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Research Article

f-statistical connections and Miao-Tam statistical manifolds

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Abstract

We introduce f-statistical connections as a family of statistical connections and study some geometric objects associated to these connections such as divergence, curvature and Ricci tensors, Hessian and Laplacian operators. We construct examples of f-statistical connections and study the introducing concepts on them. Finally we introduce Miao-Tam statistical manifolds and study properties of them.

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1. Introduction

Recently, the study of spaces consisting of probability measures is getting more attention. Information geometry as a famous theory in geometry is a tool to investigate such spaces (of course in finite dimensional sense). Nowadays, this geometry as a combination of statistics and differential geometry has effective role in science. For instance, a manifold learning theory in a hypothesis space consisting of models is developed in [15]. The semi-Riemannian metric of this hypothesis space is uniquely derived based on the information geometry of the probability distributions. In [1], Amari also combined the statistical and geometrical ideas for studying neural networks including hidden units or unobservable variables. To see more applications of this geometry in other sciences, can be referred to [6, 8, 9].

For an open subset Θ of $\mathbb{R}^n$ and a sample space Ω with parameter $\theta = (\theta^1, \cdots, \theta^n)$, we call the set of probability density functions

$$S = \{p(x;\theta) : \int_\Omega p(x;\theta) = 1, \quad p(x;\theta) > 0, \quad \theta \in \Theta \subseteq \mathbb{R}^n\},$$

as a statistical model. For a statistical model S, the semi-definite Fisher information matrix $g(\theta) = [g_{ij}(\theta)]$ is defined as

$$g_{ij}(\theta) := \int_\Omega \partial_i \ell_\theta \partial_j \ell_\theta p(x;\theta) \mathrm{d}x = E_p[\partial_i \ell_\theta \partial_j \ell_\theta], \tag{1.1}$$

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where $\ell_\theta = \ell(x;\theta) := \log p(x;\theta)$, $\partial_i := \frac{\partial}{\partial\theta^i}$, and $E_p[f]$ is the expectation of $f(x)$ with respect to $p(x;\theta)$. Equipping the space S with such information matrices, it is called a statistical manifold in literature.

Historically, Fisher was the first who introduced relation (1.1) as a mathematical intent of information in 1920 (see [10]). It is shown that if g is positive-definite and all of its components are converging to real numbers, then (S, g) will be a Riemannian manifold and g is called a Fisher metric on S. Using the Fisher metric g, an affine connection ∇ with respect to $p(x;\theta)$ is defined by

$$\Gamma_{ij,k} = g(\nabla_{\partial_i}\partial_j, \partial_k) := E_p[(\partial_i\partial_j\ell_\theta)\partial_k\ell_\theta]. \tag{1.2}$$

The study of the critical points of the volume functional associated to the space of smooth Riemannian structures is a useful problem and applicable in Riemannian geometry that has attracted the attention of many researchers (see [4, 5, 12, 13], for instance). In [12], P. Miao and L.-F. Tam proved that a Riemannian metric g on a compact manifold M of dimension at least three with the smooth boundary ∂M is a critical point of the volume functional if and only if there is a function φ on M such that $\varphi = 0$ on ∂M and

$$-\widehat{\triangle}\varphi\, g + \widehat{H}_\varphi - \varphi\widehat{Ric} = g, \tag{1.3}$$

where $\widehat{\triangle}$ and $\widehat{H}_\varphi$ are Laplacian and Hessian operators and $\widehat{Ric}$ is the Ricci tensor on M with respect to the Levi-Civita connection $\widehat{\nabla}$. The function φ is known as the potential function and (1.3) is known as Miao-Tam equation. Due to the significant role of Miao-Tam equation in the study of critical points of the volume functional on compact Riemannian manifolds with the smooth boundary, this equation is very important in Riemannian geometry.

The aim of this paper is to study the Miao-Tam equation for statistical manifolds. To achieve this goal, it is necessary to introduce and study the elements in Miao-Tam equation (Laplacian and Hessian operators and Ricci tensor) for statistical manifolds. Before introducing these concepts, we first introduce and study a family of statistical connections, which are called f-statistical connections. Then we study some geometric objects such as divergence, curvature and Ricci tensors, Hessian and Laplacian operators. We also investigate the Codazzi-coupled property of a f-statistical connection with some tensor fields. Finally we introduce Miao-Tam equation for f-statistical connections and study some properties of them. The presentation of various statistical examples covers the concepts presented in the paper.

2. f-statistical connections

Let M be an *n-dimensional* manifold and (U, x^i), $i = 1, \ldots, n$, be a local chart of the point $x \in U$. Considering the coordinates (x^i) on M, we have the local field $\frac{\partial}{\partial x^i}|_x$ as frames on T_xM.

Let ∇ be an affine connection of M. The torsion tensor of the connection ∇ is a tensor T^∇ of type $(1, 2)$ given by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for any $X, Y \in \chi(M)$. The connection ∇ is *torsion-free*, if its torsion tensor vanishes. We recall that ∇ and a symmetric tensor B of type (0,2) are Codazzi-coupled if the Codazzi equation holds that is

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z), \qquad \forall X, Y, Z \in \chi(M).$$

Assume that g is a pseudo-Riemannian metric on M. An affine connection ∇ is called *Codazzi connection* if the cubic tensor field $\mathcal{C} = \nabla g$ is totally symmetric; namely the

Codazzi equations hold:

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z), \qquad (= (\nabla_Z g)(X,Y)), \forall X,Y,Z \in \chi(M), \tag{2.1}$$

where

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z). \tag{2.2}$$

In the local coordinates, the components of $\mathcal{C}$ have the following form

$$\mathcal{C}_{ijk} = \partial_k g_{ij} - \Gamma^h_{ik} g_{jh} - \Gamma^h_{jk} g_{ih}, \qquad \mathcal{C}_{ijk} = \mathcal{C}_{jik} = \mathcal{C}_{kij}, \tag{2.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and Γ^i_{jk} are the Christoffel symbols of the Codazzi connection ∇. The triplet (M, g, ∇) also is said to be a *statistical manifold* if ∇ is a statistical connection, i.e., a torsion-free Codazzi connection. In particular, it is known that if the cubic tensor field is zero, a torsion-free Codazzi connection ∇ reduces to the Levi-Civita connection $\widehat{\nabla}$. Moreover, the affine connection ∇^* of M defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \tag{2.4}$$

is called the *(conjugate) dual connection* of ∇ with respect to g. Immediately, one can see $\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$ and

$$\mathcal{C}^*(X,Y,Z) = (\nabla^*_X g)(Y,Z) = -\mathcal{C}(X,Y,Z), \quad \forall X,Y,Z \in \chi(M).$$

Thus (M, g, ∇^*) forms a statistical manifold.

For a statistical structure (g, ∇) on M, if we consider a $(1,2)$-tensor field $K : \chi(M) \times \chi(M) \to \chi(M)$ described by

$$K_X Y = \nabla^*_X Y - \nabla_X Y, \tag{2.5}$$

it follows that K satisfies

$$K_X Y = K_Y X, \qquad g(K_X Y, Z) = g(Y, K_X Z), \quad \mathcal{C}(X,Y,Z) = g(K_X Y, Z), \tag{2.6}$$

for all $X, Y, Z \in \chi(M)$.

For an affine connection ∇, the curvature tensor R^∇ is defined as

$$R^\nabla(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad \forall X,Y,Z \in \chi(M). \tag{2.7}$$

In a statistical manifold (M, g, ∇), we denote R^∇, R^{∇^*} and $R^{\widehat{\nabla}}$ by R, R^* and $\widehat{R}$, respectively for short. It is known that the following hold

$$R(X,Y,Z,W) = -R(Y,X,Z,W), \tag{2.8}$$
$$R^*(X,Y,Z,W) = -R^*(Y,X,Z,W), \tag{2.9}$$
$$R(X,Y,Z,W) = -R^*(X,Y,W,Z), \tag{2.10}$$

where $R(X,Y,Z,W) = g(R(X,Y)Z, W)$. Moreover, M is called a *flat* statistical manifold if $R = 0$.

Let (M, g) be a pseudo-Riemannian manifold and $f \in C^\infty(M)$. The affine combination of two affine connections $\nabla^{(0)}$ and $\nabla^{(1)}$ on M is the connection $\nabla^{(f)}$ given by

$$\nabla^{(f)} = (1-f)\nabla^{(0)} + f\nabla^{(1)}.$$

Immediately, we see that

$$T^{\nabla^{(f)}} = (1-f)T^{\nabla^{(0)}} + fT^{\nabla^{(1)}}, \qquad \nabla^{(f)} g = (1-f)\nabla^{(0)} g + f\nabla^{(1)} g,$$

where $T^{\nabla^{(f)}}$, $T^{\nabla^{(0)}}$ and $T^{\nabla^{(1)}}$ are the torsion tensors of $\nabla^{(f)}$, $\nabla^{(0)}$ and $\nabla^{(1)}$, respectively [3].

Definition 2.1. Let (M, g, ∇) be a statistical manifold. The family of connections $\nabla^{(f)}$ given by affine combination of the conjugate connections $\nabla^{(0)} := \nabla$ and $\nabla^{(1)} := \nabla^*$, i.e.,

$$\nabla^{(f)} = (1-f)\nabla + f\nabla^*, \quad f \in C^\infty(M),$$

is called f-statistical connection.

Assuming $f = \frac{1}{2}, 0$ and 1 in the above definition, we obtain the connections $\widehat{\nabla}$, ∇ and ∇^*, respectively. In addition, the components of the f-statistical connection are as follows

$$\Gamma_{ij}^{(f)r} = (1-f)\Gamma_{ij}^r + f\Gamma_{ij}^{*r}, \tag{2.11}$$

where $\Gamma_{ij}^{(f)r}, \Gamma_{ij}^r$ and Γ_{ij}^{*r} are the components of $\nabla^{(f)}$, ∇ and ∇^*, respectively. From Definition 2.1, it follows that the f-statistical connection $\nabla^{(f)}$ is torsion-free, i.e., $T^{\nabla^{(f)}} = 0$ and satisfies the following condition

$$\mathcal{C}^{(f)}(X, Y, Z) := (\nabla_X^{(f)} g)(Y, Z) = (1-2f)\mathcal{C}(X, Y, Z), \quad \forall X, Y, Z \in \chi(M).$$

As (g, ∇) is a statistical structure on M, then $(g, \nabla^{(f)})$ is also a statistical structure.

Proposition 2.2. *On a statistical manifold $(M, g, \nabla^{(f)})$, we have*

$$Xg(Y, Z) = g(\nabla_X^{(f)} Y, Z) + g(Y, \nabla_X^{(1-f)} Z),$$

for any $X, Y, Z \in \chi(M)$, i.e., $\nabla^{(1-f)}$ is dual of $\nabla^{(f)}$.

Proof. Using Definition 2.1, we have

$$g(\nabla_X^{(f)} Y, Z) = (1-f)g(\nabla_X Y, Z) + fg(\nabla_X^* Y, Z).$$

The above equation and (2.4) imply

$$g(\nabla_X^{(f)} Y, Z) = (1-f)g(\nabla_X Y, Z) + fXg(Y, Z) - fg(Y, \nabla_X Z).$$

Similarly, it follows

$$g(\nabla_X^{(1-f)} Z, Y) = fg(\nabla_X Z, Y) + (1-f)Xg(Y, Z) - (1-f)g(Z, \nabla_X Y).$$

Adding the last two equations, we obtain the formula claimed by the proposition. □

Corollary 2.3. *The f-statistical connection $\nabla^{(f)}$ satisfies the following*

$$\nabla^{(f)} = \widehat{\nabla} - \frac{1-2f}{2}K, \qquad \nabla^{(f)} + \nabla^{(1-f)} = 2\widehat{\nabla}, \qquad \nabla^{(1-f)} - \nabla^{(f)} = (1-2f)K.$$

Proof. As $\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$, the f-statistical connection $\nabla^{(f)}$ can be written as

$$\nabla^{(f)} = (2-2f)\widehat{\nabla} - (1-2f)\nabla^* = \widehat{\nabla} + \frac{1-2f}{2}(\nabla - \nabla^*) = \widehat{\nabla} - \frac{1-2f}{2}K. \tag{2.12}$$

We conclude similarly that

$$\nabla^{(1-f)} = \widehat{\nabla} + \frac{1-2f}{2}K.$$

Therefore, the above equations give the following relations

$$\nabla^{(f)} + \nabla^{(1-f)} = 2\widehat{\nabla},$$

and

$$\nabla^{(1-f)} - \nabla^{(f)} = (1-2f)K. \tag{2.13}$$

□

Proposition 2.4. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold and $\omega \in \Lambda^n(M)$, where $\Lambda^n(M)$ is the space of n-forms on the manifold M. Then*

$$\nabla^{(f)}\omega = (1-f)\nabla\omega + f\nabla^*\omega.$$

Proof. Considering $X, Y_1, \dots, Y_n \in \chi(M)$, we get

$$(\nabla^{(f)}_X \omega)(Y_1, \dots, Y_n) = X(\omega(Y_1, \dots, Y_n)) - \sum_{i=1}^{n} \omega(Y_1, \dots, \nabla^{(f)}_X Y_i, \dots, Y_n).$$

The above equation and Definition 2.1 yield

$$(\nabla^{(f)}_X \omega)(Y_1, \dots, Y_n) = X(\omega(Y_1, \dots, Y_n)) - \sum_{i=1}^{n} \omega(Y_1, \dots, (1-f)\nabla_X Y_i + f\nabla^*_X Y_i, \dots, Y_n).$$

By adding and subtracting term $fX(\omega(Y_1, \dots, Y_n))$ in the last equation, it follows

$$(\nabla^{(f)}_X \omega)(Y_1, \dots, Y_n) = (1-f)(\nabla_X \omega)(Y_1, \dots, Y_n) + f(\nabla^*_X \omega)(Y_1, \dots, Y_n),$$

which completes the proof. □

Let ∇ be an affine connection of a pseudo-Riemannian manifold (M, g). The divergence of $X \in \chi(M)$ is defined as the trace of the covariant derivative ∇X, i.e.,

$$div^{\nabla} X = tr\{Y \to \nabla_Y X\},$$

which can be written locally as

$$div^{\nabla} X = \partial_i(X^i) + X^j \Gamma^i_{ij}.$$

In general for a tensor field A of type $(1, n)$ on M, $div^{\nabla} A$ is given by

$$div^{\nabla} A = tr\{Y \to (\nabla_Y A)(X_1, \dots, X_n)\}, \quad \forall Y, X_1, \dots, X_n \in \chi(M).$$

Now, suppose that $(M, g, \nabla^{(f)})$ is a statistical manifold. (2.5), (2.12) and the above equation provide the explicit formula for $div^{\nabla^{(f)}}$ of $X = X^i\partial_i \in \chi(M)$:

$$div^{\nabla^{(f)}} X = div^{\widehat{\nabla}} X - \frac{1-2f}{2}(div^{\nabla^*} X - div^{\nabla} X). \tag{2.14}$$

The last equation can be expressed in the local coordinates as

$$div^{\nabla^{(f)}} X = \partial_i(X^i) + X^j \widehat{\Gamma}^i_{ij} - \frac{1-2f}{2} X^j K^i_{ij},$$

where $\widehat{\Gamma}^i_{ij}$ and $K^i_{ij} = \Gamma^{*i}_{ij} - \Gamma^i_{ij}$ are the components of the Levi-Civita connection $\widehat{\nabla}$ and the tensor K, respectively. In addition, considering $\varphi \in C^\infty(M)$, it is easy to check that

$$div^{\nabla^{(f)}}(\varphi X) = X(\varphi) + \varphi div^{\nabla^{(f)}} X.$$

Proposition 2.5. *On a statistical manifold $(M, g, \nabla^{(f)})$, the following holds*

$$div^{\nabla^{(f)}} X = (1-f)div^{\nabla} X + f div^{\nabla^*} X, \quad \forall X \in \chi(M).$$

Proof. Using (2.14), for $f = 0$ and $f = 1$ we get

$$div^{\nabla} X = div^{\widehat{\nabla}} X - \frac{1}{2}(div^{\nabla^*} X - div^{\nabla} X), \qquad div^{\nabla^*} X = div^{\widehat{\nabla}} X + \frac{1}{2}(div^{\nabla^*} X - div^{\nabla} X).$$

Thus from the above equations, it follows

$$(1-f)div^{\nabla} X + f div^{\nabla^*} X = div^{\widehat{\nabla}} X - \frac{1-2f}{2}(div^{\nabla^*} X - div^{\nabla} X) = div^{\nabla^{(f)}} X.$$

□

On any smooth oriented manifold M of dimension n with a pseudo-Riemannian metric g, one can define a volume form ω_g associated to g as

$$\omega_g = \sqrt{|detg|} dx^1 \wedge \dots \wedge dx^n.$$

An easy computation shows that the Lie derivative $\pounds$ of the volume form ω_g with respect to a vector field $X \in \chi(M)$, satisfies

$$\pounds_X \omega_g = \nabla^{(f)}_X \omega_g + (div^{\nabla^{(f)}} X)\omega_g.$$

On the other hand, it is well known that $\pounds_X \omega_g = (div^{\widehat{\nabla}} X)\omega_g$ (see [7]). Thus we have

$$\nabla^{(f)}_X \omega_g = (div^{\widehat{\nabla}} X - div^{\nabla^{(f)}} X)\omega_g.$$

According to the above equation and (2.14), it follows

$$\nabla^{(f)}_X \omega_g = \frac{(1-2f)}{2}\tau_g(X)\omega_g, \tag{2.15}$$

where $\tau_g(X) = trK_X$.

