approach was studied in many papers (Bede at al, 2007; Nieto at al., 2009; Gültekin Çitil, 2018). Hüllermeier (1997) interpreted fuzzy differential equations as a family of differential inclusions. Some numerical methods were presented in (Gültekin Çitil, 2019; Saqib at al., 2021; Jafaria at al., 2021).

Fuzzy Laplace transform method was introduced by Allahviranloo and Barkhordary Ahmadi (2010). It gives solution satisfying the initial values of fuzzy differential equation directly. Thus, many researchers used fuzzy Laplace transform (Gültekin Çitil, 2020; Samuel and Tahir, 2021; Belhallaj at al. 2023).

The aim of this paper is to research solutions second-order fuzzy differential equation with variable coefficients via fuzzy Laplace transform method.

In section 2, some basic definitions and theorems which will be used later are provided. In section 3, the considered problem is investigated by the fuzzy Laplace transform method. Numerical example are given. In section 4, conclusions are presented.

2. Materials and Methods

Definition 1. A fuzzy number is a function $u: \mathbb{R} \to [0,1]$ verifying the following assumptions:

u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$ and $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset (Bede and Gal, 2005).

Let $\mathbb{R}_F$ be the set of all fuzzy number on $\mathbb{R}$.

Definition 2. The α-level set $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right] = \{x \in \mathbb{R} | u(x) \geq \alpha\}$ of A fuzzy number satisfty the following requirements:

(a) $\underline{u}_\alpha$ is left-continuous bounded non-decreasing on $(0,1]$, it is right-continuous for $\alpha = 0$,

(b) $\overline{u}_\alpha$ is left-continuous bounded non-increasing on $(0,1]$, it is right-continuous for $\alpha = 0$,

(c) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$ (Allahviranloo and Barkhordary Ahmadi, 2010).

Definition 3. Let be $u_1, u_2 \in \mathbb{R}_F$. If there exists $u_3 \in \mathbb{R}_F$ such that $u_1 = u_2 + u_3$ then u_3 is called the Hukuhara difference of u_1 and u_2, which we denote by $u_1 \ominus u_2$ (Khastan and Nieto, 2010).

Definition 4. Let $g: (a,b) \longrightarrow \mathbb{R}_F$ and $t_0 \in (a,b)$.

i) g is said to be (1)-differentiable at t_0, if there exists $g'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, exists $g(t_0 + h) \ominus g(t_0)$, $g(t_0) \ominus g(t_0 - h)$ and the limits hold

$$\lim_{h \to 0^+} \frac{g(t_0 + h) \ominus g(t_0)}{h} = \lim_{h \to 0^+} \frac{g(t_0) \ominus g(t_0 - h)}{h} = g'(t_0),$$

ii) g is said to be (2)-differentiable at t_0, if there exists $g'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, exists $g(t_0) \ominus g(t_0 + h)$,$g(t_0 - h) \ominus g(t_0)$ and the limits hold

$$\lim_{h \to 0^+} \frac{g(t_0) \ominus g(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{g(t_0 - h) \ominus g(t_0)}{-h} = g'(t_0)$$ (Patel and Desai, 2017).

Definition 5. Let $g: [a, b] \longrightarrow \mathbb{R}_F$ be a fuzzy function. The fuzzy Laplace transform of g is

$G(s) = L\big(g(t)\big) = \int_0^\infty e^{-st}\, g(t)dt = \left[\lim_{\rho \to \infty} \int_0^\rho e^{-st}\, \underline{g}(t)dt\, , \lim_{\rho \to \infty} \int_0^\rho e^{-st}\, \overline{g}(t)dt\right].$

$G(s, \alpha) = L([g(t)]^\alpha) = \left[L\left(\underline{g}_\alpha(t)\right), L\left(\overline{g}_\alpha(t)\right)\right]$, where

$$L\left(\underline{g}_\alpha(t)\right) = \int_0^\infty e^{-st}\, \underline{g}_\alpha(t)dt = \lim_{\rho \to \infty} \int_0^\rho e^{-st}\, \underline{g}_\alpha(t)dt,$$

$L\left(\overline{g}_\alpha(t)\right) = \int_0^\infty e^{-st}\, \overline{g}_\alpha(t)dt = \lim_{\rho \to \infty} \int_0^\rho e^{-st}\, \overline{g}_\alpha(t)dt$ (Patel and Desai, 2017).