Let M be an n-dimensional manifold and ∇ be a torsion-free affine connection on it. We say that ∇ is *equiaffine* if there is a parallel volume form on M, i.e., a nonvanishing n-form ω such that $\nabla\omega = 0$ [11, 14]. Specially if a statistical structure $(g, \nabla^{(f)})$ on M is equiaffine relative to the pseudo-Riemannian volume form ω_g, it is equivalent to condition $\tau_g(X) = 0$ for every $X \in \chi(M)$. Such structures are called *trace-free*. On the other side, we have $\nabla\omega_g = \frac{1}{2}\tau_g(X)\omega_g$ $(\nabla^*\omega_g = -\frac{1}{2}\tau_g(X)\omega_g)$. Thus $\nabla^{(f)}$ is equiaffine if and only if ∇ (∇^*) is equiaffine

3. Curvature of f-statistical connections

Let $(g, \nabla^{(f)})$ be a statistical structure on M. The f-curvature tensor $R^{\nabla^{(f)}}$ is obtained from the following formula

$$R^{(f)}(X,Y)Z = \nabla^{(f)}_X \nabla^{(f)}_Y Z - \nabla^{(f)}_Y \nabla^{(f)}_X Z - \nabla^{(f)}_{[X,Y]} Z, \tag{3.1}$$

for any $X, Y, Z \in \Im^1_0(M)$. For short, $R^{\nabla^{(f)}}$ is denoted by $R^{(f)}$. Locally, we have

$$R^{(f)r}_{ijk} = \partial_i \Gamma^{(f)r}_{jk} - \partial_j \Gamma^{(f)r}_{ik} + \Gamma^{(f)r}_{im}\Gamma^{(f)m}_{jk} - \Gamma^{(f)r}_{jm}\Gamma^{(f)m}_{ik}, \tag{3.2}$$

where $R^{(f)}(\partial_i, \partial_j)\partial_k = R^{(f)r}_{ijk}\partial_r$. Denote $R^{\nabla^{(1-f)}}$ by $R^{(1-f)}$ in the similar fashion. A statistical manifold is said to be f-*flat* if $R^{(f)} = 0$.

Proposition 3.1. *The curvature tensors $R^{(f)}$ and $R^{(1-f)}$ satisfy the following*

$$\begin{aligned} R^{(f)}(X,Y)Z =&(1-f)R(X,Y)Z + fR^*(X,Y)Z + f(1-f)[K_Y, K_X]Z \\ &+ X(f)K_Y Z - Y(f)K_X Z, \\ R^{(1-f)}(X,Y)Z =&fR(X,Y)Z + (1-f)R^*(X,Y)Z + f(1-f)[K_Y, K_X]Z \\ &- X(f)K_Y Z + Y(f)K_X Z, \end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

Proof. Applying Definition 2.1, the first term on the right of (3.1) can be obtained as

$$\begin{aligned} \nabla^{(f)}_X \nabla^{(f)}_Y Z =&\nabla^{(f)}_X ((1-f)\nabla_Y Z + f\nabla^*_Y Z) = X(f)\nabla^*_Y Z - X(f)\nabla_Y Z \\ &+ (1-f)((1-f)\nabla_X\nabla_Y Z + f\nabla^*_X\nabla_Y Z) + f((1-f)\nabla_X\nabla^*_Y Z + f\nabla^*_X\nabla^*_Y Z). \end{aligned}$$

By interchanging X and Y in the above equation, we have

$$\begin{aligned} \nabla^{(f)}_Y \nabla^{(f)}_X Z =&Y(f)\nabla^*_X Z - Y(f)\nabla_X Z + (1-f)((1-f)\nabla_Y\nabla_X Z + f\nabla^*_Y\nabla_X Z) \\ &+ f((1-f)\nabla_Y\nabla^*_X Z + f\nabla^*_Y\nabla^*_X Z). \end{aligned}$$

Again, using Definition 2.1 we obtain

$$\nabla^{(f)}_{[X,Y]} Z = (1-f)\nabla_{[X,Y]} Z + f\nabla^*_{[X,Y]} Z.$$

Setting the last three equations in (3.1) and using

$$\nabla_X \nabla^*_Y Z - \nabla_Y \nabla^*_X Z + \nabla^*_X \nabla_Y Z - \nabla^*_Y \nabla_X Z = R(X,Y)Z + R^*(X,Y)Z + \nabla_{[X,Y]}Z + \nabla^*_{[X,Y]}Z + [K_Y, K_X]Z,$$

we conclude the first formula claimed by the proposition. Similarly, the second part is proved. ☐

Applying the above proposition and (2.8)-(2.10), some properties of f-curvature tensors are contained in the following corollaries.

Corollary 3.2. *In a statistical manifold $(M, g, \nabla^{(f)})$, the following formulas hold*

$$\begin{aligned} R^{(f)}(X,Y,Z,W) &= -\, R^{(f)}(Y,X,Z,W), \\ R^{(1-f)}(X,Y,Z,W) &= -\, R^{(1-f)}(Y,X,Z,W), \\ R^{(f)}(X,Y,Z,W) &= -\, R^{(1-f)}(X,Y,W,Z), \end{aligned}$$

where $g(R^{(f)}(X,Y)Z,W) = R^{(f)}(X,Y,Z,W)$, for any $X,Y,Z,W \in \chi(M)$.

Corollary 3.3. *For a statistical manifold $(M, g, \nabla^{(f)})$, we have*

$$R^{(f)}(X,Y)Z - R^{(1-f)}(X,Y)Z = (1-2f)(R(X,Y)Z - R^*(X,Y)Z) + 2X(f)K_Y Z - 2Y(f)K_X Z,$$

for any $X,Y,Z \in \chi(M)$.

A statistical manifold (M, g, ∇) is called *conjugate symmetric* if the curvature tensors of the connections ∇ and ∇^*, are equal, i.e.,

$$R(X,Y)Z = R^*(X,Y)Z,$$

for all $X,Y,Z \in \chi(M)$.

Using the above descriptions, we obtain the following

Theorem 3.4. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. If M is conjugate symmetric, then*

$$\frac{1}{2}(R^{(f)}(X,Y)Z - R^{(1-f)}(X,Y)Z) = X(f)K_Y Z - Y(f)K_X Z, \qquad \forall X,Y,Z \in \chi(M).$$

Remark 3.5. According to the above theorem, it is worth noting that if M is conjugate symmetric with a statistical structure $(g, \nabla^{(f)})$, then $R^{(f)} = R^{(1-f)}$ does not necessarily hold.

Example 3.6. The normal distribution manifold is defined as

$$M_1 = \{p(x,\mu,\sigma) | p(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}, \mu \in \mathbb{R}, \sigma > 0\}.$$

Thus M_1 can be considered as a 2-dimensional manifold with a coordinate system $(\theta^1, \theta^2) = (\mu, \sigma)$. According to (1.1) and (1.2), the components of the Fisher metric g and the non-zero components of $\Gamma^r_{ij} := \Gamma_{ij,k} g^{rk}$ are obtained by

$$(g_{ij}) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix},$$

and

$$\Gamma^1_{12} = \Gamma^1_{21} = -\frac{2}{\sigma}, \qquad \Gamma^2_{22} = -\frac{3}{\sigma}.$$

From (2.4), it follows that $\Gamma^{*r}_{ij} = 0, i,j = 1,2$, except $\Gamma^{*2}_{11} = \Gamma^{*2}_{22} = \frac{1}{\sigma}$. Hence (2.5) yields

$$K^1_{12} = K^1_{21} = \frac{2}{\sigma}, \quad K^2_{11} = \frac{1}{\sigma}, \quad K^2_{22} = \frac{4}{\sigma}.$$

Suppose that $f = f(\mu, \sigma)$ is a function on M. We get the non-zero components of the f-statistical connection $\nabla^{(f)}$ as

$$\Gamma_{12}^{(f)1} = \Gamma_{21}^{(f)1} = -\frac{2}{\sigma}(1 - f(\mu,\sigma)), \qquad \Gamma_{11}^{(f)2} = \frac{f(\mu,\sigma)}{\sigma}, \qquad \Gamma_{22}^{(f)2} = \frac{-3+4f(\mu,\sigma)}{\sigma}.$$

The above equations and (2.3) imply

$$\mathcal{C}_{111}^{(f)} = 0, \quad \mathcal{C}_{121}^{(f)} = \mathcal{C}_{112}^{(f)} = \mathcal{C}_{211}^{(f)} = -\frac{2(2f(\mu,\sigma)-1)}{\sigma 3},$$
$$\mathcal{C}_{122}^{(f)} = \mathcal{C}_{212}^{(f)} = \mathcal{C}_{221}^{(f)} = 0, \quad \mathcal{C}_{222}^{(f)} = -\frac{8(2f(\mu,\sigma)-1)}{\sigma 3},$$

hence $(M_1, g, \nabla^{(f)})$ is a statistical manifold. Using (3.1), we compute the non-zero components of the f-curvature tensor field of M_1 as

$$R_{121}^{(f)1} = \frac{2}{\sigma}\partial_1 f(\mu,\sigma) = -R_{211}^{(f)1}, \qquad R_{122}^{(f)1} = -\frac{4}{\sigma^2} f(\mu,\sigma)(1-f(\mu,\sigma)) - \frac{2}{\sigma}\partial_2 f(\mu,\sigma) = -R_{212}^{(f)1},$$
$$R_{122}^{(f)2} = \frac{4}{\sigma}\partial_1 f(\mu,\sigma) = -R_{212}^{(f)2}, \qquad R_{121}^{(f)2} = \frac{2}{\sigma^2} f(\mu,\sigma)(1-f(\mu,\sigma)) - \frac{1}{\sigma}\partial_2 f(\mu,\sigma) = -R_{211}^{(f)2}.$$

For $f = 0$ and $f = 1$, the above equations imply $R = R^* = 0$. Thus M_1 is a conjugate symmetric manifold. We find $\frac{1}{2}(R_{ijk}^{(f)r} - R_{ijk}^{(1-f)r}) = 0 = \partial_i(f)K_{jk}^r - \partial_j(f)K_{ik}^r, i,j,k,r = 1,2,$ unless

$$\frac{1}{2}(R_{121}^{(f)1} - R_{121}^{(1-f)1}) = \frac{2}{\sigma}\partial_1 f(\mu,\sigma) = \partial_1 f(\mu,\sigma)K_{21}^1 - \partial_2 f(\mu,\sigma)K_{11}^1,$$
$$\frac{1}{2}(R_{122}^{(f)1} - R_{122}^{(1-f)1}) = -\frac{2}{\sigma}\partial_2 f(\mu,\sigma) = \partial_1 f(\mu,\sigma)K_{22}^1 - \partial_2 f(\mu,\sigma)K_{12}^1,$$
$$\frac{1}{2}(R_{121}^{(f)2} - R_{121}^{(1-f)2}) = -\frac{1}{\sigma}\partial_2 f(\mu,\sigma) = \partial_1 f(\mu,\sigma)K_{21}^2 - \partial_2 f(\mu,\sigma)K_{11}^2,$$
$$\frac{1}{2}(R_{122}^{(f)2} - R_{122}^{(1-f)2}) = \frac{4}{\sigma}\partial_1 f(\mu,\sigma) = \partial_1 f(\mu,\sigma)K_{22}^2 - \partial_2 f(\mu,\sigma)K_{12}^2,$$

and these verify Theorem 3.4.

In Proposition 3.1, we used Definition 2.1 to obtain the relationships between the curvature tensors $R^{(f)}$ ($R^{(1-f)}$), R and R^*. Now, considering the equivalent formula given by Corollary 2.3, we present the relationship between the curvature tensors $R^{(f)}$ ($R^{(1-f)}$) and $\widehat{R}$ to study the conditions under which $R^{(f)} = R^{(1-f)}$.

Lemma 3.7. *On a statistical manifold $(M, g, \nabla^{(f)})$, the following identities hold*

$$\begin{aligned} R^{(f)}(X,Y)Z =& \widehat{R}(X,Y)Z + \frac{1-2f}{2}(\widehat{\nabla}_Y K)(X,Z) - \frac{1-2f}{2}(\widehat{\nabla}_X K)(Y,Z) \\ &+ (\frac{1-2f}{2})^2[K_X, K_Y]Z + X(f)K_Y Z - Y(f)K_X Z, \\ R^{(1-f)}(X,Y)Z =& \widehat{R}(X,Y)Z - \frac{1-2f}{2}(\widehat{\nabla}_Y K)(X,Z) + \frac{1-2f}{2}(\widehat{\nabla}_X K)(Y,Z) \\ &+ (\frac{1-2f}{2})^2[K_X, K_Y]Z - X(f)K_Y Z + Y(f)K_X Z, \end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

Proof. The proof of this is similar to the proof of Proposition 3.1. □

Corollary 3.8. *For any $X, Y, Z \in \chi(M)$, we have*

$$\begin{aligned} R^{(f)}(X,Y)Z - R^{(1-f)}(X,Y)Z =& (1-2f)((\widehat{\nabla}_Y K)(X,Z) - (\widehat{\nabla}_X K)(Y,Z)) \\ &+ 2X(f)K_Y Z - 2Y(f)K_X Z, \\ R^{(f)}(X,Y)Z + R^{(1-f)}(X,Y)Z =& 2\widehat{R}(X,Y)Z + \frac{(1-2f)^2}{2}[K_X, K_Y]Z. \end{aligned}$$

Considering Corollaries 3.3 and 3.8, we have the following:

Proposition 3.9. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. Then*

$$R(X,Y)Z - R^*(X,Y)Z = (\widehat{\nabla}_Y K)(X,Z) - (\widehat{\nabla}_X K)(Y,Z),$$

for any $X, Y, Z \in \chi(M)$. Moreover, $R^{(f)} = R^{(1-f)}$ if and only if

$$(\frac{1}{2} - f)((\widehat{\nabla}_Y K)(X,Z) - (\widehat{\nabla}_X K)(Y,Z)) = Y(f)K_X Z - X(f)K_Y Z,$$

for any $X, Y, Z \in \chi(M)$.