Theorem 1. Let $g'(t)$ be an integrable fuzzy function, $g(t)$ is primitive of $g'(t)$ on $(0, \infty]$. If g is (1)-differentiable, then

$$L\big(g'(t)\big) = sL\big(g(t)\big) \ominus g(0).$$

If g is (2)-differentiable, then

$L\big(g'(t)\big) = \big(-g(0)\big) \ominus \Big(-sL\big(g(t)\big)\Big)$ (Allahviranloo and Barkhordary Ahmadi, 2010).

Theorem 2. Let $g(t)$ satisfies the condition of existence theorem of Laplace transform and

$$L\big(g(t)\big) = G(s),$$

then

$$L\big(tg(t)\big) = -G'(s).$$

If $g'(t)$ satisfies the condition of existence theorem of Laplace transform, then

$$L\big(tg'(t)\big) = -sG'(s) - G(s).$$

Similarly,

$L\big(tg''(t)\big) = -s^2G'(s) - 2sG(s) + g(0)$ (Patel and Desai, 2017).

3. Findings and Discussion

We investigate the problem with variable coefficient

$$\begin{cases} xu'' = -[\mu]^\alpha u' \\ u(0) = [\omega]^\alpha \end{cases}, \tag{1}$$

via fuzzy Laplace transform,

$$[\mu]^\alpha = \left[\underline{\mu}_\alpha, \overline{\mu}_\alpha\right], [\omega]^\alpha = \left[\underline{\omega}_\alpha, \overline{\omega}_\alpha\right]$$

are positive symmetric triangular fuzzy numbers, u is positive fuzzy function and Laplace transform of u is $L\big(u(x)\big) = U(s)$, $x > 0$.

From the equation (1), we obtain the equation

$$-s^2U'(s) - 2sU(s) + u(0) = -[\lambda]^\alpha\big(sU(s) \ominus u(0)\big)$$

via fuzzy Laplace transform method. From this, we have

$$-s^2\underline{U}'_\alpha(s) - 2s\underline{U}_\alpha(s) + \underline{u}_\alpha(0) = -\underline{\mu}_\alpha s\underline{U}_\alpha(s) + \underline{\mu}_\alpha\underline{u}_\alpha(0), \tag{2}$$

$$-s^2\overline{U}'_\alpha(s) - 2s\overline{U}_\alpha(s) + \overline{u}_\alpha(0) = -\overline{\mu}_\alpha s\overline{U}_\alpha(\alpha) + \overline{\mu}_\alpha\overline{u}_\alpha(0). \tag{3}$$

The equations (2) and (3) give the equations

$$\underline{U}'_\alpha(s) + \frac{2-\underline{\mu}_\alpha}{s}\underline{U}_\alpha(s) = \frac{\left(1-\underline{\mu}_\alpha\right)\underline{\omega}_\alpha}{s^2}, \tag{4}$$

$$\overline{U}'_\alpha(s) + \frac{2-\overline{\mu}_\alpha}{s}\overline{U}_\alpha(s) = \frac{\left(1-\overline{\mu}_\alpha\right)\overline{\omega}_\alpha}{s^2}. \tag{5}$$

Solving the equations (4) and (5), $\underline{U}_\alpha(s)$ and $\overline{U}_\alpha(s)$ are obtained as

$$\underline{U}_\alpha(s) = \frac{\underline{\omega}_\alpha}{s} + \underline{c}_\alpha\frac{1}{s^{2-\underline{\mu}_\alpha}} \tag{6}$$

$$\overline{U}_\alpha(s) = \frac{\overline{\omega}_\alpha}{s} + \overline{c}_\alpha\frac{1}{s^{2-\overline{\mu}_\alpha}}, \tag{7}$$

where $[c]^\alpha = \left[\underline{c}_\alpha, \overline{c}_\alpha\right]$ is arbitrary fuzzy number.

Taking the inverse Laplace transform of (6) and (7), we find the lower and upper solutions of the problem as

$$\underline{u}_\alpha(x) = \underline{\omega}_\alpha + \underline{c}_\alpha x^{1-\underline{\mu}_\alpha},$$

$$\overline{u}_\alpha(x) = \overline{\omega}_\alpha + \overline{c}_\alpha x^{1-\overline{\mu}_\alpha}.$$

So, the fuzzy solution of the problem (1) is

$$[u(x)]^\alpha = \left[\underline{u}_\alpha(x), \overline{u}_\alpha(x)\right].$$

This solution must be a valid fuzzy function.

Example 1. We consider the fuzzy problem

$$\begin{cases} xu'' = -[1]^\alpha u' \\ u(0) = [2]^\alpha \end{cases}, \tag{8}$$

where

$[1]^\alpha = [\alpha, 2-\alpha]$, $[2]^\alpha = [1+\alpha, 3-\alpha]$.

The solution of the problem (8) is

$$\begin{cases} \underline{u}_\alpha(x) = 1+\alpha+\underline{c}_\alpha x^{(1-\alpha)} \\ \overline{u}_\alpha(x) = 3-\alpha+\overline{c}_\alpha x^{(\alpha-1)} \end{cases}, [c]^\alpha = [\underline{c}_\alpha, \overline{c}_\alpha] \quad (9)$$

$$[u(x)]^\alpha = [\underline{u}_\alpha(x), \overline{u}_\alpha(x)]. \quad (10)$$

The solution (9)-(10) of the problem (8) must be a valid fuzzy function. So, the graphs of the solution is analyzed:

I) Let $[c]^\alpha = [\underline{c}_\alpha, \overline{c}_\alpha]$ be positive fuzzy number. For example, $[c]^\alpha$ be $[1]^\alpha = [\alpha, 2-\alpha]$. Then, the solution is

$$\begin{cases} \underline{u}_\alpha(x) = 1+\alpha+\alpha x^{(1-\alpha)} \\ \overline{u}_\alpha(x) = 3-\alpha+(2-\alpha)x^{(\alpha-1)} \end{cases}, \quad (11)$$

$$[u(x)]^\alpha = [\underline{u}_\alpha(x), \overline{u}_\alpha(x)]. \quad (12)$$

II) Let $[c]^\alpha = [\underline{c}_\alpha, \overline{c}_\alpha]$ be negative fuzzy number. For example, $[c]^\alpha$ be $[-1]^\alpha = [-2+\alpha, -\alpha]$. In this case, the solution is

$$\begin{cases} \underline{u}_\alpha(x) = 1+\alpha+(\alpha-2)x^{(1-\alpha)} \\ \overline{u}_\alpha(x) = 3-\alpha-\alpha x^{(\alpha-1)} \end{cases}, \quad (13)$$

$$[u(x)]^\alpha = [\underline{u}_\alpha(x), \overline{u}_\alpha(x)]. \quad (14)$$

From Definition 2 and since $[u(x)]^\alpha$ positive fuzzy function;

The solution (11)-(12) is a valid fuzzy function for $0 \le x \le 9$ according to Figure 1, for $0 \le x \le 0.7.8732$ according to Figure 2 and for $0 \le x \le 7.59375$ according to Figure 3.

Also, the solution (12)-(13) is a valid fuzzy function for $0.111111 \le x \le 1$ according to Figure 4, for $0.127013 \le x \le 2.44541$ according to Figure 5 and for $0.131687 \le x \le 7.59375$ according to Figure 6.