Recall that a pseudo-Riemannian manifold (M, g) with a connection ∇ has the constant curvature c if it can be expressed in the form

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \qquad \forall X, Y, Z \in \Im^1_0(M).$$

Theorem 3.10. *[7] A statistical manifold (M, g, ∇) with the constant curvature is a conjugate symmetric manifold.*

Definition 3.11. Let ∇ be an affine connection and K be a tensor of type $(1, 2)$ on M such that $K_X Y = K_Y X$. We say that ∇ and K are Codazzi-coupled if the following identity holds

$$(\nabla_X K)(Y,Z) = (\nabla_Y K)(X,Z),$$

for all $X, Y, Z \in \chi(M)$.

Corollary 3.12. *In a statistical manifold $(M, g, \nabla^{(f)})$, M is conjugate symmetric with respect to the statistical connection ∇ if and only if least one of the following holds*

(1) *$f = \frac{1}{2}$ (in this case $f = \frac{1}{2}$ is covered by the Levi-Civita connection $\widehat{\nabla}$).*
(2) *$(\widehat{\nabla}, K)$ is Codazzi-coupled.*
(3) *$\widehat{\nabla} K$ is zero.*
(4) *M is a flat statistical manifold.*
(5) *the statistical manifold (M, g, ∇) has the constant curvature.*

Applying Corollaries 3.2 and 3.8 and Theorem 3.4, we derive the following:

Theorem 3.13. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. Then $R^{(f)} = R^{(1-f)}$ if least one of the following holds*

(1) *M is a f-flat statistical manifold.*
(2) *M is conjugate symmetric and f is constant.*
(3) *M is conjugate symmetric and $Y(f)K_X Z = X(f)K_Y Z$, for any $X, Y, Z \in \chi(M)$.*

Example 3.14. Assume that M_2 is the set of gamma distributions, that is

$$M_2 = \{p(x;\mu,\nu) |\ p(x;\mu,\nu) = \mu^\nu \frac{x^{\nu-1}}{\Gamma(\nu)} e^{-x\mu},\ \mu,\nu \in \mathbb{R}^+\}.$$

Considering (μ, ν) as a local coordinate system, M_2 can be regarded as a manifold of dimensional 2. Setting $l(x, \mu, \nu) = \ln p(x; \mu, \nu)$ and $(\theta^1, \theta^2) = (\mu, \nu)$, the manifold M_2 admits a Riemannian metric, the Fisher metric g as (1.1). The Fisher metric g has the arc length function

$$ds^2 = \frac{\nu}{\mu^2} d\mu^2 - \frac{2}{\mu} d\mu d\nu + \psi'(\nu) d\nu^2,$$

where $\psi(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}$. To compute the f-statistical connection components, we consider the orthogonal coordinates $(\beta = \frac{\nu}{\mu}, \nu)$. From (1.1) and (1.2) the metric components and non-zero components of statistical connection are given by

$$(g_{ij}) = \begin{pmatrix} \frac{\nu}{\beta^2} & 0 \\ 0 & \psi'(\nu) - \frac{1}{\nu} \end{pmatrix},$$

and

$$\Gamma^1_{11} = -\frac{2}{\beta}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{\nu}.$$

Applying (2.4) and the above equations, it follows

$$\Gamma^{*2}_{11} = -\frac{\nu}{\beta^2(\psi'(\nu)\nu - 1)}, \quad \Gamma^{*2}_{22} = \frac{1+\nu^2\psi''(\nu)}{\nu(\psi'(\nu)\nu - 1)}.$$

Thus (2.5) gives us

$$K^1_{11} = \frac{2}{\beta}, \qquad K^1_{12} = K^1_{21} = -\frac{1}{\nu}, \qquad K^2_{11} = -\frac{\nu}{\beta^2(\psi'(\nu)\nu - 1)}, \qquad K^2_{22} = \frac{1+\nu^2\psi''(\nu)}{\nu(\psi'(\nu)\nu - 1)},$$

while the other independent components are zero. From Corollary 2.3, the non-zero components of $\Gamma^{(f)r}_{ij}$ are obtained by

$$\Gamma^{(f)1}_{11} = -\frac{2(1-f(\beta,\nu))}{\beta}, \qquad \Gamma^{(f)1}_{12} = \Gamma^{(f)1}_{21} = \frac{1-f(\beta,\nu)}{\nu},$$
$$\Gamma^{(f)2}_{11} = -\frac{\nu f(\beta,\nu)}{\beta^2(\psi'(\nu)\nu - 1)}, \qquad \Gamma^{(f)2}_{22} = \frac{(1+\nu^2\psi''(\nu))f(\beta,\nu)}{\nu(\psi'(\nu)\nu - 1)}.$$

The above equations yield

$$\mathcal{C}^{(f)}_{111} = -\frac{2\nu(-1+2f(\beta,\nu))}{\beta^3}, \qquad \mathcal{C}^{(f)}_{121} = \mathcal{C}^{(f)}_{112} = \mathcal{C}^{(f)}_{211} = \frac{-1+2f(\beta,\nu)}{\beta^2},$$
$$\mathcal{C}^{(f)}_{122} = \mathcal{C}^{(f)}_{212} = \mathcal{C}^{(f)}_{221} = 0, \qquad \mathcal{C}^{(f)}_{222} = \frac{(-1+2f(\beta,\nu))(1+\nu^2\psi''(\nu))}{\nu^2},$$

thus $(g, \nabla^{(f)})$ forms a statistical structure on M_2. We also obtain

$$\begin{aligned}
R^{(f)1}_{121} &= -R^{(f)1}_{211} = -\frac{1}{\nu}\partial_1 f(\beta,\nu) - \frac{2}{\beta}\partial_2 f(\beta,\nu),\\
R^{(f)1}_{122} &= -R^{(f)1}_{212} = -\frac{f(\beta,\nu)(f(\beta,\nu)-1)(\psi'(\nu)+\psi''(\nu)\nu) + \partial_2 f(\beta,\nu)(1-\psi'(\nu)\nu)}{\nu(\psi'(\nu)\nu - 1)},\\
R^{(f)2}_{121} &= -R^{(f)2}_{211} = \frac{\nu\{f(\beta,\nu)(f(\beta,\nu)-1)(\psi'(\nu)+\psi''(\nu)\nu) + \partial_2 f(\beta,\nu)(\psi'(\nu)\nu - 1)\}}{\beta^2(\psi'(\nu)\nu - 1)^2},\\
R^{(f)2}_{122} &= -R^{(f)2}_{212} = \frac{\partial_1 f(\beta,\nu)(1+\psi''(\nu)\nu^2)}{\nu(\psi'(\nu)\nu - 1)}.
\end{aligned}$$

So, it results that $\frac{1}{2}(R^{(f)r}_{ijk} - R^{(1-f)r}_{ijk}) = 0 = \partial_i(f)K^r_{jk} - \partial_j(f)K^r_{ik}, i, j, k, r = 1, 2$, except

$$\begin{aligned}
\frac{1}{2}(R^{(f)1}_{121} - R^{(1-f)1}_{121}) &= -\frac{1}{\nu}\partial_1 f(\beta,\nu) - \frac{2}{\beta}\partial_2 f(\beta,\nu) = \partial_1 f(\beta,\nu)K^1_{21} - \partial_2 f(\beta,\nu)K^1_{11},\\
\frac{1}{2}(R^{(f)1}_{122} - R^{(1-f)1}_{122}) &= \frac{1}{\nu}\partial_2 f(\beta,\nu) = \partial_1 f(\beta,\nu)K^1_{22} - \partial_2 f(\beta,\nu)K^1_{12},\\
\frac{1}{2}(R^{(f)2}_{121} - R^{(1-f)2}_{121}) &= \frac{\nu\partial_2 f(\beta,\nu)}{\beta^2(\psi'(\nu)\nu - 1)} = \partial_1 f(\beta,\nu)K^2_{21} - \partial_2 f(\beta,\nu)K^2_{11},\\
\frac{1}{2}(R^{(f)2}_{122} - R^{(1-f)2}_{122}) &= \frac{\partial_1 f(\beta,\nu)(1+\psi''(\nu)\nu^2)}{\nu(\psi'(\nu)\nu - 1)} = \partial_1 f(\beta,\nu)K^2_{22} - \partial_2 f(\beta,\nu)K^2_{12}.
\end{aligned}$$

Moreover, we find

$$\begin{aligned}
\widehat{\nabla}_{\partial_1} K^1_{11} =& - \frac{3}{2(\psi'(\nu)\nu - 1)\beta^2},\\
\widehat{\nabla}_{\partial_1} K^1_{12} =& \widehat{\nabla}_{\partial_1} K^1_{21} = \widehat{\nabla}_{\partial_2} K^1_{11} = -\frac{1}{\beta\nu},\\
\widehat{\nabla}_{\partial_1} K^2_{11} =& - \frac{\nu}{(\psi'(\nu)\nu - 1)\beta^3},\\
\widehat{\nabla}_{\partial_2} K^1_{12} =& \widehat{\nabla}_{\partial_2} K^1_{21} = \widehat{\nabla}_{\partial_1} K^1_{22} = \frac{2\psi'(\nu)\nu - 1 + \psi''(\nu)\nu^2}{2\nu^2(\psi'(\nu)\nu - 1)},\\
\widehat{\nabla}_{\partial_2} K^2_{22} =& \frac{2\psi'''(\nu)\nu^3(\psi'(\nu)\nu - 1) - 3\psi''(\nu)\nu^2(\psi''(\nu)\nu^2 + 2) - 4\psi'(\nu)\nu + 1}{2\nu^2(\psi'(\nu)\nu - 1)^2}.
\end{aligned}$$

As $R^r_{ijk} = 0 = R^{*r}_{ijk}$ we conclude M_2 is conjugate symmetric and flat manifold. Thus it follows that $R^r_{ijk} - R^{*r}_{ijk} = 0 = \widehat{\nabla}_{\partial_i} K^r_{jk} - \widehat{\nabla}_{\partial_j} K^r_{ik}$, $i, j, k, r = 1, 2$ and $(\widehat{\nabla}, K)$ is Codazzi-coupled. Considering f as a constant, we get $R^{(f)} = R^{(1-f)}$. Hence, we have Proposition 3.9, Corollary 3.12 and Theorem 3.13.

Example 3.15. The normal statistical manifold M_1 in Example 3.6, is a flat statistical manifold. It is easily seen that $\nabla_i K^r_{jk} = 0, i, j, k, r = 1, 2$, except $\nabla_1 K^1_{11} = -\frac{3}{\sigma^2}$ which give $(\widehat{\nabla}, K)$ is Codazzi-coupled. Furthermore, if f is constant, we conclude that $R^{(f)} = R^{(1-f)}$ because

$$R^{(f)1}_{122} = -\frac{4}{\sigma^2} f(\mu, \sigma)(1 - f(\mu, \sigma)) = R^{(1-f)1}_{122}, \qquad R^{(f)2}_{121} = \frac{2}{\sigma^2} f(\mu, \sigma)(1 - f(\mu, \sigma)) = R^{(1-f)2}_{121}.$$

The Ricci curvature tensor $Ric^{(f)}$ of the f-connection $\nabla^{(f)}$ is defined by

$$Ric^{(f)}(Y, Z) = tr\{X \to R^{(f)}(X, Y)Z\}.$$

Similarly, the Ricci curvature tensor $Ric^{(1-f)}$ of $\nabla^{(1-f)}$ can be described analogously.

Proposition 3.16. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. Then we have*

$$\begin{aligned}
Ric^{(f)}(Y, Z) =& \widehat{Ric}(Y, Z) + \frac{(1 - 2f)}{2}((\widehat{\nabla}_Y \tau_g)Z - (div^{\widehat{\nabla}} K)(Y, Z)) \qquad (3.3)\\
&+ (\frac{1 - 2f}{2})^2(\tau_g(K_Y Z) - g(K_Y, K_Z)) + K_Y Z(f) - Y(f)\tau_g(Z),\\
Ric^{(1-f)}(Y, Z) =& \widehat{Ric}(Y, Z) - \frac{(1 - 2f)}{2}((\widehat{\nabla}_Y \tau_g)Z - (div^{\widehat{\nabla}} K)(Y, Z)) \qquad (3.4)\\
&+ (\frac{1 - 2f}{2})^2(\tau_g(K_Y Z) - g(K_Y, K_Z)) - K_Y Z(f) + Y(f)\tau_g(Z),
\end{aligned}$$

where $\widehat{Ric}$ is the Ricci tensor on M with respect to the Levi-Civita connection $\widehat{\nabla}$, for any $Y, Z \in \chi(M)$.

Proof. Assume that $p \in M$ and $\{e_i\}^n_{i=1}$ is an orthonormal basis around p such that $\widehat{\nabla} e_i = 0$ at p. According to Lemma 3.7 and the definition of the Ricci curvature tensor $Ric^{(f)}$, we can write

$$\begin{aligned}
Ric^{(f)}(Y, Z) =& \widehat{Ric}(Y, Z) + \sum_{i=1}^{n} \{\frac{1 - 2f}{2} g((\widehat{\nabla}_Y K)(e_i, Z), e_i) - \frac{1 - 2f}{2} g((\widehat{\nabla}_{e_i} K)(Y, Z), e_i) \qquad (3.5)\\
&+ (\frac{1 - 2f}{2})^2 g([K_{e_i}, K_Y]Z, e_i) + e_i(f) g(K_Y Z, e_i) - Y(f) g(K_{e_i} Z, e_i)\}.
\end{aligned}$$

The definition of the divergence operator and 1-form τ_g lead to

$$\sum_{i=1}^{n} g((\widehat{\nabla}_{e_i}K)(Y,Z),e_i) = (div^{\widehat{\nabla}}K)(Y,Z),$$
$$\sum_{i=1}^{n} g([K_{e_i},K_Y]Z,e_i) = \tau_g(K_Y Z) - g(K_Y,K_Z),$$
$$\sum_{i=1}^{n}(e_i(f)g(K_Y Z,e_i) - Y(f)g(K_{e_i}Z,e_i)) = K_Y Z(f) - Y(f)\tau_g(Z).$$

Considering $Y,Z \in T_pM$, we can extend the vectors to vector fields, say Y,Z around p such that $\widehat{\nabla}Y = \widehat{\nabla}Z = 0$ at p. Thus we get

$$\sum_{i=1}^{n} g((\widehat{\nabla}_Y K)(e_i,Z),e_i) = \sum_{i=1}^{n} Yg(K_{e_i}Z,e_i) = Y\tau_g(Z) = (\widehat{\nabla}_Y\tau_g)Z.$$

Setting the above four equations in (3.5), we deduce (3.3). Similarly, (3.4) follows. □

From (3.3) and (3.4), it follows

$$Ric^{(f)}(Y,Z) + Ric^{(1-f)}(Y,Z) = 2\widehat{Ric}(Y,Z) + \frac{(1-2f)^2}{2}\big(\tau_g(K_Y Z) - g(K_Y,K_Z)\big). \quad (3.6)$$

Assume that $(g,\nabla^{(f)})$ is trace-free and g is positive definite. The above equation implies

$$Ric^{(f)}(X,X) + Ric^{(1-f)}(X,X) \leq 2\widehat{Ric}(X,X).$$

Moreover, we get

$$Ric^{(f)}(Y,Z) - Ric^{(f)}(Z,Y) = \frac{(1-2f)}{2}d\tau_g(Y,Z) + Z(f)\tau_g(Y) - Y(f)\tau_g(Z), \quad (3.7)$$

where $d\tau_g(Y,Z) = (\widehat{\nabla}_Y\tau_g)Z - (\widehat{\nabla}_Z\tau_g)Y$. Therefore, $Ric^{(f)}$ is symmetric if and only if

$$\frac{(1-2f)}{2}d\tau_g(Y,Z) = Y(f)\tau_g(Z) - Z(f)\tau_g(Y). \quad (3.8)$$

Lemma 3.17. *Let $(M,g,\nabla^{(f)})$ be a statistical manifold and f be a constant function. Then the Ricci curvature tensor $Ric^{(f)}$ is symmetric if and only if at least one of the following holds*

(1) $f = \frac{1}{2}$ *(in this case $\nabla^{(f)}\omega_g = \widehat{\nabla}\omega_g = 0$);*
(2) $\nabla^{(f)}$ *is equiaffine.*

Proof. To prove, we first let $Ric^{(f)}$ be symmetric. (3.8) implies $f = \frac{1}{2}$ or $d\tau_g = 0$. If $d\tau_g = 0$, one can find a function ψ such that $d\log\psi = -\tau_g$. It is fairly easy to see that the volume element $\psi\omega_g$ satisfies $\nabla^{(f)}\psi\omega_g = 0$. Thus in this case, $\nabla^{(f)}$ is equiaffine. The converse is obvious. □

Proposition 3.18. *For a statistical manifold $(M,g,\nabla^{(f)})$, we have*

$$Ric(Y,Z) - Ric(Z,Y) = \frac{1}{2}d\tau_g(Y,Z), \quad (3.9)$$
$$Ric^*(Y,Z) - Ric^*(Z,Y) = -\frac{1}{2}d\tau_g(Y,Z), \quad (3.10)$$

where Ric and Ric^* are the Ricci tensors associated with the statistical connections ∇ and ∇^*, for any $Y,Z \in \chi(M)$.

Proof. From Proposition 3.1 and the definition of the Ricci curvature tensor $Ric^{(f)}$, we get

$$\begin{aligned} Ric^{(f)}(Y,Z) =&(1-f)Ric(Y,Z) + fRic^*(Y,Z) - f(1-f)\big(\tau_g(K_Y Z) - g(K_Y, K_Z)\big) \\ &+ K_Y Z(f) - Y(f)\tau_g(Z), \end{aligned} \tag{3.11}$$

which gives

$$\begin{aligned} Ric^{(f)}(Y,Z) - Ric^{(f)}(Z,Y) =&(1-f)(Ric(Y,Z) - Ric(Z,Y)) + f(Ric^*(Y,Z) - Ric^*(Z,Y)) \\ &+ Z(f)\tau_g(Y) - Y(f)\tau_g(Z). \end{aligned}$$

Considering $f = \frac{1}{2}$ in the above equation it follows

$$Ric(Y,Z) - Ric(Z,Y) = Ric^*(Z,Y) - Ric^*(Y,Z). \tag{3.12}$$

The last two equations imply

$$Ric^{(f)}(Y,Z) - Ric^{(f)}(Z,Y) =(1-2f)(Ric(Y,Z) - Ric(Z,Y)) + Z(f)\tau_g(Y) - Y(f)\tau_g(Z).$$

(3.7) and the above equation yield (3.9). From (3.9) and (3.12), we have (3.10). □

Corollary 3.19. *In a statistical manifold $(M, g, \nabla^{(f)})$, the following conditions are equivalent:*

(1) *the Ricci tensor Ric is symmetric;*
(2) *the Ricci tensor Ric^* is symmetric.*

Moreover, if f is constant, the Ricci tensor $Ric^{(f)}$ is symmetric.

Proposition 3.20. *The Ricci tensors $Ric^{(f)}$ and $Ric^{(1-f)}$ are related by*

$$Ric^{(f)}(Y,Z) - Ric^{(1-f)}(Y,Z) =(1-2f)\big(Ric(Y,Z) - Ric^*(Y,Z)\big) + 2K_Y Z(f) - 2Y(f)\tau_g(Z),$$

for any $Y, Z \in \chi(M)$.

Proof. Similar to (3.11), from Proposition 3.1, it follows

$$\begin{aligned} Ric^{(1-f)}(Y,Z) =&fRic(Y,Z) + (1-f)Ric^*(Y,Z) - f(1-f)\big(\tau_g(K_Y Z) - g(K_Y, K_Z)\big) \\ &- K_Y Z(f) + Y(f)\tau_g(Z). \end{aligned}$$

Subtracting (3.11) and the above equation, we obtain the assertion. □

From Propositions 3.16 and 3.20, we deduce the following:

Corollary 3.21. *For a statistical manifold $(M, g, \nabla^{(f)})$, the following holds*

$$Ric(Y,Z) - Ric^*(Y,Z) = (\widehat{\nabla}_Y \tau_g)Z - (div^{\widehat{\nabla}} K)(Y,Z),$$

for any $Y, Z \in \chi(M)$.

A statistical manifold (M, g, ∇) is called *conjugate Ricci-symmetric* if

$$Ric(Y,Z) = Ric^*(Y,Z), \qquad \forall Y, Z \in \chi(M).$$

Proposition 3.22. *Let $(g, \nabla^{(f)})$ be a statistical structure on a manifold M. If(M, g, ∇) is a conjugate Ricci-symmetric manifold, then we have*

(1) *the Ricci curvature tensors Ric and Ric^* are symmetric;*
(2) *$\nabla^{(f)}$ is equiaffine;*
(3) *$\widehat{\nabla}\tau_g = div^{\widehat{\nabla}} K$;*
(4) *for any $X, Y, Z \in \chi(M)$,*

$$Ric^{(f)}(Y,Z) - Ric^{(1-f)}(Y,Z) =2K_Y Z(f) - 2Y(f)\tau_g(Z).$$

Moreover, if f is constant, $Ric^{(f)} = Ric^{(1-f)}$.

Proof. As $Ric = Ric^*$, (3.12) implies (1). From (3.9) and (1), we get $d\tau_g = 0$ which is equivalent to (2). Using Proposition 3.20 and Corollary 3.21, (3) and (4) follow. □

Example 3.23. Consider the statistical manifold $(M_1, g, \nabla^{(f)})$ in Example 3.6. As $R = R^* = 0$, it follows $Ric = Ric^* = 0$, thus (M_1, g, ∇) is a conjugate Ricci-symmetric manifold. We see that $\tau_g(\partial_1) = trK_{\partial_1} = 0$ and $\tau_g(\partial_2) = trK_{\partial_2} = \frac{6}{\sigma}$, so $(\widehat{\nabla}_{\partial_i}\tau_g)\partial_j = 0 = (div^{\widehat{\nabla}}K)(\partial_i, \partial_j), i, j = 1, 2$, except

$$(\widehat{\nabla}_{\partial_1}\tau_g)\partial_1 = -\frac{3}{\sigma^2} = (div^{\widehat{\nabla}}K)(\partial_1, \partial_1).$$

We also conclude $d\tau_g = 0$. Hence $\nabla^{(f)}$ is equiaffine. The Ricci tensor $Ric^{(f)}$ is given by

$$(Ric^{(f)}(\partial_i, \partial_j)) = \begin{pmatrix} -\frac{2}{\sigma^2}f(\mu,\sigma)(1-f(\mu,\sigma)) + \frac{1}{\sigma}\partial_2 f(\mu,\sigma) & -\frac{4}{\sigma}\partial_1 f(\mu,\sigma) \\ \frac{2}{\sigma}\partial_1 f(\mu,\sigma) & -\frac{4}{\sigma^2}f(\mu,\sigma)(1-f(\mu,\sigma)) - \frac{2}{\sigma}\partial_2 f(\mu,\sigma) \end{pmatrix}.$$

It is easy to check that

$$\begin{aligned}
Ric^{(f)}(\partial_1,\partial_1) - Ric^{(1-f)}(\partial_1,\partial_1) &= \frac{2}{\sigma}\partial_2 f(\mu,\sigma) = 2(K_{\partial_1}\partial_1)f(\mu,\sigma) - 2\partial_1 f(\mu,\sigma)\tau_g(\partial_1),\\
Ric^{(f)}(\partial_1,\partial_2) - Ric^{(1-f)}(\partial_1,\partial_2) &= -\frac{8}{\sigma}\partial_1 f(\mu,\sigma) = 2(K_{\partial_1}\partial_2)f(\mu,\sigma) - 2\partial_1 f(\mu,\sigma)\tau_g(\partial_2),\\
Ric^{(f)}(\partial_2,\partial_1) - Ric^{(1-f)}(\partial_2,\partial_1) &= \frac{4}{\sigma}\partial_1 f(\mu,\sigma) = 2(K_{\partial_2}\partial_1)f(\mu,\sigma) - 2\partial_2 f(\mu,\sigma)\tau_g(\partial_1),\\
Ric^{(f)}(\partial_2,\partial_2) - Ric^{(1-f)}(\partial_2,\partial_2) &= -\frac{4}{\sigma}\partial_2 f(\mu,\sigma) = 2(K_{\partial_2}\partial_2)f(\mu,\sigma) - 2\partial_2 f(\mu,\sigma)\tau_g(\partial_2).
\end{aligned}$$

Therefore $Ric^{(f)} = Ric^{(1-f)}$ if f is constant. Thus we have Proposition 3.22.

Example 3.24. The Ricci curvature tensor $Ric^{(f)}$ of the statistical manifold $(M_2, g, \nabla^{(f)})$ described in Example 3.14 is obtained by

$$\begin{aligned}
Ric^{(f)}(\partial_1,\partial_1) &= -\frac{\nu\{f(\beta,\nu)(f(\beta,\nu)-1)(\psi'(\nu)+\psi''(\nu)\nu) + \partial_2 f(\beta,\nu)(\psi'(\nu)\nu-1)\}}{\beta^2(\psi'(\nu)\nu-1)^2},\\
Ric^{(f)}(\partial_1,\partial_2) &= -\frac{\partial_1 f(\beta,\nu)(1+\psi''(\nu)\nu^2)}{\nu(\psi'(\nu)\nu-1)},\\
Ric^{(f)}(\partial_2,\partial_1) &= -\frac{1}{\nu}\partial_1 f(\beta,\nu) - \frac{2}{\beta}\partial_2 f(\beta,\nu),\\
Ric^{(f)}(\partial_2,\partial_2) &= -\frac{f(\beta,\nu)(f(\beta,\nu)-1)(\psi'(\nu)+\psi''(\nu)\nu) + \partial_2 f(\beta,\nu)(1-\psi'(\nu)\nu)}{\nu(\psi'(\nu)\nu-1)}.
\end{aligned}$$

For $f = 0$ and $f = 1$, the above equations imply $Ric = Ric^* = 0$. Thus (M_2, g, ∇) is a conjugate Ricci-symmetric manifold. We also obtain

$$\tau_g(\partial_1) = \frac{2}{\beta} = trK_{\partial_1}, \qquad \tau_g(\partial_2) = -\frac{\psi'(\nu)\nu - 2 - \psi''(\nu)\nu^2}{\nu(\psi'(\nu)\nu-1)} = trK_{\partial_2}.$$

Hence we see that

$$\begin{aligned}
(\widehat{\nabla}_{\partial_1}\tau_g)\partial_1 &= -\frac{\psi'(\nu)\nu - 2 - \psi''(\nu)\nu^2}{2\beta^2(\psi'(\nu)\nu-1)} = (div^{\widehat{\nabla}}K)(\partial_1,\partial_1),\\
(\widehat{\nabla}_{\partial_1}\tau_g)\partial_2 &= (\widehat{\nabla}_{\partial_2}\tau_g)\partial_1 = -\frac{1}{\nu\beta} = (div^{\widehat{\nabla}}K)(\partial_1,\partial_2) = (div^{\widehat{\nabla}}K)(\partial_2,\partial_1),\\
(\widehat{\nabla}_{\partial_2}\tau_g)\partial_2 &= -\frac{\{\psi''(\nu)\nu^2(7-\psi'(\nu)\nu+3\psi''(\nu)\nu^2)+\psi'(\nu)\nu(7-2\psi'(\nu)\nu)+2\psi'''(\nu)\nu^3(1-\psi'(\nu)\nu)-2\}}{2\nu^2(\psi'(\nu)\nu-1)^2}\\
&= (div^{\widehat{\nabla}}K)(\partial_2,\partial_2).
\end{aligned}$$

So it follows that $d\tau_g = 0$ and $\nabla^{(f)}$ is equiaffine. Moreover, we find

$$Ric^{(f)}(\partial_1,\partial_1) - Ric^{(1-f)}(\partial_1,\partial_1) = -\frac{2\nu\partial_2 f(\beta,\nu)}{\beta^2(\psi'(\nu)\nu-1)} = 2(K_{\partial_1}\partial_1)f(\beta,\nu) - 2\partial_1 f(\beta,\nu)\tau_g(\partial_1),$$

$$Ric^{(f)}(\partial_1,\partial_2) - Ric^{(1-f)}(\partial_1,\partial_2) = -\frac{2\partial_1 f(\beta,\nu)(\psi''(\nu)\nu^2+1)}{\nu(\psi'(\nu)\nu-1)} = 2(K_{\partial_1}\partial_2)f(\beta,\nu) - 2\partial_1 f(\beta,\nu)\tau_g(\partial_2),$$

$$Ric^{(f)}(\partial_2,\partial_1) - Ric^{(1-f)}(\partial_2,\partial_1) = -\frac{2}{\nu}\partial_1 f(\beta,\nu) - \frac{2}{\beta}\partial_2 f(\beta,\nu) = 2(K_{\partial_2}\partial_1)f(\beta,\nu) - 2\partial_2 f(\beta,\nu)\tau_g(\partial_1),$$

$$Ric^{(f)}(\partial_2,\partial_2) - Ric^{(1-f)}(\partial_2,\partial_2) = \frac{2}{\nu}\partial_2 f(\beta,\nu) = 2(K_{\partial_2}\partial_2)f(\beta,\nu) - 2\partial_2 f(\beta,\nu)\tau_g(\partial_2).$$

Considering f as a constant in the last equations, we deduce $Ric^{(f)} = Ric^{(1-f)}$.