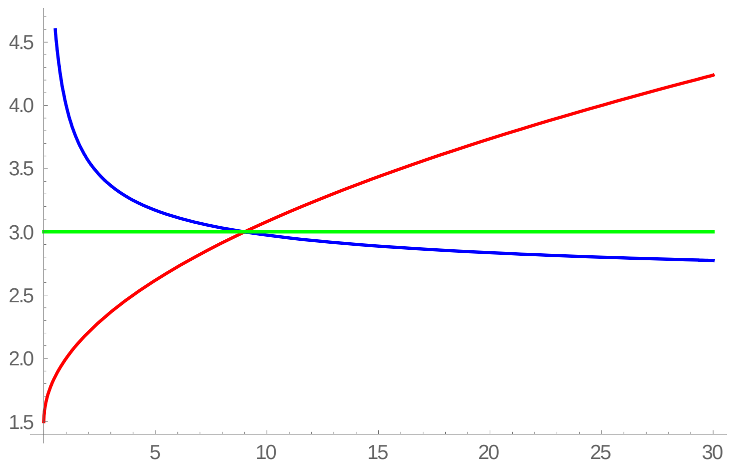

Figure 1. Graphic of the solution (11)-(12) for $\alpha = 0.5$

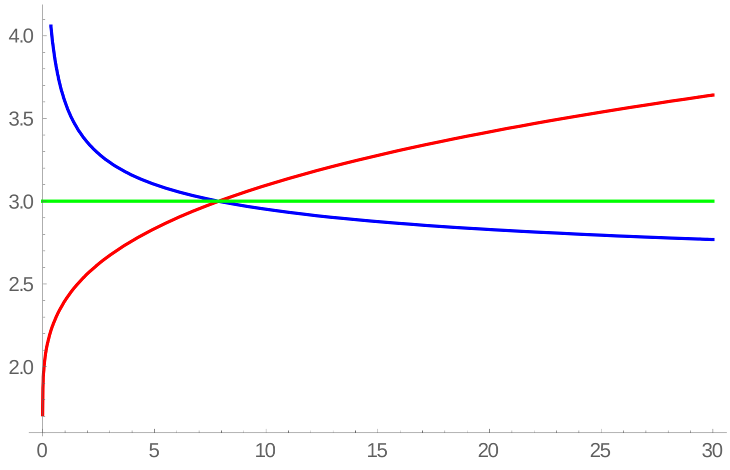

Figure 2. Graphic of the solution (11)-(12) for $\alpha = 0.7$

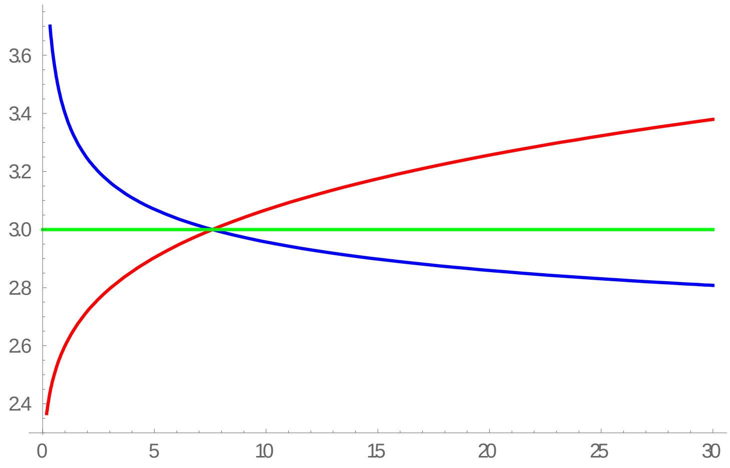

Figure 3. Graphic of the solution (11)-(12) for $\alpha = 0.8$

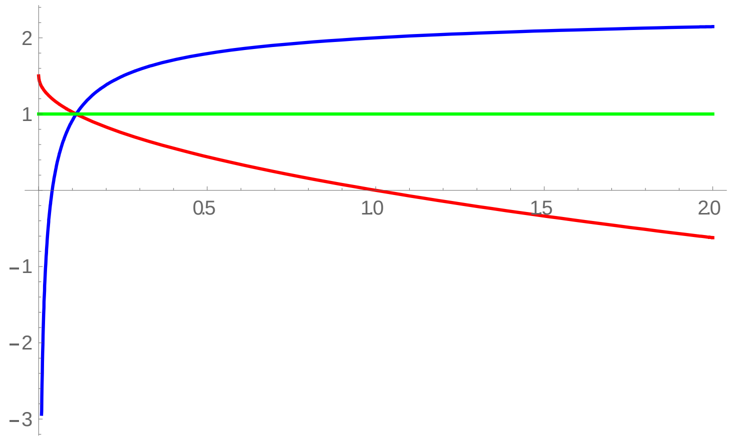

Figure 4. Graphic of the solution (13)-(14) for $\alpha = 0.5$

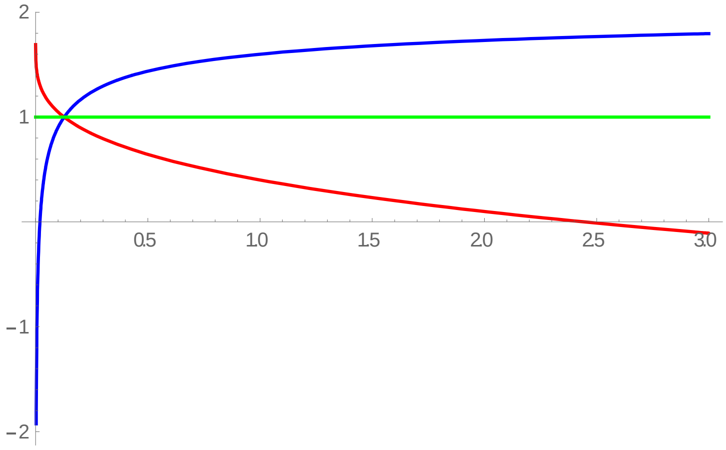

Figure 5. Graphic of the solution (13)-(14) for $\alpha = 0.7$

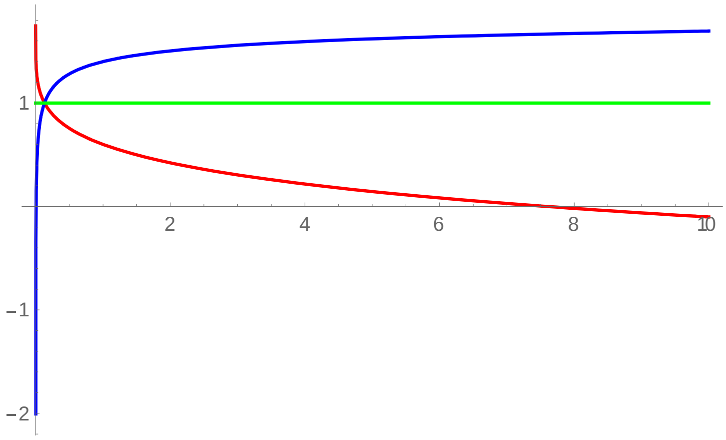

Figure 6. Graphic of the solution (13)-(14) for $\alpha = 0.8$

Red $\rightarrow \underline{u}_{\alpha}(x)$, Blue $\rightarrow \overline{u}_{\alpha}(x)$, Green $\rightarrow \overline{u}_{1}(x) = \underline{u}_{1}(x)$.

4. Conclusions and Recommendations

In this study, we research an initial value problem for second-order fuzzy differential equation with variable coefficients using the fuzzy Laplace transform method. We give numerical example to illustrate the problem. The graphics of the solutions are drawn for each alpha level set. We search whether the solutions are valid fuzzy functions or not.

It is seen that if the arbitrary fuzzy number in the solution is positive, the interval in which the solution is a valid fuzzy function decreases as the alpha level set increases. Also, if the arbitrary fuzzy number in the solution is negative, the interval in which the solution is a valid fuzzy function decreases as the alpha level set increases.

Statement of Research and Publication Ethics

The author declares that all the rules required to be followed within the scope of "Higher Education Institutions Scientific Research and Publication Ethics Directive" have been complied with in all processes of the article, that The Black Sea Journal of Science and the editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than The Black Sea Journal of Science.

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