Let $(M, g, \nabla^{(f)})$ be a statistical manifold. We consider a tensor field $S^{(f)}$ of type $(1,3)$ on M given by

$$S^{(f)}(X,Y)Z = \frac{1}{2}\{R^{(f)}(X,Y)Z + R^{(1-f)}(X,Y)Z\}.$$

The tensor field $S^{(f)}$ is called *the statistical curvature tensor field* of $(g, \nabla^{(f)})$. Corollary 3.8 implies

$$S^{(f)}(X,Y)Z = \widehat{R}(X,Y)Z + \frac{(1-2f)^2}{4}[K_X, K_Y]Z.$$

From the above equation, one can see that

$$\begin{aligned} S^{(f)}(X,Y,Z,W) &= -S^{(f)}(Y,X,Z,W),\\ S^{(f)}(X,Y,Z,W) &= -S^{(f)}(X,Y,W,Z),\\ S^{(f)}(X,Y,Z,W) &= S^{(f)}(Z,W,X,Y), \end{aligned}$$

where $S^{(f)}(X,Y,Z,W) = g(S^{(f)}(X,Y)Z,W)$, for any $X,Y,Z,W \in \chi(M)$. We set

$$L^{(f)}(X,Y) = tr\{X \to S^{(f)}(X,Y)Z\} = \frac{1}{2}\{Ric^{(f)}(X,Y)Z + Ric^{(1-f)}(X,Y)Z\},$$

which is called *the statistical Ricci curvature tensor*. Shortly, we denote $S^{(0)}$, $S^{(1)}$, $L^{(0)}$ and $L^{(1)}$ by S, S^*, L and L^*, respectively. (3.6) leads to

$$L^{(f)}(X,Y) = \widehat{Ric}(X,Y) + \frac{(1-2f)^2}{4}(\tau_g(K_X Y) - g(K_X, K_Y)). \tag{3.13}$$

The last equation implies that the statistical Ricci curvature tensor $L^{(f)}$ is symmetric, i.e., $L^{(f)}(X,Y) = L^{(f)}(Y,X)$. It is also obvious that $L = L^*$. Moreover, (3.11) implies

$$L^{(f)}(X,Y) = L(X,Y) - f(1-f)(\tau_g(K_X Y) - g(K_X, K_Y)). \tag{3.14}$$

Example 3.25. Let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 | \Pi_{i=1}^2 x_i > 0\}$ and $\mathbb{R}^2_+ = \{x = (x_1,x_2) \in \mathbb{R}^2 | x_i > 0, i = 1,2\}$. A 2-dimensional statistical manifold is defined by

$$M_3 = \Big\{ f(\mathbf{x};\lambda) | f(\mathbf{x};\lambda) = 2\Pi_{i=1}^2 \frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} e^{-\frac{\lambda_i x_i^2}{2}}, \mathbf{x} \in \Omega,\ \lambda \in \mathbb{R}^2_+ \Big\}.$$

The distribution in M_3 can be rewrite as

$$f(\mathbf{x};\lambda) = e^{\frac{1}{2}} \sum_{i=1}^{2} \log(-\theta_i) + \sum_{i=1}^{2} \theta_i x_i^2 + \log 2 - \log\sqrt{2\pi},$$

where $\theta_i = -\frac{1}{2}\lambda_i$. This is one member of the exponential family with the natural coordinates (θ_1, θ_2) and the potential function $\psi = -\frac{1}{2}\sum_{i=1}^{2} \log(-\theta_i)$. It is known that for the exponential family, the Fisher information is just the second derivative of the potential function

$$g_{ij} = \frac{\partial^2 \psi}{\partial\theta_i \partial\theta_j} = -\frac{1}{2}\frac{1}{\theta_i\theta_j}\delta_{ij}.$$

The matrix expression of metric g given by the above equation is as follows:

$$g = (g_{ij}) = \begin{pmatrix} -\frac{1}{2\theta_1^2} & 0 \\ 0 & -\frac{1}{2\theta_2^2} \end{pmatrix}. \tag{3.15}$$

The non-zero components $\widehat{\Gamma}_{ij}^k$ of the the Levi-Civita connection $\widehat{\nabla}$ are given by

$$\widehat{\Gamma}_{11}^1 = -\frac{1}{\theta_1}, \quad \widehat{\Gamma}_{22}^2 = -\frac{1}{\theta_2}. \tag{3.16}$$

Considering $K_{ij}^r = 0, r, i, j = 1, 2$, except $K_{11}^1 = -\frac{2}{\theta_1}$ and $K_{22}^2 = -\frac{2}{\theta_2}$, we get

$$\Gamma_{11}^{(f)1} = -\frac{2}{\theta_1} f(\theta_1, \theta_2), \qquad \Gamma_{22}^{(f)2} = -\frac{2}{\theta_2} f(\theta_1, \theta_2).$$

It is easy to check that

$$\mathfrak{C}_{111}^{(f)} = \frac{1 - 2f(\theta_1, \theta_2)}{\theta_1^3}, \quad \mathfrak{C}_{121}^{(f)} = \mathfrak{C}_{112}^{(f)} = \mathfrak{C}_{211}^{(f)} = \mathfrak{C}_{122}^{(f)} = \mathfrak{C}_{212}^{(f)} = \mathfrak{C}_{221}^{(f)} = 0, \quad \mathfrak{C}_{222}^{(f)} = \frac{1 - 2f(\theta_1, \theta_2)}{\theta_2^3},$$

thus $(M, g, \nabla^{(f)})$ forms a statistical manifold. By definition, the non-zero components of the f-curvature tensor are determined by

$$R_{121}^{(f)1} = \frac{2}{\theta_1}\partial_2 f(\theta_1, \theta_2) = -R_{211}^{(f)1}, \qquad R_{122}^{(f)2} = -\frac{2}{\theta_2}\partial_1 f(\theta_1, \theta_2) = -R_{212}^{(f)2},$$

which gives $\frac{1}{2}(R_{ijk}^{(f)r} - R_{ijk}^{(1-f)r}) = 0 = \partial_i(f)K_{jk}^r - \partial_j(f)K_{ik}^r, i, j, k, r = 1, 2$, except

$$\frac{1}{2}(R_{121}^{(f)1} - R_{121}^{(1-f)1}) = \frac{2}{\theta_1}\partial_2 f(\theta_1, \theta_2) = \partial_1 f(\theta_1, \theta_2)K_{21}^1 - \partial_2 f(\theta_1, \theta_2)K_{11}^1,$$
$$\frac{1}{2}(R_{122}^{(f)2} - R_{122}^{(1-f)2}) = -\frac{2}{\theta_2}\partial_1 f(\theta_1, \theta_2) = \partial_1 f(\theta_1, \theta_2)K_{22}^2 - \partial_2 f(\theta_1, \theta_2)K_{12}^2.$$

We get the components of the Ricci curvature tensor $Ric^{(f)}$ as

$$(Ric^{(f)}(\partial_i, \partial_j)) = \begin{pmatrix} 0 & \frac{2}{\theta_2}\partial_1 f(\theta_1, \theta_2) \\ \frac{2}{\theta_1}\partial_2 f(\theta_1, \theta_2) & 0 \end{pmatrix}.$$

Since $\tau_g(\partial_1) = -\frac{2}{\beta}$ and $\tau_g(\partial_2) = -\frac{2}{\nu}$, it follows that

$$Ric^{(f)}(\partial_i, \partial_j) - Ric^{(1-f)}(\partial_i, \partial_j) = 0 = 2K_{ij}^r \partial_r f(\theta_1, \theta_2) - 2\partial_i f(\theta_1, \theta_2)\tau_g(\partial_j), i, j, r = 1, 2,$$

unless

$$Ric^{(f)}(\partial_1, \partial_2) - Ric^{(1-f)}(\partial_1, \partial_2) = \frac{4}{\theta_2}\partial_1 f(\theta_1, \theta_2) = 2(K_{\partial_1}\partial_2)f(\theta_1, \theta_2) - 2\partial_1 f(\theta_1, \theta_2)\tau_g(\partial_2),$$
$$Ric^{(f)}(\partial_2, \partial_1) - Ric^{(1-f)}(\partial_2, \partial_1) = \frac{4}{\theta_1}\partial_2 f(\theta_1, \theta_2) = 2(K_{\partial_2}\partial_1)f(\theta_1, \theta_2) - 2\partial_2 f(\theta_1, \theta_2)\tau_g(\partial_1).$$

As (M_3, g, ∇) is flat, we deduce that it is a conjugate symmetric and a conjugate Ricci-symmetric manifold. We obtain $\widehat{\nabla}K = 0$, $\widehat{\nabla}\tau_g = 0 = div^{\widehat{\nabla}}K$. According to the above description, if f is constant, we have $R^{(f)} = R^{(1-f)}$ and $Ric^{(f)} = Ric^{(1-f)}$. Moreover, we conclude that $S^{(f)} = 0$ and $L^{(f)} = 0$.

4. Hessian and Laplacian operators associated with f-statistical connections

Assume that ∇ be an affine connection on a manifold M. A tensor field H^{∇}_{φ} of type $(0,2)$ on M is called *Hessian* of a function $\varphi \in C^{\infty}(M)$ with respect to the connection ∇ if

$$H^{\nabla}_{\varphi}(X,Y) = (\nabla_X d\varphi)Y, \qquad \forall X,Y \in \chi(M), \tag{4.1}$$

where

$$(\nabla_X d\varphi)Y = Xd\varphi(Y) - d\varphi(\nabla_X Y) = XY(\varphi) - (\nabla_X Y)\varphi. \tag{4.2}$$

Let $(M,g,\nabla^{(f)})$ be a statistical manifold. We denote the Hessian $H^{\nabla^{(f)}}_{\varphi}$ by $H^{(f)}_{\varphi}$. For $f = 0,1$ and $f = \frac{1}{2}$, we use the notations $H_{\varphi}, H^{*}_{\varphi}$ and $\widehat{H}_{\varphi}$, respectively. Applying Corollary 2.3 and (4.2), the tensor field $H^{(f)}_{\varphi}$ can be expressed as

$$H^{(f)}_{\varphi}(X,Y) = XY(\varphi) - (\widehat{\nabla}_X Y)\varphi + \frac{1-2f}{2}(K_X Y)\varphi.$$

In the local coordinates this becomes

$$H^{(f)}_{\varphi\, ij} = \partial_i\partial_j\varphi - \widehat{\Gamma}^k_{ij}\partial_k\varphi + \frac{1-2f}{2}K^k_{ij}\partial_k\varphi. \tag{4.3}$$

It is clear that $H^{(f)}_{\varphi\, ij}$ is symmetric. In addition, setting $\varphi = f$ in the above equation, it follows

$$H^{(f)}_{f\, ij} = \partial_i\partial_j f - \widehat{\Gamma}^k_{ij}\partial_k f - \frac{\partial_k(1-2f)^2}{8}K^k_{ij}.$$

According to (4.1), the dual Hessian $H^{(1-f)}_{\varphi}$ is given by

$$H^{(1-f)}_{\varphi}(X,Y) = (\nabla^{(1-f)}_X d\varphi)Y.$$

Corollary 4.1. *The Hessians $H^{(f)}_{\varphi}$ and $H^{(1-f)}_{\varphi}$ satisfy the following*

$$H^{(f)}_{\varphi}(X,Y) = (1-f)H^{(0)}_{\varphi}(X,Y) + fH^{(1)}_{\varphi}(X,Y), \tag{4.4}$$

$$H^{(1-f)}_{\varphi}(X,Y) = fH^{(0)}_{\varphi}(X,Y) + (1-f)H^{(1)}_{\varphi}(X,Y), \tag{4.5}$$

for any $X,Y \in \chi(M)$.

Proposition 4.2. *Let $(M,g,\nabla^{(f)})$ be a statistical manifold. Then we have*

$$H^{(f)}_{\varphi}(X,Y) = g(\nabla^{(1-f)}_X(grade\varphi),Y), \tag{4.6}$$

$$H^{(1-f)}_{\varphi}(X,Y) = g(\nabla^{(f)}_X(grade\varphi),Y), \tag{4.7}$$

where $grade\varphi$ is the gradient vector field of φ, for all $X,Y \in \chi(M)$.

Proof. As $g(grade\varphi, X) = X(\varphi)$ for any $X \in \chi(M)$, so (4.2) yields

$$H^{(f)}_{\varphi}(X,Y) = XY(\varphi) - (\nabla^{(f)}_X Y)\varphi = Xg(grade\varphi,Y) - g(grade\varphi, \nabla^{(f)}_X Y).$$

Applying Proposition 2.2 in the above equation, we get

$$\begin{aligned} H^{(f)}_{\varphi}(X,Y) &= g(\nabla^{(1-f)}_X grade\varphi, Y) + g(grade\varphi, \nabla^{(f)}_X Y) - g(grade\varphi, \nabla^{(f)}_X Y) \\ &= g(\nabla^{(1-f)}_X grade\varphi, Y), \end{aligned}$$

which gives (4.6). Similarly, we get (4.7). □

The above expressions, (2.5) and Definition 2.1 lead to the following formulas of $H^{(f)}_{\varphi}$ and $H^{(1-f)}_{\varphi}$:

Corollary 4.3. *The tensors $H_\varphi^{(f)}$ and $H_\varphi^{(1-f)}$ can be written in the following forms*

$$H_\varphi^{(f)}(X,Y) = g(\nabla_X(grade\varphi), Y) + (1-f)(K_XY)\varphi,$$
$$H_\varphi^{(1-f)}(X,Y) = g(\nabla_X(grade\varphi), Y) + f(K_XY)\varphi,$$

for any $X, Y \in \chi(M)$.

Proposition 4.4. *On a statistical manifold $(M, g, \nabla^{(f)})$, the following holds*

$$H_\varphi^{(f)}(X,Y) - H_\varphi^{(1-f)}(X,Y) = (1-2f)(\nabla_X g)(grade\varphi, Y), \quad \forall X, Y \in \chi(M).$$

Proof. Substituting the two terms in Corollary 4.3, we obtain

$$H_\varphi^{(f)}(X,Y) - H_\varphi^{(1-f)}(X,Y) = (1-2f)(K_XY)\varphi = (1-2f)g(K_XY, grade\varphi). \tag{4.8}$$

As $g(K_XY, grade\varphi) = (\nabla_X g)(grade\varphi, Y)$, we conclude the assertion. □

Corollary 4.5. *We have*

$$H_\varphi^{(f)}(X,Y) + H_\varphi^{(1-f)}(X,Y) = 2H_\varphi^*(X,Y) + (K_XY)\varphi = 2H_\varphi(X,Y) - (K_XY)\varphi,$$
$$\widehat{H}_\varphi(X,Y) = H_\varphi^*(X,Y) + \frac{1}{2}(K_XY)\varphi = H_\varphi(X,Y) - \frac{1}{2}(K_XY)\varphi,$$

for any $X, Y \in \chi(M)$.

4.1. Codazzi Coupling of f-statistical connections $\nabla^{(f)}$ with $H_\varphi^{(f)}$

Let (M, g, ∇) be a statistical manifold and $\nabla^{(f)}$ be the f-statistical connection induced by ∇. For any $X, Y, Z \in \chi(M)$, we have

$$(\nabla_X^{(f)} H_\varphi^{(f)})(Y,Z) = XH_\varphi^{(f)}(Y,Z) - H_\varphi^{(f)}(\nabla_X^{(f)}Y, Z) - H_\varphi^{(f)}(Y, \nabla_X^{(f)}Z).$$

In the local coordinates, the above equation has the following form

$$\nabla_{\partial_i}^{(f)} H_{\varphi\ jk}^{(f)} = \partial_i H_{\varphi\ jk}^{(f)} - \Gamma_{ij}^{(f)s} H_{\varphi\ sk}^{(f)} - \Gamma_{ik}^{(f)s} H_{\varphi\ js}^{(f)}.$$

Applying (4.3) in the above equation, it follows

$$\begin{aligned}\nabla_{\partial_i}^{(f)} H_{\varphi\ jk}^{(f)} =& \partial_i\partial_j\partial_k\varphi - \partial_i\Gamma_{jk}^{(f)r}\partial_r\varphi - \Gamma_{jk}^{(f)r}\partial_i\partial_r\varphi - \Gamma_{ij}^{(f)s}(\partial_s\partial_k\varphi - \Gamma_{sk}^{(f)r}\partial_r\varphi) - \Gamma_{ik}^{(f)s}(\partial_s\partial_j\varphi \\ &- \Gamma_{js}^{(f)r}\partial_r\varphi),\end{aligned}$$

which gives us

$$\nabla_{\partial_i}^{(f)} H_{\varphi\ jk}^{(f)} - \nabla_{\partial_j}^{(f)} H_{\varphi\ ik}^{(f)} = -\partial_i\Gamma_{jk}^{(f)r}\partial_r\varphi + \Gamma_{ik}^{(f)s}\Gamma_{js}^{(f)r}\partial_r\varphi + \partial_j\Gamma_{ik}^{(f)r}\partial_r\varphi - \Gamma_{jk}^{(f)s}\Gamma_{is}^{(f)r}\partial_r\varphi.$$

The above equation and (3.2) imply

$$\nabla_{\partial_i}^{(f)} H_{\varphi\ jk}^{(f)} - \nabla_{\partial_j}^{(f)} H_{\varphi\ ik}^{(f)} = -R_{ijk}^{(f)r}\partial_r\varphi.$$

We summarize the above discussions by the following lemma and theorem.

Lemma 4.6. *In a statistical manifold $(M, g, \nabla^{(f)})$, the following holds*

$$(\nabla_X^{(f)} H_\varphi^{(f)})(Y,Z) - (\nabla_Y^{(f)} H_\varphi^{(f)})(X,Z) = -(R^{(f)}(X,Y)Z)(\varphi), \tag{4.9}$$

for any $X, Y, Z \in \chi(M)$. In particular, for $f = 0$ and $f = 1$ we have

$$(\nabla_X H_\varphi)(Y,Z) - (\nabla_Y H_\varphi)(X,Z) = -\,(R(X,Y)Z)(\varphi), \tag{4.10}$$
$$(\nabla_X^* H_\varphi^*)(Y,Z) - (\nabla_Y^* H_\varphi^*)(X,Z) = -\,(R^*(X,Y)Z)(\varphi). \tag{4.11}$$

Theorem 4.7. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. Then $(\nabla^{(f)}, H_\varphi^{(f)})$ is Codazzi-coupled if and only if*

$$(R^{(f)}(X,Y)Z)(\varphi) = 0,$$

for all $X, Y, Z \in \chi(M)$.

From Theorem 4.7, it is obvious that if M is f-flat or φ is constant, then $(\nabla^{(f)}, H_\varphi^{(f)})$ is Codazzi-coupled. The first question that arises is whether the converse is true?

The second question that can arise for anyone here is that if $(\nabla^{(f)}, H_\varphi^{(f)})$ is Codazzi-coupled in a statistical manifold $(M, g, \nabla^{(f)})$, does the pair $(\nabla^{(1-f)}, H_\varphi^{(1-f)})$ carry the same property? In the example below we see that in general the answer to these questions are negative.

Example 4.8. Consider $f(\mu,\sigma) = \frac{\sigma^2}{\sigma^2+c}$ and $\varphi(\mu,\sigma) = \sigma$, for the normal statistical manifold $(M_1, g, \nabla^{(f)})$ described in Example 3.6. Note that, in this case M_1 is not f-flat and φ is not constant. We obtain

$$H_\varphi^{(f)} = (H_{\varphi\ ij}^{(f)}) = \begin{pmatrix} -\frac{\sigma}{\sigma^2+c} & 0 \\ 0 & \frac{-\sigma^2+3c}{\sigma(\sigma^2+c)} \end{pmatrix}.$$

It is a simple matter to check that

$$\nabla_{\partial_1}^{(f)} H_{\varphi\ 11}^{(f)} = 0, \qquad \nabla_{\partial_1}^{(f)} H_{\varphi\ 12}^{(f)} = \nabla_{\partial_1}^{(f)} H_{\varphi\ 21}^{(f)} = \nabla_{\partial_2}^{(f)} H_{\varphi\ 11}^{(f)} = \frac{\sigma^2-5c}{(\sigma^2+c)^2},$$

$$\nabla_{\partial_2}^{(f)} H_{\varphi\ 22}^{(f)} = \frac{3\sigma^4-22\sigma^2c+15c^2}{\sigma^2(\sigma^2+c)^2}, \qquad \nabla_{\partial_1}^{(f)} H_{\varphi\ 22}^{(f)} = \nabla_{\partial_2}^{(f)} H_{\varphi\ 12}^{(f)} = \nabla_{\partial_2}^{(f)} H_{\varphi\ 21}^{(f)} = 0.$$

Thus $(\nabla^{(f)}, H_\varphi^{(f)})$ is Codazzi-coupled. We also see that $R_{ijk}^{(f)r}\partial_r\varphi = 0$, so Theorem 4.7 holds. Moreover, it follows easily that

$$H_\varphi^{(1-f)} = (H_{\varphi\ ij}^{(1-f)}) = \begin{pmatrix} -\frac{c}{\sigma(\sigma^2+c)} & 0 \\ 0 & -\frac{-3\sigma^2+c}{\sigma(\sigma^2+c)} \end{pmatrix}.$$

Hence we get

$$\nabla_{\partial_1}^{(1-f)} H_{\varphi\ 11}^{(1-f)} = 0, \quad \nabla_{\partial_1}^{(1-f)} H_{\varphi\ 12}^{(1-f)} = \nabla_{\partial_1}^{(1-f)} H_{\varphi\ 21}^{(1-f)} = \frac{c(-5\sigma^2+c)}{\sigma^2(\sigma^2+c)^2}, \quad \nabla_{\partial_2}^{(1-f)} H_{\varphi\ 11}^{(1-f)} = \frac{c(-\sigma^2+c)}{\sigma^2(\sigma^2+c)^2},$$

$$\nabla_{\partial_2}^{(1-f)} H_{\varphi\ 22}^{(1-f)} = \frac{3(5\sigma^4-2\sigma^2c+c^2)}{\sigma^2(\sigma^2+c)^2}, \qquad \nabla_{\partial_1}^{(1-f)} H_{\varphi\ 22}^{(1-f)} = \nabla_{\partial_2}^{(1-f)} H_{\varphi\ 12}^{(1-f)} = \nabla_{\partial_2}^{(1-f)} H_{\varphi\ 21}^{(1-f)} = 0.$$

The above equations imply

$$\nabla_{\partial_1}^{(1-f)} H_{\varphi\ 12}^{(1-f)} - \nabla_{\partial_2}^{(1-f)} H_{\varphi\ 11}^{(1-f)} = -\frac{4c}{(\sigma^2+c)^2} = -R_{121}^{(1-f)2}(\partial_2\varphi),$$

which gives us that the pair $(\nabla^{(1-f)}, H_\varphi^{(1-f)})$ isn't Codazzi-coupled.

Corollary 4.9. *The pairs $(\nabla^{(f)}, H^{(f)})$ and $(\nabla^{(1-f)}, H^{(1-f)})$ are Codazzi-coupled in a statistical manifold $(M, g, \nabla^{(f)})$, if least one of the following holds*

(1) *M is a f-flat statistical manifold.*
(2) *the function φ is constant.*

Theorem 4.10. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. If (∇, H) and (∇^*, H^*) are Codazzi-coupleds, then $(\nabla^{(f)}, H_\varphi^{(f)})$ is Codazzi-coupled if and only if*

$$f(1-f)([K_Y, K_X]Z)(\varphi) = Y(f)(K_XZ)(\varphi) - X(f)(K_YZ)(\varphi),$$

for any $X, Y, Z \in \chi(M)$.

Proof. Applying Proposition 3.1 in (4.9), it follows

$$\begin{aligned}(\nabla_X^{(f)} H_\varphi^{(f)})(Y,Z) - (\nabla_Y^{(f)} H_\varphi^{(f)})(X,Z) = &-(1-f)(R(X,Y)Z)(\varphi) - f(R^*(X,Y)Z)(\varphi) \\ &- f(1-f)([K_Y, K_X])Z(\varphi) - X(f)(K_YZ)(\varphi) \\ &+ Y(f)(K_XZ)(\varphi).\end{aligned}$$

Setting (4.10) and (4.11) in the above equation, we have

$$\begin{aligned}(\nabla^{(f)}_X H^{(f)}_\varphi)(Y,Z) - (\nabla^{(f)}_Y H^{(f)}_\varphi)(X,Z) =&(1-f)\{(\nabla_X H_\varphi)(Y,Z) - (\nabla_Y H_\varphi)(X,Z)\}\\ &+ f\{(\nabla^*_X H^*_\varphi)(Y,Z) - (\nabla^*_Y H^*_\varphi)(X,Z)\}\\ &- f(1-f)([K_Y, K_X])Z(\varphi) - X(f)(K_Y Z)(\varphi)\\ &+ Y(f)(K_X Z)(\varphi).\end{aligned}$$

If (∇, H) and (∇^*, H^*) are Codazzi-coupleds, the last equation leads to

$$\begin{aligned}(\nabla^{(f)}_X H^{(f)}_\varphi)(Y,Z) - (\nabla^{(f)}_Y H^{(f)}_\varphi)(X,Z) =&- f(1-f)([K_Y, K_X])Z(\varphi) - X(f)(K_Y Z)(\varphi)\\ &+ Y(f)(K_X Z)(\varphi),\end{aligned}$$

which gives us the assertion. □

Let (M, g) be a pseudo-Riemannian manifold with an affine connection ∇. The operator

$$\triangle^{\nabla}\varphi = div^{\nabla}(grade\varphi), \qquad \forall f, \varphi \in C^\infty(M), \tag{4.12}$$

is called *Laplacian*. In a statistical manifold $(M, g, \nabla^{(f)})$, to simplify we denote by $\triangle^{(f)}$ the operator Laplacian $\triangle^{\nabla^{(f)}}$ with respect to the connection $\nabla^{(f)}$. For $f = 0, 1$ and $f = \frac{1}{2}$, we use the notations $\triangle, \triangle^*$ and $\widehat{\triangle}$, respectively.

Proposition 4.11. *The Laplacian $\triangle^{(f)}$ is obtained by*

$$\triangle^{(f)}\varphi = tr((X,Y) \to H^{(1-f)}_\varphi(X,Y)),$$

for any $X, Y \in \chi(M)$.

Proof. Applying (4.7), it follows

$$tr((X,Y) \to H^{(1-f)}_\varphi(X,Y) = g(\nabla^{(f)}_X(grade\varphi), Y)) = div^{\nabla^{(f)}}(grade\varphi) = \triangle^{(f)}\varphi.$$

□

Applying (4.3), the Laplacian $\triangle^{(f)}\varphi$ can be written locally as

$$\triangle^{(f)}\varphi = g^{ij} H^{(1-f)}_{\varphi\ ij} = g^{ij}(\partial_i\partial_j\varphi - \widehat{\Gamma}^k_{ij}\partial_k\varphi - \frac{1-2f}{2}K^k_{ij}\partial_k\varphi).$$

(4.12) induces the dual Laplacian $\triangle^{(1-f)}\varphi$ as

$$\triangle^{(1-f)}\varphi = div^{\nabla^{(1-f)}}(grade\varphi) = tr((X,Y) \to H^{(f)}_\varphi(X,Y)).$$

Therefore, we can see

$$\triangle^{(f)}\varphi(X,Y) = (1-f)\triangle\varphi + f\triangle^*\varphi, \tag{4.13}$$

$$\triangle^{(1-f)}\varphi(X,Y) = f\triangle\varphi + (1-f)\triangle^*\varphi. \tag{4.14}$$

Proposition 4.12. *The operators $\triangle^{(f)}\varphi$ and $\triangle^{(1-f)}\varphi$ are related by*

$$\triangle^{(1-f)}\varphi - \triangle^{(f)}\varphi = (1-2f)\widetilde{K}(\varphi),$$

where $\widetilde{K} = tr((X,Y) \to (K_X Y))$, for any $X, Y \in \chi(M)$.

Proof. From Proposition 4.11, we have

$$\triangle^{(1-f)}\varphi - \triangle^{(f)}\varphi = tr((X,Y) \to H^{(f)}_\varphi(X,Y) - H^{(1-f)}_\varphi(X,Y)).$$

The above equation and (4.8) imply

$$\triangle^{(1-f)}\varphi - \triangle^{(f)}\varphi = tr((X,Y) \to (1-2f)(K_X Y)\varphi) = (1-2f)\widetilde{K}(\varphi). \tag{4.15}$$

□

Corollary 4.13. *On a statistical manifold $(M, g, \nabla^{(f)})$, we have*

(1) $\triangle^{(f)}\varphi = \triangle^{(1-f)}\varphi$ *if and only if* $\widetilde{K} = 0$.
(2) $\widehat{\triangle}\varphi = \frac{1}{2}(\triangle\varphi + \triangle^*\varphi)$.
(3) $\widetilde{K}(\varphi) = \triangle^*\varphi - \triangle\varphi$.
(4) $\triangle^{(f)}\varphi = \widehat{\triangle}\varphi - \frac{(1-2f)}{2}\widetilde{K}(\varphi)$.

Proof. The proof (1) is a consequence of Proposition 4.12. To prove (2), we have

$$\triangle\varphi + \triangle^*\varphi = div^{\nabla}(grade\varphi) + div^{\nabla^*}(grade\varphi). \tag{4.16}$$

On the other hand, considering $f = 0$ and $f = 1$ in (2.14) we get

$$\begin{aligned} div^{\nabla}(grade\varphi) &= div^{\widehat{\nabla}}(grade\varphi) - \frac{1}{2}(div^{\nabla^*}(grade\varphi) - div^{\nabla}(grade\varphi)),\\ div^{\nabla^*}(grade\varphi) &= div^{\widehat{\nabla}}(grade\varphi) + \frac{1}{2}(div^{\nabla^*}(grade\varphi) - div^{\nabla}(grade\varphi)). \end{aligned}$$

Setting the above two equations in (4.16), it follows

$$\triangle\varphi + \triangle^*\varphi = 2div^{\widehat{\nabla}}(grade\varphi) = 2\widehat{\triangle}\varphi.$$

Putting $f = 0$ in (4.15), (3) follows. Applying (2.14), we have

$$div^{\nabla^{(f)}}(grade\varphi) = div^{\widehat{\nabla}}(grade\varphi) - \frac{1-2f}{2}(div^{\nabla^*}(grade\varphi) - div^{\nabla}(grade\varphi)),$$

which gives

$$\triangle^{(f)}\varphi = \widehat{\triangle}\varphi - \frac{(1-2f)}{2}(\triangle^*\varphi - \triangle\varphi).$$

So (3) and the last equation imply (4). ☐

Example 4.14. For the normal statistical manifold $(M_1, g, \nabla^{(f)})$ described in Example 3.6, we consider $\varphi(\mu,\sigma) = \frac{1}{2}(\mu^2 + \sigma^2)$. So, we get

$$H_{\varphi}^{(f)} = (H_{\varphi\ ij}^{(f)}) = \begin{pmatrix} 1 - f(\mu,\sigma) & \frac{2\mu(1-f(\mu,\sigma))}{\sigma} \\ \frac{2\mu(1-f(\mu,\sigma))}{\sigma} & 4(1 - f(\mu,\sigma)) \end{pmatrix}, \quad H_{\varphi}^{(1-f)} = (H_{\varphi\ ij}^{(1-f)}) = \begin{pmatrix} f(\mu,\sigma) & \frac{2\mu f(\mu,\sigma)}{\sigma} \\ \frac{2\mu f(\mu,\sigma)}{\sigma} & 4f(\mu,\sigma) \end{pmatrix}.$$

It is easily seen that

$$\begin{aligned} \triangle^{(f)}\varphi &= 3f(\mu,\sigma)\sigma^2 = g^{11}H_{\varphi\ 11}^{(1-f)} + g^{22}H_{\varphi\ 22}^{(1-f)},\\ \triangle^{(1-f)}\varphi &= 3(1 - f(\mu,\sigma))\sigma^2 = g^{11}H_{\varphi\ 11}^{(f)} + g^{22}H_{\varphi\ 22}^{(f)}. \end{aligned}$$

As $\widetilde{K}(\varphi) = g^{11}K_{11}^2\partial_2(\varphi) + g^{22}K_{22}^2\partial_2(\varphi) = 3\sigma^2$, thus

$$\begin{aligned} &\triangle^{(1-f)}\varphi - \triangle^{(f)}\varphi = 3(1 - 2f(\mu,\sigma))\sigma^2 = (1 - 2f(\mu,\sigma))\widetilde{K}(\varphi),\\ &\widetilde{K}(\varphi) = 3\sigma^2 = \triangle^*\varphi - \triangle\varphi,\\ &\widehat{\triangle}\varphi = \frac{3\sigma^2}{2} = \frac{1}{2}(\triangle\varphi + \triangle^*\varphi),\\ &\triangle^{(f)}\varphi = 3f(\mu,\sigma)\sigma^2 = \widehat{\triangle}\varphi - \frac{(1 - 2f(\mu,\sigma))}{2}\widetilde{K}(\varphi). \end{aligned}$$

Therefore, we have Proposition 4.12 and Corollary 4.13.

Example 4.15. Considering $\varphi(\beta,\nu) = e^{\beta+\nu}$ in Example 3.14, we obtain

$$\begin{aligned} H_{\varphi}^{(f)} &= (H_{\varphi\ ij}^{(f)})\\ &= \begin{pmatrix} \frac{e^{\beta+\nu}\{\beta(\psi'(\nu)\nu-1)(\beta+2(1-f(\beta,\nu)))+f(\beta,\nu)\nu\}}{\beta^2(\psi'(\nu)\nu-1)} & \frac{e^{\beta+\nu}(\nu-1+f(\beta,\nu))}{\nu} \\ \frac{e^{\beta+\nu}(\nu-1+f(\beta,\nu))}{\nu} & -\frac{e^{\beta+\nu}\{\nu(-\psi'(\nu)\nu+1)+f(\beta,\nu)(\psi''^2+1)\}}{\nu(\psi'(\nu)\nu-1)} \end{pmatrix}. \end{aligned}$$

It is easily seen that

$$\triangle^{(f)}\varphi=\frac{e^{\beta+\nu}\{\beta(\beta+2f(\beta,\nu))(\psi'(\nu)\nu(\psi'(\nu)\nu-2)+1)+\nu(1-f(\beta,\nu))(\psi'(\nu)\nu-2-\psi''(\nu)\nu^2)+\psi'(\nu)\nu^3-\nu^2\}}{\nu(\psi'(\nu)\nu-1)^2},$$

and hence

$$\begin{aligned}\triangle^{(1-f)}\varphi-\triangle^{(f)}\varphi&=\frac{e^{\beta+\nu}(1-2f(\beta,\nu))\{\nu(2\beta\psi'(\nu)-1)(\psi'(\nu)\nu-2)+2\beta+\psi''(\nu)\nu^3\}}{\nu(\psi'(\nu)\nu-1)^2}\\&=(1-2f(\beta,\nu))\widetilde{K}(\varphi).\end{aligned}$$

In particular, for $f=0$, it follows

$$\widetilde{K}(\varphi)=\frac{e^{\beta+\nu}\{\nu(2\beta\psi'(\nu)-1)(\psi'(\nu)\nu-2)+2\beta+\psi''(\nu)\nu^3\}}{\nu(\psi'(\nu)\nu-1)^2}=\triangle^*\varphi-\triangle\varphi.$$

In addition, we find

$$\begin{aligned}\widehat{\triangle}\varphi&=\frac{e^{\beta+\nu}\{\nu(\psi'(\nu)\nu-2)(2\beta\psi'(\nu)(\beta+1)+1)+2\nu^2(\psi'(\nu)\nu-1)+2\beta(\beta+1)-\psi''(\nu)\nu^3\}}{\nu(\psi'(\nu)\nu-1)^2}\\&=\frac{1}{2}(\triangle\varphi+\triangle^*\varphi),\end{aligned}$$

and

$$\begin{aligned}\triangle^{(f)}\varphi&=\frac{e^{\beta+\nu}\{\beta(\beta+2f(\beta,\nu))(\psi'(\nu)\nu(\psi'(\nu)\nu-2)+1)+\nu(1-f(\beta,\nu))(\psi'(\nu)\nu-2-\psi''(\nu)\nu^2)+\psi'(\nu)\nu^3-\nu^2\}}{\nu(\psi'(\nu)\nu-1)^2}\\&=\widehat{\triangle}\varphi-\frac{(1-2f(\beta,\nu))}{2}\widetilde{K}(\varphi).\end{aligned}$$

Definition 4.16. Let $(M,g,\nabla^{(f)})$ be a statistical manifold and $\varphi\in C^\infty(M)$. The function φ is called f-harmonic if $\triangle^{(f)}\varphi=0$.

Corollary 4.17. *For a statistical manifold $(M,g,\nabla^{(f)})$ equipped with a f-harmonic function $\varphi\in C^\infty(M)$, we have*

(1) $\widetilde{K}(\varphi)=\widetilde{K}(\varphi^2)=0$.

(2) $\triangle^{(f)}(\varphi^2)=2||grade\varphi||^2$, *where* $||grade\varphi||^2=g(grade\varphi,grade\varphi)$.

Proof. Since $\triangle^{(f)}\varphi=0$, it follows $\triangle^{(1-f)}\varphi=0$. Thus (4.15) yields $\widetilde{K}(\varphi)=0$. On the other hand, we have $\widetilde{K}(\varphi^2)=\widetilde{K}^r\partial_r(\varphi^2)=2\varphi\widetilde{K}^r\partial_r(\varphi)=2\varphi\widetilde{K}(\varphi)=0$, which gives us (1). Using (4.12), we have

$$\triangle^{(f)}(\varphi^2)=div^{\nabla^{(f)}}(grade\varphi^2).$$

As $grade\varphi^2=2\varphi grade\varphi$, the above equation implies

$$\triangle^{(f)}(\varphi^2)=2\varphi div^{\nabla^{(f)}}(grade\varphi)+2g(grade\varphi,grade\varphi)=2\varphi\triangle^{(f)}\varphi+2||grade\varphi||^2.$$

Considering $\triangle^{(f)}\varphi=0$ in the last equation, (2) is obtained. □

Theorem 4.18. *Let $(g,\nabla^{(f)})$ be a statistical structure on a compact oriented manifold M such that $\partial M=0$. If $\varphi\in C^\infty(M)$ is f-harmonic, then φ is constant.*

Proof. The part (4) of Corollary 4.13 implies

$$\triangle^{(f)}(\varphi^2)=\widehat{\triangle}(\varphi^2)-\frac{(1-2f)}{2}\widetilde{K}(\varphi^2).$$

Applying Corollary 4.17, it follows

$$\widehat{\triangle}(\varphi^2)=2||grade\varphi||^2.$$

Integrating we get

$$\int_M\widehat{\triangle}(\varphi^2)\omega=2\int_M||grade\varphi||^2\omega,$$

where ω is a volume element on M. As $\partial M = 0$, the divergence theorem leads to

$$\int_M \widehat{\triangle}(\varphi^2)\omega = \int_M div^{\widehat{\nabla}}(grade\varphi)\omega = 0.$$

Therefore, we conclude $\int_M ||grade\varphi||^2\omega = 0$, and consequently $grade\varphi = 0$. Hence φ is constant. □

Theorem 4.19. *On a compact oriented statistical manifold $(M, g, \nabla^{(f)})$ with $\partial M = 0$ and a volume element ω, if $\varphi \in C^\infty(M)$ is non-constant and*

(1) 0-*harmonic, then*

$$\int_M \widetilde{K}(\varphi^2)\omega < 0.$$

(2) 1-*harmonic, then*

$$\int_M \widetilde{K}(\varphi^2)\omega > 0.$$

Proof. To prove (1), Corollary 4.13 shows that $\triangle(\varphi^2) = \widehat{\triangle}(\varphi^2) - \frac{1}{2}\widetilde{K}(\varphi^2)$. This and Corollary 4.17 together the divergence theorem give

$$-\frac{1}{2}\int_M \widetilde{K}(\varphi^2) = 2\int_M ||grade\varphi||^2\omega.$$

Since $||grade\varphi||^2 > 0$, the above equation leads to (1). By a similar argument, we get (2). □

5. Miao-Tam statistical manifolds

Proposition 5.1. *Let (M, g, ∇) be a statistical manifold. If $\varphi \in C^\infty(M)$, then we have*

$$\begin{aligned}-\triangle\varphi g(X,Y) + H^*_\varphi(X,Y) - \varphi L(X,Y) = &- \widehat{\triangle}\varphi g(X,Y) + \widehat{H}_\varphi(X,Y) - \varphi\widehat{Ric}(X,Y) \\ &+ \frac{1}{2}\{\widetilde{K}(\varphi)g(X,Y) - K_XY(\varphi) - \frac{1}{2}\varphi(\tau_g(K_XY) \\ &- g(K_X, K_Y))\},\end{aligned}$$

for any $X, Y \in \chi(M)$.

Proof. Applying (3.14) and Corollaries 4.5 and 4.13, we obtain the assertion. □

In the above proposition, we note that if (g, φ) satisfies the Miao-Tam equation, then

$$\begin{aligned}-\triangle\varphi g(X,Y) + H^*_\varphi(X,Y) - \varphi L(X,Y) = &g(X,Y) + \frac{1}{2}\{\widetilde{K}(\varphi)g(X,Y) - K_XY(\varphi) \\ &- \frac{1}{2}\varphi(\tau_g(K_XY) - g(K_X, K_Y))\}.\end{aligned}$$

Definition 5.2. A triplet (g, ∇, φ) is called a Miao-Tam statistical structure on M if (g, ∇) is a statistical structure, (g, φ) satisfies the Miao-Tam equation and the following condition holds

$$\widetilde{K}(\varphi)g(X,Y) - K_XY(\varphi) - \frac{1}{2}\varphi(\tau_g(K_XY) - g(K_X, K_Y)) = 0, \tag{5.1}$$

for any $X, Y \in \chi(M)$.

Example 5.3. We consider the four-dimensional bivariate Gaussian manifold

$$M_4 = \{p(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2) | p(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2) = \frac{1}{2\pi\sqrt{\sigma_1\sigma_2}} e^{-\frac{1}{2\sigma_1\sigma_2}\left(\sigma_2(x-\mu_1)^2 + \sigma_1(y-\mu_2)^2\right)}\},$$

defined on $-\infty < x, y < \infty$, where $-\infty < \mu_1, \mu_2 < \infty$ and $0 < \sigma_1, \sigma_2 < \infty$. M_4 forms an exponential family with natural coordinate system

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\frac{\mu_1}{\sigma_1}, \frac{\mu_2}{\sigma_2}, -\frac{1}{2\sigma_1}, -\frac{1}{2\sigma_2}),$$

and potential function $\psi = log(2\pi\sqrt{D}) - D(\theta_2^2\theta_3 + \theta_1^2\theta_4)$, where $D = \frac{1}{4\theta_3\theta_4}$ (see [2]). The Fisher metric on M_4 is determined by

$$(g_{ij}) = \begin{pmatrix} \sigma_1 & 0 & 2\mu_1\sigma_1 & 0 \\ 0 & \sigma_2 & 0 & 2\mu_2\sigma_2 \\ 2\mu_1\sigma_1 & 0 & 2\sigma_1(2\mu_1^2+\sigma_1) & 0 \\ 0 & 2\mu_2\sigma_2 & 0 & 2\sigma_2(2\mu_2^2+\sigma_2) \end{pmatrix}.$$

The non-zero components $\widehat{\Gamma}_{ij}^k$ of the Levi-Civita connection $\widehat{\nabla}$ on M_4 are given by

$$\widehat{\Gamma}_{11}^1 = -\widehat{\Gamma}_{13}^3 = -\widehat{\Gamma}_{31}^3 = -\mu_1,\ \widehat{\Gamma}_{13}^1 = \widehat{\Gamma}_{31}^1 = \sigma_1 - 2\mu_1^2,\ \widehat{\Gamma}_{33}^1 = -4\mu_1^3,\ \widehat{\Gamma}_{11}^3 = \frac{1}{2},\quad \widehat{\Gamma}_{33}^3 = 2(\mu_1^2+\sigma_1),$$

$$\widehat{\Gamma}_{22}^2 = -\widehat{\Gamma}_{24}^4 = -\widehat{\Gamma}_{42}^4 = -\mu_2,\ \widehat{\Gamma}_{24}^2 = \widehat{\Gamma}_{42}^2 = \sigma_2 - 2\mu_2^2,\ \ \widehat{\Gamma}_{44}^2 = -4\mu_2^3,\ \widehat{\Gamma}_{22}^4 = \frac{1}{2},\ \widehat{\Gamma}_{44}^4 = 2(\mu_2^2+\sigma_2).$$

The curvature tensor $\widehat{R}$ satisfies the following

$$\widehat{R}_{1313} = \sigma_1^3, \qquad \widehat{R}_{2424} = \sigma_2^3,$$

while the other independent components are zero. In addition, the Ricci tensor $\widehat{Ric}$ is described by

$$(\widehat{Ric}(\partial_i,\partial_j)) = -\begin{pmatrix} \frac{\sigma_1}{2} & 0 & \mu_1\sigma_1 & 0 \\ 0 & \frac{\sigma_2}{2} & 0 & \mu_2\sigma_2 \\ \mu_1\sigma_1 & 0 & \sigma_1(2\mu_1^2+\sigma_1) & 0 \\ 0 & \mu_2\sigma_2 & 0 & \sigma_2(2\mu_2^2+\sigma_2) \end{pmatrix}. \tag{5.2}$$

Let $\varphi = ae^{\theta_1+\theta_2+\theta_3+\theta_4} + b$, where a and b are constants. So, it follows

$$-\widehat{\triangle}\varphi g(\partial_1,\partial_2) + \widehat{H}_\varphi(\partial_1,\partial_2) - \varphi\widehat{Ric}(\partial_1,\partial_2) = ae^{\theta_1+\theta_2+\theta_3+\theta_4}.$$

Thus $ae^{\theta_1+\theta_2+\theta_3+\theta_4} = 0 = g(\partial_1,\partial_2)$ if and only if $a = 0$. In this case, we also obtain

$$\begin{aligned} -\widehat{\triangle}\varphi g(\partial_1,\partial_1) + \widehat{H}_\varphi(\partial_1,\partial_1) - \varphi\widehat{Ric}(\partial_1,\partial_1) &= \frac{b}{2}\sigma_1, \\ -\widehat{\triangle}\varphi g(\partial_1,\partial_3) + \widehat{H}_\varphi(\partial_1,\partial_3) - \varphi\widehat{Ric}(\partial_1,\partial_3) &= b\mu_1\sigma_1, \\ -\widehat{\triangle}\varphi g(\partial_2,\partial_2) + \widehat{H}_\varphi(\partial_2,\partial_2) - \varphi\widehat{Ric}(\partial_2,\partial_2) &= \frac{b}{2}\sigma_2, \\ -\widehat{\triangle}\varphi g(\partial_2,\partial_4) + \widehat{H}_\varphi(\partial_2,\partial_4) - \varphi\widehat{Ric}(\partial_2,\partial_4) &= b\mu_2\sigma_2, \\ -\widehat{\triangle}\varphi g(\partial_3,\partial_3) + \widehat{H}_\varphi(\partial_3,\partial_3) - \varphi\widehat{Ric}(\partial_3,\partial_3) &= b\sigma_1(\mu_1^2+\sigma_1), \\ -\widehat{\triangle}\varphi g(\partial_4,\partial_4) + \widehat{H}_\varphi(\partial_4,\partial_4) - \varphi\widehat{Ric}(\partial_4,\partial_4) &= b\sigma_2(\mu_2^2+\sigma_2). \end{aligned}$$

According to (1.3) and the above equations, (g,φ) satisfies the Miao-Tam equation if and only if $b = 2$. Setting the non-zero components of a $(1,2)$-tensor field K on M_4 as

$$K_{11}^1 = K_{13}^3 = K_{31}^3 = 1, \quad K_{33}^1 = 2(2\mu_1^2+\sigma_1),$$

we can see that $(M_4, g, \nabla = \widehat{\nabla} - \frac{1}{2}K)$ is a statistical manifold. On the other hand, we get

$$g^{mn}K_{mn}^r\partial_r\varphi\ g_{ij} - K_{ij}^r\partial_r\varphi - \frac{1}{2}\varphi(K_{ij}^rK_{lr}^l - K_{il}^rK_{jr}^l) = 0, \qquad \forall i,j = 1,2,3,4,$$

i.e., (5.1) holds and $(M_4, g, \nabla, \varphi)$ is a Miao-Tam statistical manifold.

Lemma 5.4. *For a Miao-Tam statistical manifold (M, g, ∇, φ), we have*

$$\widetilde{K}(\varphi) = \frac{\varphi}{2(n-1)}tr\{(X,Y) \to (\tau_g(K_XY) - g(K_X,K_Y))\}, \tag{5.3}$$

$$\triangle\varphi = -\frac{1}{2}\widetilde{K}(\varphi) + \frac{1}{1-n}(n+\varphi\widehat{\sigma}), \tag{5.4}$$

where $\widehat{\sigma} = tr\{(X,Y) \to \widehat{Ric}(X,Y)\}$ is scalar curvature for any $X, Y \in \chi(M)$.

Proof. According to (5.1), we can write

$$tr\{(X,Y) \to \widetilde{K}(\varphi)g(X,Y) - K_X Y(\varphi)\} = \frac{\varphi}{2} tr\{(X,Y) \to (\tau_g(K_X Y) - g(K_X, K_Y))\},$$

which gives

$$(n-1)\widetilde{K}(\varphi) = \frac{\varphi}{2} tr\{(X,Y) \to (\tau_g(K_X Y) - g(K_X, K_Y))\}.$$

Thus (5.3) holds. Tracing

$$-\triangle\varphi\, g + H^*_\varphi - \varphi L = g, \tag{5.5}$$

we get

$$(1-n)\triangle\varphi - \varphi\widehat{\sigma} + \frac{1-n}{2}\widetilde{K}(\varphi) = n.$$

From the above equation, (5.4) follows. □

Proposition 5.5. *If the pair (∇^*, L) is Codazzi-coupled in a Miao-Tam statistical manifold (M, g, ∇, φ), then we have*

$$\begin{aligned}(R^*(X,Y)Z)(\varphi) =& \frac{\widehat{\sigma}}{n-1}(X(\varphi)g(Y,Z) - Y(\varphi)g(X,Z)) + Y(\varphi)L(X,Z) - X(\varphi)L(Y,Z)\\ &+ \frac{1}{2}(X(\widetilde{K}(\varphi))g(Y,Z) - Y(\widetilde{K}(\varphi))g(X,Z)),\end{aligned}$$

for any $X, Y, Z \in \chi(M)$.

Proof. Effecting ${\nabla_X}^*$ on both sides of (5.5), we get

$$\begin{aligned}&-(\nabla^*_X\triangle\varphi)g(Y,Z) - (\triangle\varphi)\, \mathfrak{C}^*(X,Y,Z) + (\nabla^*_X H^*_\varphi)(Y,Z) - (\nabla^*_X\varphi)L(Y,Z) - \varphi(\nabla^*_X L)(Y,Z)\\ &= \mathfrak{C}^*(X,Y,Z).\end{aligned}$$

Switching X and Y in the above equation and subtract the result from it, we obtain

$$\begin{aligned}\varphi((\nabla^*_X L)(Y,Z) - (\nabla^*_Y L)(X,Z)) =& ((\nabla^*_Y\triangle\varphi)g(X,Z) - (\nabla^*_X\triangle\varphi)g(Y,Z)) + (\nabla^*_X H^*_\varphi)(Y,Z)\\ &- (\nabla^*_Y H^*_\varphi)(X,Z) + (\nabla^*_Y\varphi)L(X,Z) - (\nabla^*_X\varphi)L(Y,Z).\end{aligned}$$

As the scalar curvature $\widehat{\sigma}$ is constant [12], (5.4) yields

$$\nabla^*_X\triangle\varphi = -\frac{1}{2}\nabla^*_X\widetilde{K}(\varphi) + \frac{\widehat{\sigma}}{1-n}\nabla^*_X\varphi.$$

The above two equations and (4.11) imply

$$\begin{aligned}\varphi((\nabla^*_X L)(Y,Z) - (\nabla^*_Y L)(X,Z)) =& -(R^*(X,Y)Z)(\varphi) + \frac{\widehat{\sigma}}{n-1}(X(\varphi)g(Y,Z) - Y(\varphi)g(X,Z))\\ &+ Y(\varphi)L(X,Z) - X(\varphi)L(Y,Z) + \frac{1}{2}(X(\widetilde{K}(\varphi))g(Y,Z)\\ &- Y(\widetilde{K}(\varphi))g(X,Z)).\end{aligned}$$

Since (∇^*, L) is Codazzi-coupled, the last equation gives the assertion. □

Proposition 5.6. *Let (M, g, ∇, φ) be a Miao-Tam statistical manifold. Then $(M, g, \nabla^*, \varphi)$ is a Miao-Tam statistical manifold if and only if*

$$\widetilde{K}(\varphi)X = K_X(grade\varphi), \tag{5.6}$$

for any $X \in \chi(M)$. Moreover, the following holds

$$\tau_g(K_X Y) = g(K_X, K_Y). \tag{5.7}$$

Proof. To prove, we first let that $(M, g, \nabla^*, \varphi)$ is a Miao-Tam statistical manifold, i.e.,

$$-\triangle^*\varphi\; g + H_\varphi - \varphi L^* = g. \tag{5.8}$$

According to (4.8) and (4.15), we have

$$H_\varphi(X,Y) = H^*_\varphi(X,Y) + (K_XY)\varphi, \qquad \triangle^*\varphi = \triangle\varphi + \widetilde{K}(\varphi).$$

The above equations imply

$$-\triangle\varphi g(X,Y) - \widetilde{K}(\varphi)g(X,Y) + H^*_\varphi(X,Y) + (K_XY)\varphi - \varphi L^*(X,Y) = g(X,Y).$$

Thus from $L = L^*$ and using (5.5), it follows

$$\widetilde{K}(\varphi)g(X,Y) - (K_XY)\varphi = 0.$$

Applying the non-degenerate property of g, the above equation yields (5.6). On the other hand, since the above equations are invertible, we have (5.8). (5.1) implies (5.7). Thus the proof is complete. □

Proposition 5.7. *Let (g, ∇, φ) and (g, ∇^*, φ) be Miao-Tam statistical structures on a manifold M and $f \in C^\infty(M)$. Then the quadruple $(M, g, \nabla^{(f)}, \varphi)$ is a Miao-Tam statistical manifold.*

Proof. Using Corollaries 4.5, 4.13 and (3.13), we have

$$-\triangle^{(f)}\varphi\; g(X,Y) + H^{(1-f)}_\varphi(X,Y) - \varphi L^{(f)}(X,Y) = -\widehat{\triangle}\varphi g(X,Y) + \widehat{H}_\varphi(X,Y) - \varphi\widehat{Ric}(X,Y)$$
$$+ \frac{1-2f}{2}\{\widetilde{K}(\varphi)g(X,Y) - K_XY(\varphi) - \frac{1-2f}{2}\varphi(\tau_g(K_XY) - g(K_X, K_Y))\}.$$

Applying (1.3), (5.6) and (5.7), the above equation yields

$$-\triangle^{(f)}\varphi\; g(X,Y) + H^{(1-f)}_\varphi(X,Y) - \varphi L^{(f)}(X,Y) = g(X,Y).$$

Thus this completes the proof. □

Theorem 5.8. *Let $(M, g, \nabla^{(f)})$ be a statistical manifold. If (g, ∇, φ) and (g, ∇^*, φ) are the Miao-Tam statistical structures on M, then we have*

$$L^{(f)} = L = \widehat{Ric}.$$

Proof. Setting (5.7) in (3.13) and (3.14), we deduce the assertion. □

Example 5.9. For the bivariate Gaussian manifold M_4 with the Miao-Tam statistical structure (g, ∇, φ), it follows that (5.6) holds. Hence $(M_4, g, \nabla^*, \varphi)$ forms a Miao-Tam statistical manifold. One can see that

$$\tau_g(K_{\partial_i}\partial_j) = g(K_{\partial_i}, K_{\partial_j}), \qquad \forall i,j = 1,2,3,4,$$

except

$$\tau_g(K_{\partial_1}\partial_1) = 2 = g(K_{\partial_1}, K_{\partial_1}),$$
$$\tau_g(K_{\partial_3}\partial_3) = 4(2\mu_1^2 + \sigma_1) = g(K_{\partial_3}, K_{\partial_3}),$$

i.e., (5.7) holds. For any $f := f(\theta_1, \theta_2, \theta_3, \theta_4)$ on M_4, using Corollary 2.3, we obtain

$$\Gamma^{(f)1}_{11} = -\Gamma^{(f)3}_{13} = -\Gamma^{(f)3}_{31} = -\mu_1 - \frac{1}{2} + f, \quad \Gamma^{(f)1}_{13} = \Gamma^{(f)1}_{31} = \sigma_1 - 2\mu_1^2, \quad \Gamma^{(f)3}_{11} = \Gamma^{(f)4}_{22} = \frac{1}{2},$$
$$\Gamma^{(f)1}_{33} = -4\mu_1^3 + (-1+2f)(2\mu_1^2 + \sigma_1), \quad \Gamma^{(f)2}_{24} = \Gamma^{(f)2}_{42} = \sigma_2 - 2\mu_2^2, \quad \Gamma^{(f)2}_{44} = -4\mu_2^3,$$
$$\Gamma^{(f)2}_{22} = -\Gamma^{4}_{24} = -\Gamma^{4}_{42} = -\mu_2, \quad \Gamma^{(f)3}_{33} = 2(\mu_1^2 + \sigma_1), \quad \Gamma^{(f)4}_{44} = 2(\mu_2^2 + \sigma_2),$$

and other components are zero. It is obvious that $\mathcal{C}^{(f)}_{ijk} = 0, i,j,k = 1,2,3,4,$ unless

$$\mathcal{C}^{(f)}_{111} = (1-2f)\sigma_1, \qquad \mathcal{C}^{(f)}_{113} = \mathcal{C}^{(f)}_{131} = \mathcal{C}^{(f)}_{311} = 2(1-2f)\mu_1\sigma_1,$$
$$\mathcal{C}^{(f)}_{333} = 4(1-2f)\mu_1\sigma_1(\mu_1^2 + \sigma_1), \qquad \mathcal{C}^{(f)}_{133} = \mathcal{C}^{(f)}_{313} = \mathcal{C}^{(f)}_{331} = 2(1-2f)\sigma_1(\mu_1^2 + \sigma_1),$$

so $(M, g, \nabla^{(f)})$ is a statistical manifold. The non-zero components of f-curvature tensor $R^{(f)}$ are obtained as

$$R^{(f)1}_{121} = R^{(f)3}_{123} = R^{(f)3}_{321} = -\partial_2 f, \quad R^{(f)1}_{233} = 2(2\mu_1^2 + \sigma_1)\partial_2 f, \quad R^{(f)1}_{131} = -\mu_1\sigma_1 - \partial_3 f + 2\mu_1^2(1-2f),$$

$$R^{(f)1}_{141} = R^{(f)3}_{143} = R^{(f)3}_{341} = -\partial_4 f, \quad R^{(f)1}_{433} = 2(2\mu_1^2 + \sigma_1)\partial_4 f, \quad R^{(f)3}_{131} = \frac{1}{2}\sigma_1 + \partial_1 f + \mu_1(-1+2f),$$

$$R^{(f)2}_{422} = R^{(f)4}_{244} = pq, \qquad R^{(f)2}_{244} = -\sigma_2(2\mu_2^2 + \sigma_2), \quad R^{(f)3}_{133} = \mu_1\sigma_1 - \partial_3 f + 2\mu_1^2(-1+2f),$$

$$R^{(f)4}_{242} = \frac{1}{2}\sigma_2, \qquad R^{(f)1}_{133} = (\sigma_1 + 2\mu_1^2)(2\partial_1 f + 2\mu_1(1-2f) - \sigma_1),$$

where $R^{(f)r}_{ijk} = R^{(f)r}_{jik}, i, j, k, r = 1, 2, 3, 4$. Thus, it follows that

$$(Ric^{(f)}(\partial_i, \partial_j)) = \begin{pmatrix} -\frac{1}{2}\sigma_1 - \partial_1 f + \mu_1(1-2f) & 0 & -\mu_1\sigma_1 + \partial_3 f + 2\mu_1^2(1-2f) & 0 \\ -2\partial_2 f & -\frac{1}{2}\sigma_2 & 0 & -\mu_2\sigma_2 \\ -\mu_1\sigma_1 - \partial_3 f + 2\mu_1^2(1-2f) & 0 & (\sigma_1 + 2\mu_1^2)(2\partial_1 f + 2\mu_1(1-2f) - \sigma_1) & 0 \\ -2\partial_4 f & -\mu_2\sigma_2 & 0 & -\sigma_2(2\mu_2^2 + \sigma_2) \end{pmatrix}.$$

As $L^{(f)}(\partial_i, \partial_j) = \frac{1}{2}(Ric^{(f)} + Ric^{(1-f)})(\partial_i, \partial_j)$, we get

$$(L^{(f)}(\partial_i, \partial_j)) = -\begin{pmatrix} \frac{\sigma_1}{2} & 0 & \mu_1\sigma_1 & 0 \\ 0 & \frac{\sigma_2}{2} & 0 & \mu_2\sigma_2 \\ \mu_1\sigma_1 & 0 & \sigma_1(2\mu_1^2 + \sigma_1) & 0 \\ 0 & \mu_2\sigma_2 & 0 & \sigma_2(2\mu_2^2 + \sigma_2) \end{pmatrix} = (L(\partial_i, \partial_j)).$$

This and (5.2) imply $L^{(f)} = L = \widehat{Ric}$. Hence for $\varphi = 2$, we see that $-\triangle^{(f)}\varphi\, g(\partial_i, \partial_j) + H^{(1-f)}_{\varphi}(\partial_i, \partial_j) - \varphi L^{(f)}(\partial_i, \partial_j) = g(\partial_i, \partial_j), i, j = 1, 2, 3, 4$. Therefore, $(M_4, g, \nabla^{(f)}, 2)$ is a Miao-Tam statistical manifold.

Conflict of interest

The authors declare no conflict of interest in this paper.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study

References

[1] S. Amari, *Information geometry of the EM and em algorithms for neural networks*, Neural Networks **8** (9), 1379–1408, 1995.

[2] K. Arwini and C.T.J. Dodson, *N*eighbourhoods of independence and associated geometry in manifolds of bivariate Gaussian and Freund distributions, Open Math. **5** (1), 50–83, 2007.

[3] B. Balcerzak, *Linear connection and secondary characteristic classes of Lie algebroids*, Monographs of Lodz University of Technology, 2021. Doi:10.34658/9788366741287.

[4] H. Baltazar and A. Da Silva, *On static manifolds and related critical spaces with cyclic parallel Ricci tensor*, Advances in Geometry **21** (3), 443–450, 2021.

[5] A. Barros, R. Diógenes and E. Ribeiro Jr, *Bach-Flat Critical Metrics of the Volume Functional on 4-Dimensional Manifolds with Boundary*, J. Geom. Anal. **25**, 2698–2715, 2015.

[6] M. Belkin, P. Niyogi and V. Sindhwani, *Manifold regularization: a geometric framework for learning from labeled and unlabeled examples*, Journal of Machine Learning Research **7**, 2399–2434, 2006.

[7] O. Calin and C. Udrişte, *Geometric Modeling in Probability and Statistics*, Springer, Cham, Switzerland, 2014.

[8] A. Caticha, *Geometry from information geometry*, arxiv.org/abs/1512.09076v1.
[9] A. Caticha, *The information geometry of space and time*, Proceedings, **33** (1), 15, 2019.
[10] R. A. Fisher, *On the mathematical foundations of theoretical statistics*, Phil. Trans. Roy. Soc. London. **222**, 309–368, 1922.
[11] S. Kobayashi and Y. Ohno ,*On a constant curvature statistical manifold*, Inf. Geom. **5** (1), 31-46, 2022.
[12] P. Miao and L.F. Tam, *On the volume functional of compact manifolds with boundary with constant scalar curvature*, Calc. Var. PDE. **36**, 141–171, 2009.
[13] P. Miao and L.F. Tam, *Einstein and conformally flat critical metrics of the volume functional*, Trans. Amer. Math. Soc. **363**, 2907–2937, 2011.
[14] B. Opozda, *B*ochners technique for statistical structures, Ann. Global Anal. Geom. **48**, 357–395, 2015.
[15] K. Sun and S. Marchand-Maillet, *An information geometry of statistical manifold learning*, Proceedings of the 31st International Conference on Machine Learning (ICML-14), 1–9, 2014.